INCOMPLETENESS OF A FREE ARITHMETIC

Ermanno BENCIVENGA

For every natural number n, let $\mathbf{n}$ be the corresponding numeral s ... $s(0)$. The rules of the theory *FQ* are Modus Ponens and

$$\text{(R1)}\quad \frac{\vdash A(0/x) \qquad \vdash \forall x(A \supset A(s(x)/x)}{\vdash \forall x A}$$

The axiom-schemata of *FQ* are (A0)-(A11) in [1], plus

(A12) $(t = \mathbf{i} \vee t < \mathbf{i}) \supset \exists x(x = t)$
(A13) $t < \mathbf{i} \supset (t = 0 \vee ... \vee t = \mathbf{i-1})$
(A14) $t < \mathbf{i} \vee t = \mathbf{i} \vee \mathbf{i} < t$

The consistency of *FQ* is a simple consequence of the following

Lemma 1. Let *FQn* be the subtheory of *FQ* which results from eliminating all the axioms of the form (A12), where $n < i$. *FQn* is consistent.

The proof of Lemma 1 proceeds as in [1], with the following addition to the definition of the model M:

$$f(<) = \{<m, n>: m < n\}$$

For the sake of illustration, consider an instance of (A12). Suppose that $V^{\mathrm{v}}_M(\exists x(x = t)) = \mathrm{F}$. Then $W^{\mathrm{v}}_M(t) \notin D$, and hence $= n + 1$. It is easy to see that $W^{\mathrm{v}}_M(\mathbf{i}) = i$ for all i such that it is not the case that $n < i$, and hence that, for all such i, $W^{\mathrm{v}}_M(\mathbf{i}) < W^{\mathrm{v}}_M(t)$. Therefore, $V^{\mathrm{v}}_M(t = \mathbf{i}) = V^{\mathrm{v}}_M(t < \mathbf{i}) = \mathrm{F}$.

Let an n-place arithmetical function f be *representable* in *FQ* iff there is a formula A of the language of *FQ*, containing exactly $n+1$ variables $x_1, ..., x_n, y$ free and such that, whenever $f(i_1, ..., i_n) = j$, $\vdash_{FQ} A(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n t/y) \equiv t = \mathbf{j}$.

Lemma 2. All recursive functions are representable in *FQ*.

Proof. Given the results in [2], we can limit ourselves to proving that all Recursive functions are representable in *FQ*. There are six parts to this proof.

Part I: All identity functions are representable in *FQ*. Trivial.

Part II: Addition is representable in *FQ*, by the formula $x + y = z$. For suppose that $i + j = k$. We prove by induction on j that $\vdash_{FQ} \mathbf{i} + \mathbf{j} = \mathbf{k}$. If $j = 0$ then $\vdash_{FQ} \mathbf{i} + \mathbf{j} = \mathbf{k}$ by (A8). If $j = s(m)$ then $k = s(n)$ and $i + m = n$. By the induction hypothesis $\vdash_{FQ} \mathbf{i} + \mathbf{m} = \mathbf{n}$, and hence by the logic of identity $\vdash_{FQ} s(\mathbf{i} + \mathbf{m}) = \mathbf{k}$. But by (A9) $\vdash_{FQ} s(\mathbf{i} + \mathbf{m}) = \mathbf{i} + \mathbf{j}$. Thus $\vdash_{FQ} \mathbf{i} + \mathbf{j} = \mathbf{k}$, and by the logic of identity $\vdash_{FQ} \mathbf{i} + \mathbf{j} = t \equiv t = \mathbf{k}$.

Part III: Multiplication is representable in *FQ*, by the formula $x \cdot y = z$. Proof analogous to Part II.

Part IV: The characteristic function of identity $f_=$ is representable in *FQ*, by the formula $(x = y \mathbin{\&} z = \mathbf{1}) \vee (x \neq y \mathbin{\&} z = 0)$. First, let $f_=(i, j) = 1$. Then $i = j$, and hence $\vdash_{FQ} \mathbf{i} = \mathbf{j} \mathbin{\&} \mathbf{1} = \mathbf{1}$. Second, let $f_=(i, j) = 0$. Then $i \neq j$, and we prove by induction on i (assuming, without loss of generality, that $i < j$) that $\vdash_{FQ} \mathbf{i} \neq \mathbf{j}$. If $i = 0$ then, for some m, $j = s(m)$. By (A12), $\vdash_{FQ} \exists x(x = \mathbf{m})$; hence, by (A6), $\vdash_{FQ} \mathbf{j} \neq 0$. If $i = s(m)$ for some m then $j = s(n)$ for some n, and $m \neq n$. By the induction hypothesis $\vdash_{FQ} \mathbf{m} \neq \mathbf{n}$, and hence by (A7) $\vdash_{FQ} \mathbf{i} \neq \mathbf{j}$. Then $\vdash_{FQ} \mathbf{i} \neq \mathbf{j} \mathbin{\&} 0 = 0$.

Part V: Let A represent the m-ary function f in *FQ*, and let $B_1, \ldots, B_m$ represent the n-ary functions $g_1, \ldots, g_m$, respectively. Let h be obtained by composition from $f, g_1, \ldots, g_m$. Then $C = \exists y_1 \ldots \exists y_m(B_1(y_1/y) \mathbin{\&} \ldots \mathbin{\&} B_m(y_m/y) \mathbin{\&} A(y_1/x_1 .. y_m/x_m))$ represents h. (Through all this part, we assume that $x_1, \ldots, x_n, y_1, \ldots, y_m$ are pairwise distinct.) For suppose that $h(i_1, \ldots, i_n) = f(g_1(i_1, \ldots, i_n), \ldots, g_m(i_1, \ldots, i_n)) = j$, and let $g_1(i_1, \ldots, i_n) = k_1, \ldots, g_m(i_1, \ldots, i_n) = k_m$. Then

$$\vdash_{FQ} B_1(\mathbf{i}_1/x_1 \ldots \mathbf{i}_n/x_n t/y) \equiv t = \mathbf{k}_1$$
$$\vdots$$
$$\vdash_{FQ} B_m(\mathbf{i}_1/x_1 \ldots \mathbf{i}_n/x_n t/y) \equiv t = \mathbf{k}_m$$
$$\vdash_{FQ} A(\mathbf{k}_1/x_1 \ldots \mathbf{k}_m/x_m t/y) \equiv t = \mathbf{j}$$

Consequently

$$\vdash_{FQ} B_1(\mathbf{i}_1/x_1 \ldots \mathbf{i}_n/x_n \mathbf{k}_1/y)$$
$$\vdots$$
$$\vdash_{FQ} B_m(\mathbf{i}_1/x_1 \ldots \mathbf{i}_n/x_n \mathbf{k}_m/y)$$
$$\vdash_{FQ} A(\mathbf{k}_1/x_1 \ldots k_m/x_m \mathbf{j}/y)$$

Thus

$$\vdash_{FQ} (B_1(\mathbf{k}_1/y) \mathbin{\&} \ldots \mathbin{\&} B_m(\mathbf{k}_m/y))(\mathbf{i}_1/x_1 \ldots \mathbf{i}_n/x_n) \mathbin{\&} A(\mathbf{k}_1/x_1 \ldots \mathbf{k}_m/x_m \mathbf{j}/y)$$

But

$$\vdash_{FQ} \exists x(x = \mathbf{k}_1)$$
$$\vdots$$
$$\vdash_{FQ} \exists x(x = \mathbf{k}_m)$$

and hence

(1) $\vdash_{FQ} \exists y_1 \ldots \exists y_m(B_1(y_1/y) \mathbin{\&} \ldots \mathbin{\&} B_m(y_m/y) \mathbin{\&} A(y_1/x_1 \ldots y_m/x_m))$
$(\mathbf{i}_1/x_1 \ldots \mathbf{i}_n/x_n \mathbf{j}/y)$

On the other hand, suppose that

$$\exists y_1 \ldots \exists y_m(B_1(y_1/y) \mathbin{\&} \ldots \mathbin{\&} B_m(y_m/y) \mathbin{\&} A(y_1/x_1 \ldots y_m/x_m))$$
$$(\mathbf{i}_1/x_1 \ldots \mathbf{i}_n/x_n t/y)$$

To utilize the free logic equivalent of Existential Instantiation, we assume

$$B_1(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n a_1/y)$$
$$\exists x(x = a_1)$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$B_m(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n a_m/y)$$
$$\exists x(x = a_m)$$
$$A(a_1/x_1 ... a_m/x_m t/y)$$

From these assumptions and earlier stated theorems of *FQ*,

$$t = \mathbf{j}$$

clearly follows, and hence, since none of $a_1, ..., a_n$ occur in $t = \mathbf{j}$,

(2) $\vdash_{FQ} \exists y_1 ... \exists y_m(B_1(y_1/y) \ \& \ ... \ \& \ B_m(y_m/y) \ \& \ A(y_1/x_1 .. y_m/x_m)$
$(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n t/y)) \supset t = \mathbf{j}$.

The desired result follows easily from the conjunction of (1) and (2).

Part VI: Let A represent the $n+1$ regular function f, and let g be obtained by minimization on f. Then $B = A(0/y) \ \& \ \forall w(w < x_{n+1} \supset \sim A(w/x_{n+1}0/y))$ represents g. For suppose that $g(i_1, ..., i_n) = \mu x_{n+1} f(i_1, ..., i_n, x_{n+1}) = j$. Then $f(i_1, ..., i_n, j) = 0$ and, for all $k < j$, $f(i_1, ..., i_n, k) \neq 0$. Since A represents f,

$$\vdash_{FQ} \sim A(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n 0/x_{n+1} 0/y)$$
$$\cdot$$
$$\cdot$$
$$\cdot$$
$$\vdash_{FQ} \sim A(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n \mathbf{j}-1/x_{n+1} 0/y)$$
$$\vdash_{FQ} A(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n \mathbf{j}/x_{n+1} 0/y)$$

Then, by (A13),

(3) $\vdash_{FQ} (A(0/y) \ \& \ \forall w(w < x_{n+1} \supset \sim A(w/x_{n+1}0/y))$
$(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n \mathbf{j}/x_{n+1})$

Now assume

(4) $(A(0/y) \ \& \ \forall w(w < t \supset \sim A(w/x_{n+1}0/y)))(\mathbf{i}_1/x_1 ... \mathbf{i}_n/x_n t/x_{n+1})$

Since

$$\vdash_{FQ} \exists x(x = \mathbf{j})$$

(4) entails $\sim \mathbf{j} < t$. On the other hand, by (A12) $t < \mathbf{j}$ entails $\exists x(x = t)$, and hence (3) and (4) entail $\sim t < \mathbf{j}$. Therefore, by (A14), (4) entails $t = \mathbf{j}$, and in conclusion

$$(5) \vdash_{FQ} (A(0/y) \ \& \ \forall w(w < t \supset \ \sim A(w/x_{n+1}0/y)))(\mathbf{i}_1/x_1 \ldots \mathbf{i}_n/x_n t/x_{n+1}) \supset t = \mathbf{j}$$

The desired result follows easily from the conjunction of (3) and (5).

If n is the gödel number of A, let A^* be $\mathbf{n}$. Let the function *diag* be as in [2], p. 172. Since *diag* is recursive, it is represented in *FQ* by a formula B containing exactly two variables free.

Lemma 3. For every formula C, containing exactly the variable y free, there is a sentence G such that

$$\vdash_{FQ} G \equiv C(G^*/y)$$

Proof. Let F be $\exists y(B \ \& \ C)$. Let n be the gödel number of F. Let $G = \exists x(x = \mathbf{n} \ \& \ F)$. In view of (A12), G is provably equivalent to $\exists y(B(\mathbf{n}/x) \ \& \ C)$. Let k be the gödel number of G. Then $diag(n) = k$ and $\mathbf{k} = G^*$. So

$$\vdash_{FQ} B(\mathbf{n}/x t/y) \equiv t = \mathbf{k}$$
$$\vdash_{FQ} G \equiv \exists y(y = \mathbf{k} \ \& \ C)$$

and in view of (A12)

$$\vdash_{FQ} G \equiv C(\mathbf{k}/y) \qquad \text{[that is, } \vdash_{FQ} G \equiv C(G^*/y)\text{]}$$

Let a set u of natural numbers be *definable* in a theory T if there is a formula A of the language of T, containing exactly the variable x free, and such that, for any number k, $\vdash_T A(\mathbf{k}/x)$ if $k \in u$, and $\vdash_T \sim A(\mathbf{k}/x)$ otherwise.

Lemma 4. If T is a consistent extension of *FQ*, then the set of gödel numbers of theorems of T is not definable in T.
Proof is as in [2], p. 174.

Theorem 2 (Gödel's first incompleteness theorem). There is no consistent, complete, axiomatizable extension of *FQ*.
Proof is as in [2], p. 179.

University of California at Irvine Ermanno BENCIVENGA

REFERENCES

[1] Bencivenga, E., "Finitary Consistency of a Free Arithmetic," *Notre Dame Journal of Formal Logic* 25 (1984), pp. 224-26.

[2] Boolos, G., and R. Jeffrey, *Computability and Logic*, 2nd edition, Cambridge University Press, Cambridge, 1980.