PARACONSISTENCY AND THE J-SYSTEMS OF ARRUDA AND DA COSTA

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Abstract

Da Costa's two conditions for paraconsistency are discussed. It is noted that condition (I), which requires that arbitrary formulas not be derivable from inconsistencies, is universally accepted; but (II), which requires substantial containment of classical logic, is less generally endorsed. In particular, it is suggested that (II) be weakened to (II′), which allows substantial containment of intuitionistic logic as an alternative. It is noted that J_2 to J_5 explicitly or substantively violate (I), leaving only J_1 for further consideration. While J_1 does not satisfy (II) as well as J_2 to J_5 do, it comes closer to satisfying (II′). However, certain anomalies are pointed out, e.g. while $\rightarrow \neg(A \,\&\, \neg A)$ is a postulate, $\rightarrow \neg(\neg A \,\&\, A)$ and $\rightarrow \neg((A \,\&\, \neg A) \,\&\, (A \,\&\, \neg A))$ are not derivable. Consequently, J_1 does not enjoy SE, the property of intersubstitutivity of provable equivalents. It is investigated whether SE can be secured by extending J_1, with the negative result that the weakest extension of J_1 enjoying SE is J_5. Weaker versions of SE are proposed, but it is shown that any extension of J_1 enjoying these also substantively violates (I). Finally, it is suggested that weaker systems, i.e. subsystems of J_1, be investigated.

1. Conditions for paraconsistency

Da Costa and Alves state in [8] that, in general, systems of paraconsistent logic must satisfy the following conditions:

(I) from two contradictory formulas A and $\neg$A, it must not be possible in general to deduce an arbitrary formula B; and

(II) such systems should contain most of the schemata and deduction rules of classical logic that do not interfere with (I).

Although further requirements have occasionally been added for particular paraconsistent systems (see, for example, [7]), these two conditions have consistently operated as the primary guiding principles for da Costa and his collaborators in the construction of their paraconsistent logics.

However, this approach is not beyond controversy, for while (I) is universally accepted as a necessary condition for paraconsistent systems, (II) is less generally endorsed (see, for example, [4], [5] and [11]. Indeed, dissent over (II) is sufficiently widespread that adherence to this condition has come to be regarded as a distinguishing feature of the Brazilian approach to paraconsistency (see [11]). Without rehearsing the arguments of others in detail, a number of grounds for dissatisfaction with (II) are worth mentioning.

Firstly, da Costa himself notes that this condition is "vague" ([7], p. 498). Certainly, it is not clear exactly what measure of containment of classical schemata and rules constitutes satisfaction of the condition. But more significant is the fact that it is not *determinative*. For it is conceivable – indeed, this will emerge in later sections – that two different schemata or rules could be singly but not jointly incorporated into a paraconsistent system without compromising condition (I). In such a case, (II) suggests that one of the pair ought to be incorporated, or at least considered for incorporation, but no means of deciding between them is suggested. This indicates that, for (II) to be coherently applied in the construction of paraconsistent systems, it must be coupled with some account of the relative merits of competing candidate schemata and rules. One suggestion for such an account will be advanced in Section 5.

A second objection to condition (II) is that it needlessly places on paraconsistent logics the burden of ensuring that inconsistent theories based on these logics sufficiently resemble their classical competitors to be considered as serious rival theories. For example, it is plausible that inconsistent set theories (such as those constructed on the basis of the J-systems in [2]) should sufficiently approximate classically based set theories for them to be considered to be genuine rival formal models of the same informal intuitions concerning sets. But it does not follow that this overall similarity should extend also to the components of the respective theories, and in particular, to their logical bases.

Finally, even if the general tenor of (II) is accepted, a case still exists for slightly modifying it. For it is noteworthy that, as a matter of practice, da Costa and his collaborators consider worth mentioning not only substantial containment of classical logic, but also containment of intuitionistic logic. (For example, Theorem 1 of [1] states that C_ω, the weakest of the C-systems, contains the theorems of the intuitionistic positive calculus). This practice has led some authors to read into (II) the proviso that substantial containment of intuitionistic logic is a next-best alternative, or even a no worse alternative, to substantial containment of classical logic (see, for example, [4] and [13]). This suggests that condition (II) be modified as follows: (II′) paraconsistent systems should contain most of the schemata and deduction rules of classical or intuitionistic logic that do not interfere with (I).

2. *The J-systems and the paraconsistency conditions*

The systems J_1 to J_5, introduced by Arruda and da Costa in [2], are among the early contributions of Brazilian logicians to the paraconsistent logic programme. On the basis of each J_i, Arruda and da Costa construct a set theory ZF_i in which the postulate of separation occurs without the restrictions which accompany it in Zermelo-Fraenkel set theory as a guard against paradox.

It was hoped that the theories ZF_i, though containing such paradoxical sequents as $\rightarrow \exists x(x \in x \equiv x \notin x)$, would nonetheless prove to be non-trivial (in the sense that not every sequent is derivable). Unfortunately, it turned out that $A \equiv \neg A \rightarrow B \supset C$ is derivable in J_2 to J_5 (by Theorem 1 of [3]), and hence that ZF_2 to ZF_5 contain every sequent of the form $\rightarrow B \supset C$ (Theorem 3). A particularly unwanted consequence (Theorem 4) is that in these theories all sets are identical: $\rightarrow \forall x \forall y (x=y)$. Moreover, from $A \equiv \neg A \rightarrow B \supset C$ it is easy to derive $A, \neg A \rightarrow B \supset C$, which makes it clear that the systems J_2 to J_5 satisfy condition (I) at most in letter but not in spirit. In fact, J_5 fails to do even this, as the sequent $A, \neg A \rightarrow B$ is derivable in this system (Theorem 18 of [2]).

This leaves only J_1, the weakest of the J-systems. As far as condition (I) is concerned, J_1 fares better than its stronger siblings: $A, \neg A \rightarrow B$ is not derivable in J_1 (by Theorem 3 of [2]), nor is $A, \neg A \rightarrow B \supset C$ (by Theorem 2 of [3]). It is rather because of its apparent failure to satisfy (II) that Arruda and da Costa take their leave not only of the stronger J-systems, but also of J_1 (see [3], p. 186).

Certainly, J_1 fails to contain some classically derivable sequents which would not obviously interfere with its satisfaction of (I) if they were incorporated. For example, it does not contain even all of the *theorems* of *positive* classical logic, since $\to ((A \supset B) \supset A) \supset A$ is not derivable in J_1 (by Theorem 3 of [2]). But this constitutes only *prima facie* evidence that (II) is not satisfied; the case is not settled until it is demonstrated that significant portions of classical logic not contained by J_1 can be added without compromising (I). And in any case, (II) is itself open to question, as noted in Section 1.

Of particular interest in this regard is the suggested replacement of (II) by the weaker (II′). For when it comes to containing substantial parts of intuitionistic rather than classical logic, J_1 fares somewhat better. Theorem 1 of [5] shows that the theorems of J_1^+, the subsystem generated by only the negation-free postulates of J_1, are precisely the theorems of positive intuitionistic logic; and the conservative extension result (Theorem 3) of [14] shows that these are precisely the negation-free theorems of J_1.

It is where negation is involved that J_1 diverges more spectacularly from intuitionistic logic, possibly to a greater extent than condition (II′) envisages. Some of the intuitionistically derivable sequents involving negation that are not derivable in J_1 are listed in Theorem 3 of [2], but one which is particularly striking, and which would least obviously interfere with the satisfaction of (I) were it incorporated into J_1, is cited in Theorem 5 of that paper. The (infinite) matrices supporting Theorem 5 are needlessly complex for present purposes, so we restate the result in question in greater generality, and supply simpler matrices for its proof.

Theorem 1. In J_1, the following sequents are not derivable:

$\to \neg((A \,\&\, \neg A) \,\&\, B)$;
$\to \neg X$, where X is any reassociation and/or permutation of $(A \,\&\, \neg A) \,\&\, B$.

Proof: The following matrices validate the postulates of J_1, but invalidate these sequents when A is assigned the value 0 and B is assigned the value 1.

→/⊃	0	1	2	¬
*0	0	1	2	0
*1	0	0	2	2
2	0	0	0	1

∧/&	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

∨	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

(The values 0 and 1 are designated, and the value of → A is taken to be the same as that of A).

Even more striking than Theorem 1 is the following.

Theorem 2. In J_1, the following sequents are not derivable:

→ ⌝(⌝A & A);
→ ⌝((A & ⌝A) & (A & ⌝A));
→ ⌝((A & ⌝A) ∨ (A & ⌝A)).

Proof: The following matrices validate the postulates of J_1, but invalidate these sequents when A is assigned the value 0.

→/⊃	0	1	2	3	4	⌝
*0	0	0	0	0	4	1
*1	0	0	0	0	4	2
*2	0	0	0	0	4	1
*3	0	0	0	0	4	4
4	0	0	0	0	0	3

∧/&	0	1	2	3	4
0	3	1	3	3	4
1	3	3	2	3	4
2	3	2	3	2	4
3	3	3	3	3	4
4	4	4	4	4	4

∨	0	1	2	3	4
0	3	3	3	3	3
1	3	3	3	3	3
2	3	3	3	3	3
3	3	3	3	3	3
4	3	3	3	3	4

(Only the value 4 is not designated).

The above results provide a number of reasons for dissatisfaction with J_1. Firstly, it is counterintuitive that the sequents of Theorems 1 and 2 are not derivable in J_1. This is not because of their underivability *per se*, but because the sequent of which they would ordinarily be taken to be merely syntactic variants, → ⌝(A & ⌝A), is explicitly incorporated as a postulate. It may be possible to provide a plausible motivation for such fine discrimination, but no such motivation is to be discerned in the discussion accompanying the construction of the J-systems in [2], nor in the sequel [3]. In the absence of any illuminating motivation, such fine discrimination is simply anomalous.

Secondly, it may be that the absence of the (intuitionistically derivable) sequents of Theorems 1 and 2 from the stock of derivable sequents of J_1 constitutes an infringement of (II′). For the following result indicates that these sequents can be added to J_1 without endangering the satisfaction of (I).

Theorem 3. In the system formed by adding the sequents of Theorems 1 and 2 to the postulates of J_1, the sequents A, ⌝A → B and A, ⌝A → B ⊃ C are not derivable.

Proof: The following matrices validate the postulates of J_1 and the sequents of Theorems 1 and 2, but invalidate A, ¬A → B when A assigned the value 1 and B is assigned the value 3, and A, ¬A → B ⊃ C when A is assigned the value 1, B is assigned the value 0 and C is assigned the value 3.

→/⊃	0	1	2	3	¬
*0	0	0	2	3	3
*1	0	0	2	3	2
*2	0	0	0	3	1
3	0	0	0	0	0

∧/&	0	1	2	3
0	0	1	2	3
1	1	1	2	3
2	2	2	2	3
3	3	3	3	3

∨	0	1	2	3
0	0	0	0	0
1	0	1	1	1
2	0	1	2	2
3	0	1	2	3

(The values 0 and 1 are designated).

Even if the sequents of Theorems 1 and 2 were added to J_1, however, there is no guarantee that further deficiencies could not be exhibited. In particular, it is likely that other syntactic variants of → ¬(A & ¬A) would still prove to be underivable. Obviously, a more systematic strategy is needed; and this in turn requires that we establish which more general property is shown by Theorems 1 and 2 to be lacking from J_1.

3. J_1 and the property of intersubstitutivity of provable equivalents

The underivability of the sequents of Theorem 2 in J_1 is symptomatic of a more general deficiency, namely, that this system lacks SE, the property of intersubstitutivity of provable equivalents. The most natural way of defining provable equivalence in the J-systems is as follows: two formulas C and D are *provably equivalent* just in case the pair of sequents C → D and D → C (abbreviated C ↔ D) is derivable. SE, then, is the property that, if Γ → A is a derivable sequent, B a subformula of some member(s) of Γ and/or of A, and C a formula which is provably equivalent to B, then the sequent obtained by substituting C for some or all occurrence(s) of B in Γ → A is also derivable. Equivalent in this context is the property that, if B is a subformula of A, and B and C are provable equivalent, then so are A and the formula obtained by substituting C for some or all occurrence(s) of B in A.

Theorem 4. J_1 does not enjoy SE.

Proof: Easily derived in J_1 are the sequents $A \& \neg A \rightarrow \neg A \& A$ and $\neg A \& A \rightarrow A \& \neg A$. If J_1 enjoyed SE, then $\rightarrow \neg(\neg A \& A)$, the result of substituting $\neg A \& A$ for $A \& \neg A$ in postulate $\neg_3$), would also be derivable. But this sequent is not derivable, by Theorem 2; hence, J_1 does not enjoy SE.

The proof of Theorem 4 illustrates how unsystematically the connective $\neg$ behaves in J_1; two formulas are provably equivalent and yet their putative negations are not (indeed, one is a postulate while the other is underivable). Of course, it may be possible to interpret $\neg$ in such a way that this result is acceptable, but there is nothing in [2] and [3] to indicate that $\neg$ is to be interpreted even as a special kind of negation, let alone as something other than negation. Again, in the absence of any such illumination, the behaviour of this connective in J_1 is simply anomalous. (In [11]), a similar view is expressed about the behaviour of $\neg$ in the C-systems of da Costa, which similarly fail to enjoy SE; and in [10], it is argued that no reasonable conditional or biconditional can be expressed in the C-systems, again because of their failure to enjoy SE).

In general, the absence of SE makes it difficult to provide a natural and uniform interpretation of the connectives of a logic and the relations thereby definable. Technically, this tends to be reflected in the complexity of formal semantical and algebraic perspectives (again, see [10] and [11], and also [9] and [13]).

The desired general strategy for removing the deficiencies of J_1 exhibited in (at least) Theorem 2, then, is to attempt to secure the property SE. In [13], the parallel problem of securing SE for the C-systems is adressed, and the respective addition of two rules is proposed. The appropriate versions of these rules in the present context are:

$$\text{RC} \quad \frac{C \rightarrow D}{\neg D \rightarrow \neg C} \qquad \text{and} \qquad \text{EC} \quad \frac{C \leftrightarrow D}{\neg D \rightarrow \neg C}.$$

By Lemma 2 of [14], J_1 enjoys SE^+, the property of intersubstitutivity of provable equivalents in negation-free contexts. It follows that, for any extension of J_1 (in the same vocabulary), the admissibility of RC or of EC is sufficient to guarantee SE in full, and the admissibility of EC is also evidently necessary. (A rule is *admissible* in a sequent-based system just in case the system formed by adding that rule is a conservative extension of, i.e. has the same stock of derivable sequents as, the original system.

Thus, every derivable rule is admissible, but the converse does not generally hold).

We now investigate the result of respectively adding RC and EC to J_1. That neither rule is admissible in J_1 follows from Theorem 4; it is to be expected, therefore, that their addition will strengthen J_1 in an interesting way.

The strength imparted by the addition of RC proves to be somewhat excessive.

Theorem 5. In J_1 + RC, the sequent A, $\neg$A $\rightarrow$ B is derivable.

Proof: Application of $\rightarrow_2$) to postulate $\neg_3$) of J_1 yields $\neg$B $\rightarrow$ $\neg$(A & $\neg$A), from which $\neg\neg$(A & $\neg$A) $\rightarrow$ $\neg\neg$B follows by RC. Together with A & $\neg$A $\rightarrow$ $\neg\neg$(A & $\neg$A), which is an instance of $\neg_1$), this yields A & $\neg$A $\rightarrow$ $\neg\neg$B by $\rightarrow_5$). From this, A & $\neg$A $\rightarrow$ B follows by $\neg_2$) and $\rightarrow_5$), and from this A, $\neg$A $\rightarrow$ B is easily derived using $\&_1$) and $\rightarrow_5$).

Unfortunately, the addition of the ostensibly weaker EC has precisely the same result.

Theorem 6. In J_1 + EC, the sequent A, $\neg$A $\rightarrow$ B is derivable.

Proof: Application of $\rightarrow_2$) to postulate $\neg_3$) yields $\neg$(B & $\neg$B) $\rightarrow$ $\neg$(A & $\neg$A). A parallel derivation yields the converse, $\neg$(A & $\neg$A) $\rightarrow$ $\neg$(B & $\neg$B). From these, EC delivers $\neg\neg$(A & $\neg$A) $\rightarrow$ $\neg\neg$(B & $\neg$B), which quickly reduces to A & $\neg$A $\rightarrow$ B & $\neg$B using $\neg_1$), $\neg_2$) and $\rightarrow_5$). An instance of $\&_2$) is B & $\neg$B $\rightarrow$ B, whence $\rightarrow_5$) yields A & $\neg$A $\rightarrow$ B. As in the proof of Theorem 5, this suffices to deliver A, $\neg$A $\rightarrow$ B.

As with J_5, the derivability of A, $\neg$A $\rightarrow$ B in J_1 + RC and J_1 + EC constitutes an explicit violation of condition (I), effectively disqualifying these systems from contention as paraconsistent logics. In fact, these three systems are equivalent.

Theorem 7. $J_1 + EC = J_1 + RC = J_5$.

Note: In this and subsequent proofs, we will make use of the following rules and sequent, which are easily shown to be derivable in J_1:

$$\text{Transitivity (of } \supset\text{):} \quad \frac{\Gamma \rightarrow C \supset D \quad \Delta \rightarrow D \supset E}{\Gamma, \Delta \rightarrow C \supset E}$$

Permutation of antecedents: $\dfrac{\rightarrow C \supset (D \supset E)}{\rightarrow D \supset (C \supset E)}$.

Restricted *modus ponens* (sequent): $C, C \supset (D \supset E) \rightarrow D \supset E$

Restricted *modus ponens* (rule): $\dfrac{\Gamma \rightarrow C \supset (D \supset E)}{\Gamma, C \rightarrow D \supset E}$

The sequent form of restricted *modus ponens* is shown to be derivable in J_1 in [5] (p. 45); from this, $\rightarrow_4$) and $\rightarrow_5$) deliver the rule form.

Proof: That J_1 + EC is a subsystem of J_1 + RC is evident, since the derivability of RC ensures the derivability of the weaker EC. Moreover, J_1 + RC is a subsystem of J_5, since J_1 is a subsystem of J_5 to begin with, and RC is easily derived in J_5 as follows. Assume C $\rightarrow$ D. From this, $\neg$D, C $\rightarrow$ D follows by $\rightarrow_2$). An instance of $\rightarrow_1$) is $\neg$D $\rightarrow$ $\neg$D. But the last two sequents yield $\neg$D $\rightarrow$ $\neg$C by postulate $\rightarrow_7$) of J_5.

To complete the proof of Theorem 7, it suffices to show that J_5 is a subsystem of J_1 + EC, i.e. that those postulates which are added to J_1 in the construction of J_5 are derivable in J_1 + EC. These are $\rightarrow_7$), $\neg_6$) and $\neg_5$).

Postulate $\rightarrow_7$) is the rule $\dfrac{\Delta, A \rightarrow B \qquad \Delta \rightarrow \neg B}{\Delta \rightarrow \neg A}$.

This is derived in J_1 + EC as follows. Assume Δ, A $\rightarrow$ B and Δ $\rightarrow$ $\neg$B. Applications of $\rightarrow_2$) and $\rightarrow_4$) to the second sequent give Δ, A $\rightarrow$ $\neg$B, which can be combined with the first to yield Δ, A $\rightarrow$ B & $\neg$B (see Theorem 2 of [2]). But B & $\neg$B $\rightarrow$ $\neg$A is derivable in J_1 + EC by Theorem 6; hence by $\rightarrow_5$), we get Δ, A $\rightarrow$ $\neg$A. Easily derived by $\rightarrow_1$) and $\rightarrow_2$) is Δ, $\neg$A $\rightarrow$ $\neg$A; hence $\vee_4$) yields Δ, A $\vee$ $\neg$A $\rightarrow$ $\neg$A. Applications of $\rightarrow_4$) transform this into A $\vee$ $\neg$A, Δ $\rightarrow$ $\neg$A, from which Δ $\rightarrow$ $\neg$A follows by $\neg_4$) and $\rightarrow_5$). Thus, $\rightarrow_7$) is derivable in J_1 + EC.

Postulate $\neg_6$) is the sequent $\neg$A, $\neg$B $\rightarrow$ $\neg$(A $\vee$ B). This is derived in J_1 + EC as follows. Firstly, we have $\neg$A, $\neg$B, A $\vee$ B $\rightarrow$ ($\neg$A & $\neg$B) & (A $\vee$ B) by $\&_1$) and $\rightarrow_5$). By distribution (see Theorem 2 of [2]), we have ($\neg$A & $\neg$B) & (A $\vee$ B) $\rightarrow$ (($\neg$A & $\neg$B) & A) $\vee$ (($\neg$A & $\neg$B) & B), whence $\neg$A, $\neg$B, A $\vee$ B $\rightarrow$ (($\neg$A & $\neg$B) & A) $\vee$ (($\neg$A & $\neg$B) & B) by $\rightarrow_5$). But ($\neg$A & $\neg$B) & A $\rightarrow$ A & $\neg$A is easily derived, and ($\neg$A & $\neg$B) & B $\rightarrow$ A & $\neg$A is derivable in J_1 + EC by virtue of Theorem 6; hence, we have (($\neg$A & $\neg$B) & A) $\vee$ (($\neg$A & $\neg$B) & B) $\rightarrow$ A & $\neg$A by $\vee_4$), and therefore $\neg$A, $\neg$B, A $\vee$ B $\rightarrow$ A & $\neg$A by $\rightarrow_5$). From $\neg_3$), $\neg$A, $\neg$B $\rightarrow$ $\neg$(A & $\neg$A) follows by $\rightarrow_2$). From the last two sequents, the above-derived $\rightarrow_7$) delivers the desired $\neg$A, $\neg$B $\rightarrow$ $\neg$(A $\vee$ B).

Postulate $\neg_5$) is the sequent $\rightarrow$ (A $\supset$ B) $\supset$ ((A $\supset$ $\neg$B) $\supset$ $\neg$A). This is derived in J_1 + EC as follows. Firstly, we have A $\supset$ B, A $\supset$ $\neg$B $\rightarrow$ A $\supset$ (B & $\neg$B) by $\&_1$) and $\&_4$). But by Theorem 6, B & $\neg$B $\rightarrow$ $\neg$A is derivable in J_1 + EC, and hence so is $\rightarrow$ (B & $\neg$B) $\supset$ $\neg$A by $\supset_1$). By transitivity, we therefore have A $\supset$ B, A $\supset$ $\neg$B $\rightarrow$ A $\supset$ $\neg$A. Easily derived using $\vee_3$) is A $\supset$ $\neg$A $\rightarrow$ (A $\vee$ $\neg$A) $\supset$ $\neg$A, so by $\rightarrow_5$), we get A $\supset$ B, A $\supset$ $\neg$B $\rightarrow$ (A $\vee$ $\neg$A) $\supset$ $\neg$A. Applying $\supset_1$) and permuting antecedents gives $\rightarrow$ (A $\vee$ $\neg$A) $\supset$ ((A $\supset$ B) $\supset$ ((A $\supset$ $\neg$B) $\supset$ $\neg$A)), from which A $\vee$ $\neg$A $\rightarrow$ (A $\supset$ B) $\supset$ ((A $\supset$ $\neg$B) $\supset$ $\neg$A) follows by the restricted *modus ponens* rule. The desired $\neg_5$) then follows by $\neg_4$) and $\rightarrow_5$).

This concludes the proof of Theorem 7.

We note that the proofs of Theorems 6 and 7 do not rely upon the actual derivability of EC in J_1 + EC; it suffices merely that this rule is admissible. But the admissibility of EC in any extension of J_1 is a necessary condition for SE. We can therefore state the following more general result.

Theorem 8. There is no extension of J_1 which enjoys SE but which is weaker than J_5.

We conclude that the deficiencies of J_1 in Theorems 1 and 2 cannot be remedied by extending this system so as to secure SE without thereby rendering it too strong to satisfy paraconsistency condition (I). Two alternative strategies present themselves: (i) to extend J_1 so as to secure not SE but some weaker version of this property which would nonetheless at least mitigate these deficiencies; and (ii) to explore methods of variation other than extension. These strategies will be investigated in the next two sections.

4. *J_1 and weaker versions of the property of intersubstitutivity of provable equivalents*

Two weaker versions of SE suggest themselves, each a restricted variant of one of the equivalent statements of SE presented in Section 3. Firstly, let SE′ be the property that, if A is a theorem (i.e. $\rightarrow$ A is derivable), B is a subformula of A, and C is provably equivalent to B, then the formula obtained by substituting C for some or all occurrence(s) of B in A is also a theorem. SE′ is the property of intersubstitutivity of provable equivalents in theorems, and its obtaining would at least avoid the deficiencies exhibited in Theorem 2. A second variant is SE″, which is the property that, if B is a subformula of A, and A(C) is the result of substituting a formula C which is provably equivalent to B for some or all occurrence(s) of B in A, then $\rightarrow$ A $\equiv$ A(C) is a derivable sequent. (C $\equiv$ D abbreviates the formula (C $\supset$ D) & (D $\supset$ C)).

Theorem 9. J_1 does not enjoy SE′.

Proof: As for Theorem 4.

Theorem 10. J_1 does not enjoy SE″.

Proof: As in the proof of Theorem 4, A & $\neg$A and $\neg$A & A are provably equivalent in J_1. If J_1 enjoyed SE″, then $\rightarrow$ $\neg$(A & $\neg$A) $\equiv$ $\neg$($\neg$A & A) would be derivable. But the matrices of Theorem 2, which validate the postulates of J_1, invalidate this sequent when A is assigned the value 0; hence, J_1 does not enjoy SE″.

Theorem 11. Every extension of J_1 which enjoys SE′ also enjoys SE″, but not conversely.

Proof: To establish the first part, it suffices to note that A $\equiv$ A is a theorem of J_1 for any formula A. If SE′ holds, then any occurrence of a subformula B of A can be replaced by a provably equivalent formula C; hence, A $\equiv$ A(C) is also a theorem, and SE″ holds.

To establish the second part, it suffices to exhibit an extension of J_1 which enjoys SE″ but not SE′. Consider the system J_1' constructed by adding the postulate $\rightarrow$ A $\equiv$ B to J_1. Trivially, J_1' enjoys SE″. However, the matrices in the proof of Theorem 2, with the modification that A $\supset$ B is assigned the value 0 always, validate the postulates of J_1' but continue to invalidate the sequents of Theorem 2; hence J_1' does not enjoy SE′.

It follows from Theorem 11 that any addition to/of J_1 which is necessary to secure SE″ is necessary also for SE′. We therefore proceed to establish which additions are necessary for the weaker property.

It is easy to verify that the admissibility of the following variant of EC in any extension of J_1 is a necessary condition for SE″:

$$\text{EC} \supset \quad \frac{C \leftrightarrow D}{\rightarrow \neg D \supset \neg C}.$$

Moreover, a straightforward inductive argument can be used to establish that the admissibility of EC $\supset$ is also sufficient for SE″, since the following rules are already derivable in J_1:

$$\frac{C \leftrightarrow D}{\rightarrow B^* C \equiv B^* D} \qquad \frac{C \leftrightarrow D}{\rightarrow C^* B \equiv C^* D},$$

where * represents any of the connectives &, $\vee$ or $\supset$.

The admissibility of any of the following rules is evidently also sufficient:

$$\text{RC} \supset \quad \frac{C \rightarrow D}{\rightarrow \neg D \supset \neg C}$$

$$\supset \text{EC} \supset \quad \frac{\rightarrow C \equiv D}{\rightarrow \neg D \supset \neg C}$$

$$\supset \text{RC} \supset \quad \frac{\rightarrow C \supset D}{\rightarrow \neg D \supset \neg C}$$

We now consider the systems formed by respectively adding these rules to J_1. In fact, these systems are all equivalent.

Theorem 12. $J_1 + \text{EC}\supset \; = \; J_1 + \text{RC}\supset \; = \; J_1 + \supset\text{EC}\supset \; = \; J_1 + \supset\text{RC}\supset$.

Proof: Evidently, J_1 + EC $\supset$ is a subsystem of both J_1 + RC and J_1 + $\supset$ EC $\supset$, and both of these are subsystems of J_1 + $\supset$ RC $\supset$. We need only show, therefore, that J_1 + $\supset$ RC $\supset$ is a subsystem of J_1 + EC $\supset$, i.e. that $\supset$ RC $\supset$ is derivable in J_1 + EC $\supset$.

By $\neg_3$) and $\rightarrow_2$), we have $\neg(C \& \neg C) \leftrightarrow \neg(D \& \neg D)$ in J_1. Applying EC $\supset$ yields $\rightarrow \neg\neg(D \& \neg D) \supset \neg\neg(C \& \neg C)$, which, using $\neg_1$), $\neg_2$), $\supset_1$) and transitivity, is easily reduced to $\rightarrow (D \& \neg D) \supset (C \& \neg C)$. By $\&_3$), $\supset_1$) and transitivity, this further reduces to $\rightarrow (D \& \neg D) \supset \neg C$. Applying $\supset_1$) to an instance of $\&_1$) gives $D \rightarrow \neg D \supset (D \& \neg D)$, which together with the preceding sequent, yields by transitivity, $D \rightarrow \neg D \supset \neg C$. Applying $\supset_1$) gives $\rightarrow D \supset (\neg D \supset \neg C)$.

An instance of $\vee_3$) is $(C \supset \neg C) \& (\neg C \supset \neg C) \rightarrow (C \vee \neg C) \supset \neg C$, which is transformed into $C \supset \neg C, \neg C \supset \neg C \rightarrow (C \vee \neg C) \supset \neg C$ by $\&_1$) and $\rightarrow_5$), and further into $\neg C \supset \neg C \rightarrow (C \supset \neg C) \supset ((C \vee \neg C) \supset \neg C)$ by $\rightarrow_4$) and $\supset_1$). But $\rightarrow \neg C \supset \neg C$ is derivable by $\rightarrow_1$) and $\supset_1$), so $\rightarrow_5$) delivers $\rightarrow (C \supset \neg C) \supset ((C \vee \neg C) \supset \neg C)$. Permutation of antecedents transforms this into $\rightarrow (C \vee \neg C) \supset ((C \supset \neg C) \supset \neg C)$, from which the restricted *modus ponens* rule gives $C \vee \neg C \rightarrow (C \supset \neg C) \supset \neg C$. From this, $\rightarrow(C \supset \neg C) \supset \neg C$ follows by $\neg_4$) and $\rightarrow_5$).

We now consider $\supset$ RC $\supset$. Assume $\rightarrow C \supset D$. Together with the sequent $\rightarrow D \supset (\neg D \supset \neg C)$ derived above, this yields by transitivity, $\rightarrow C \supset (\neg D \supset \neg C)$. Permuting antecedents gives $\rightarrow \neg D \supset (C \supset \neg C)$, which, together with the last sequent of the preceding paragraph, yields by transitivity, $\rightarrow \neg D \supset \neg C$. Thus, $\supset$ RC $\supset$ is derivable in J_1 + EC $\supset$.

This concludes the proof of Theorem 12.

Interestingly, J_1 + EC $\supset$ also has an equivalent formulation purely in terms of the postulates of the J-systems.

Theorem 13. Let $J_{1.5}$ be the system formed by adding to J_1 postulate $\neg_5$) of J_2. Then J_1 + EC $\supset$ = $J_{1.5}$.

Proof: To show that J_1 + EC $\supset$ is a subsystem of $J_{1.5}$, it suffices to derive EC $\supset$ in the latter system. This is derived as follows. An instance of postulate $\neg_5$) is $\rightarrow (C \supset D) \supset ((C \supset \neg D) \supset \neg C)$. Using the restricted version of *modus ponens* cited in the proof of Theorem 7, this is transformed into $C \supset D \rightarrow (C \supset \neg D) \supset \neg C$. Easily derived using $\rightarrow_1$), $\neg_2$) and $\supset_1$) is $\rightarrow \neg D \supset (C \supset \neg D)$, whence transitivity yields $C \supset D \rightarrow \neg D \supset \neg C$. But $\rightarrow C \supset D$ follows from the premise of EC $\supset$ (or of any of the other three rules); whence $\rightarrow_5$) yields the conclusion, $\rightarrow \neg D \supset \neg C$. Thus, EC $\supset$ is derivable in $J_{1.5}$.

To show conversely that $J_{1.5}$ is a subsystem of J_1 + EC $\supset$, it suffices to derive $\neg_5$) in J_1 + EC $\supset$. An instance of $\&_4$) is $(A \supset B) \& (A \supset \neg B) \rightarrow A \supset (B \& \neg B)$, which by $\&_1$) and $\rightarrow_5$) is transformed into $A \supset B, A \supset \neg B \rightarrow A \supset (B \& \neg B)$. As in the proof of Theorem 12, the sequent $\rightarrow (B \& \neg B) \supset \neg A$ is derivable in J_1 + EC $\supset$; hence transitivity yields $A \supset B, A \supset \neg B \rightarrow A \supset \neg A$. Applying $\supset_1$) gives

$A \supset B \rightarrow (A \supset \neg B) \supset (A \supset \neg A)$. Again as in the proof of Theorem 12, $\rightarrow (A \supset \neg A) \supset \neg A$ is derivable in J_1 + EC $\supset$, so transitivity delivers $A \supset B \rightarrow (A \supset \neg B) \supset \neg A$, from which the desired $\neg_5$) follows by $\supset_1$). Thus, $\neg_5$) is derivable in J_1 + EC $\supset$.

This concludes the proof of Theorem 13.

$J_{1.5}$ lies between J_1 and J_2, but it is equivalent to neither.

Theorem 14. $J_1 \neq J_{1.5}$.

Proof: That postulate $\neg_5$) of $J_{1.5}$ is not derivable in J_1 is shown in Theorem 3 of [2]. However, the result also follows from the fact that $J_{1.5}$ enjoys SE″, while J_1 does not, by Theorem 10 above.

Theorem 15. $J_{1.5} \neq J_2$.

Proof: The matrices in the proof of Theorem 2, as modified in the proof of Theorem 11, validate the postulates of $J_{1.5}$ but invalidate postulate $\neg_6$) of J_2 when A and B are both assigned the value 0.

Unfortunately, even though J_1 + EC $\supset$ (= $J_{1.5}$) is weaker than J_2, it similarly fails to substantively satisfy paraconsistency condition (I).

Theorem 16. In J_1 + EC $\supset$, the sequent $A, \neg A \rightarrow B \supset C$ is derivable.

Proof: As in the proof of Theorem 12, $\rightarrow (A \mathbin{\&} \neg A) \supset (C \mathbin{\&} \neg C)$ is derivable in J_1 + EC $\supset$. By $\&_2$), $\supset_1$) and transitivity, this yields $\rightarrow (A \mathbin{\&} \neg A) \supset C$. But $\rightarrow C \supset (B \supset C)$ is easily derived using $\rightarrow_2$), $\rightarrow_2$) and $\supset_1$); whence transitivity again yields $\rightarrow (A \mathbin{\&} \neg A) \supset (B \supset C)$. Using the restricted version of *modus ponens* cited in the proof of Theorem 7, this is transformed into $A \mathbin{\&} \neg A \rightarrow B \supset C$, from which $A, \neg A \rightarrow B \supset C$ follows by $\&_1$) and $\rightarrow_5$).

We note that the proofs of Theorems 13 and 16 (and those parts of the proof of Theorem 12 which they presuppose) do not rely upon the actual derivability of EC $\supset$ in J_1 + EC $\supset$; it suffices merely that this rule is admissible. But the admissibility of EC $\supset$ in any extension of J_1 is a necessary condition for SE″. We can therefore state the following more general result.

Theorem 17. There is no extension of J_1 which enjoys SE″ but which is weaker than $J_{1.5}$.

From Theorems 11 and 17, the following is also immediate.

Theorem 18. There is no extension of J_1 which enjoys SE′ but which is weaker than $J_{1.5}$.

We conclude that even the weaker versions of SE considered in this section cannot be secured by extending J_1 without substantially compromising its satisfaction of (I). We turn instead to methods of variation other than extension.

5. *Other methods of variation*

The obvious alternative to extension is subtraction. In particular, it may be that systems obtained by deleting some of the postulates of J_1, i.e. subsystems of J_1, could be shown to either enjoy SE naturally or be amenable to extension so as to secure this property without infringing condition (I).

A likely candidate for deletion is postulate $\neg_3$). The motivation for extending J_1 in the first place was to remove the anomalies exhibited in Theorems 1 and 2; but these results are anomalous only because J_1 incorporates $\neg_3$) – the anomalies might just as well be removed by deleting this postulate as by adding its variants. The subsystem so obtained would still enjoy SE^+, since the rules required to guarantee this property would not be affected by the deletion of a negation postulate, and it would also evidently satisfy (I), since the sequents $A, \neg A \rightarrow B$ and $A, \neg A \rightarrow B \supset C$ would still not be derivable. If, in addition, the admissibility of RC or EC in this subsystem could be established, then it would enjoy SE naturally; and even if not, it may be that these rules could be added without compromising its satisfaction of (I).

Similar considerations apply also to the removal of any of the other negation postulates of J_1. Of course, an obvious constraint on this strategy of subtraction is condition (II). At first glance, it would appear that the deletion of any of the postulates of J_1 would increase the degree to which (II) is not satisfied. However, there are several considerations which indicate that the matter is not so straightforward.

Firstly, there is the suggested replacement of (II) by (II′). It may be that the deletion from J_1 of the intuitionistically underivable postulates $\neg_2$) and $\neg_4$) would leave unaffected its stock of intuitionistically derivable rules and sequents, in which case the satisfaction of (II′) would not be diminished. Of course, the subsystem so obtained would fail to enjoy SE, since it would still incorporate $\neg_3$) but not its variants listed in Theorem 2; but again it may be that the addition of RC or EC in order to secure SE would not entail the undesirable consequences of the corresponding additions to J_1.

Even without resorting to the intuitionistic escape clause of (II′), however, there are good reasons for not being too concerned by the deletion of some of the negation postulates of J_1, especially if this allows the addition of such a rule as RC without harm to the satisfaction of (I). This is the case foreshadowed in Section I; we are assuming that, for some subsystem of J_1, the addition of a missing negation postulate results in J_1, which satisfies (I), and the addition of RC results in some other system which also satisfies (I), but the addition of both the postulate and RC results in J_5, which does not satisfy (I). Condition (II) suggests that one or the other ought to be added, but it does not suggest which.

An argument which weighs heavily in favour of adding RC in this case is that this addition would guarantee the systematic behaviour of $\neg$, while the addition of the competing postulate would only result in the unsystematic behaviour documented in Theorems 1 and 2. This suggests that RC is more general, or expresses a more fundamental property of negation, than any of the negation postulates of J_1. Certainly, this rule is incorporated in a very broad range of logics, including all of the negation systems of [6] (among them, classical and intuitionistic logic, and Johansson's "minimal calculus"), and also all of the main relevant logics (see [12]). On the other hand, postulates $\neg_2$) and $\neg_4$) are not so universally incorporated, which suggests that they express properties of a somewhat special (and strong) type of negation; and, perhaps more interestingly, (the appropriate versions of) $\neg_1$) and $\neg_3$) are notably absent from the C-systems of da Costa.

The widespread inclusion of RC is hardly surprising, for this rule expresses little more than that negation reverses the order of strength among propositions: the weaker a proposition, the stronger is its rejection or denial; and the stronger the proposition, the weaker its denial. Indeed, it is difficult to see how a connective not conforming to this rule can be interpreted as negation at all, rather than as some more enigmatic functor. These considerations apply also to the weaker EC; indeed, this rule expresses the even less arresting precept that a logic which identifies two propositions should not distinguish between their denials. Again, it is difficult to see how a connective which does not conform to this rule can be interpreted as anything other than a very selective type of negation, if as negation at all.

The considerations expressed in this section indicate that an investigation into the subsystems of J_1, augmented by RC or EC if required, is well warranted. Accordingly, a detailed investigation is undertaken in [15] and [16].

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