TRUTH-LOGICS

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Introduction

This essay is a continuation to and generalization of results presented in two earlier published papers of mine: "Truth and Logic" (included in my *Philosophical Papers, Volume III, Truth, Knowledge and Modality*, Basil Blackwell, Oxford 1984) and "Truth, Negation, and Contradiction" (*Synthese '66*, 1986). Ultimately, however, it is the result of research which started with my paper "On the Logic of Negation" published in *Societas Scientiarum Fennica, Commentationes Physico-Mathematicae* XXII 4, 1959. Preliminary versions of the present essay appeared in Hungarian in *Doxa 5*, Budapest 1985 and in Russian in G.H. von Wright, "Logiko-filosofskie issledovanija" (Logico-Philosophical Investigations), "Progress", Moscow 1986.

In working out this final version I have much profited from discussions with Professor Carlos Alchourrón. I am particularly indebted to him for some decisive insights and proofs concerning the status of what I call "mixed" formulas.

1. A variety of logical calculi will be developed and studied. Their vocabulary is that of traditional propositional logic (PL) embellished with one new symbol, an operator *T*. It is called the *truth operator* and should be read "it is true that". Formally, its rôle in the calculus is similar to the rôle of the modal operators in a propositional modal logic.

Use of the operator enables us to distinguish between two ways of negating a proposition. In the formal language the distinction is one between $\sim T$ and $T\sim$, between "it is not true that" and "it is true that not". It is not obvious that the two phrases have different meanings. But they can be *given* different meanings – and this evidently is of some interest.

The two meanings I shall also refer to as "not-truth" and "truth of the negation". The first is a denial, the second an affirmation. But, to repeat, it is not obvious that denial of truth must be something else than the affirmation of truth of the negation. One *can*, however, distinguish them. Truth of the negation I shall also call *falsehood*. I thus distinguish between falsehood and (mere) not-truth.

2. Another device which can be used for similar purposes as the ones which are pursued in this paper is to introduce two symbols for negation instead of the truth operator. One symbol might be the tilde $\sim$. The other might be the hook $\neg$.

This way of proceeding has some disadvantages, however. One is that the two negations have no handy equivalents in ordinary language. For both we have to use the word "not". $\neg$ would correspond to $T\sim$ (our symbol for falsehood). For T itself one would have to write $\neg\sim$ and for $\sim T$ (not-truth) accordingly $\sim\neg\sim$.

If the system which I shall call TL is built using two signs for negation instead of a combination of a negation sign and a truth operator, there is good reason to call the negation denoted "$\sim$" *weak*, and the negation "$\neg$" signifying falsehood *strong*. But in the system T′L which answers dually to TL it would be natural to reverse the terminology. It depends upon the particular system of truth-logic which one has in mind whether not-truth should be said to be a weaker concept than falsehood, or the other way round. In Classical Logic (CL) they are equally strong (identical).

3. A good many systems of so-called non-classical logic essentially depend upon the fact that they use a concept of negation which is different from the classical one. Thus one can secure, for example, that the Law of Excluded Middle is not (unrestrictedly) valid, as is the case in Intuitionist Logic, or that the Law of Contradiction does not necessarily hold good, as in systems of Paraconsistent Logic. In both types of logic another classical principle, *viz.* the Law of Double Negation, is subject to restrictions.

It seems to me that a logic which employs *only* a non-classical negation is "crippled". This does not necessarily mean that such a "crippled" or "mutilated" logic may not be interesting to study or of good use – for example in reconstructing proofs in mathematics. But it seems to me that one gets a deepened understanding of these non-classical systems and their relation to classical logic if one tries to embody their peculiar features within a broader logical frame where the *one* notion of negation can "behave" both classically and non-classically. The truth-logics which are studied in this paper satisfy this requirement.

4. The vocabulary of the systems of truth-logic which we are building consists, as already indicated (p. 1), of sentential variables p, q, ... (an unlimited multitude), the connectives $\sim$, &, $\vee$, $\rightarrow$, $\leftrightarrow$, the truth operator T, and brackets (). The variables are schematic representations of sentences which express propositions. (A sentence expresses a proposition, I shall say, if it is grammatically well-formed and the sentence which we get by prefixing to it the phrase "it is true that" is also well-formed.)

Only the connectives $\sim$ and & for negation and conjunction respectively are primitive concepts of the systems. $\vee$, $\rightarrow$, and $\leftrightarrow$ are defined in terms of the primitives in the "classical" manner:

$$\text{“}p \vee q\text{”} =_{\text{df}} \text{“}\sim(\sim p \,\&\, \sim q)\text{”}$$
$$\text{“}p \rightarrow q\text{”} =_{\text{df}} \text{“}\sim(p \,\&\, \sim q)\text{”}$$
$$\text{“}p \leftrightarrow q\text{”} =_{\text{df}} \text{“}\sim(p \,\&\, \sim q) \,\&\, \sim(\sim p \,\&\, q)\text{”}.$$

(Here "p" and "q" are used as meta-variables for arbitrary formulas of the systems.)

The recursive definition of well-formed formulas we need not state explicitly here. Be it only observed in passing that the truth operator is iterable and that formulas or parts of formulas of the forms given in the above definitions can be replaced by their defined equivalents in the derivation of theorems from the axioms.

The conventions adopted for introducing and omitting brackets are likewise assumed to be self-explanatory.

A formula consisting of the letter T followed by a well-formed formula (in the simplest case a variable) and molecular compounds of such formulas will be called T-formulas or T-expressions. A T-expression is of the first order when there is no occurrence of an operator T within the scope of another T.

A formula which is a variable or a molecular compound of variables and does not contain any occurrence of the operator T will be called a ("classical") proposition logic formula or PL-formula.

A formula which is a molecular compound of PL-formulas and T-formulas will be called *mixed*. $Tp \rightarrow p$, for example, is a mixed formula.

5. $Tp \,\&\, \sim T \sim p$ says that the proposition that p is univocally true. $\sim Tp \,\&\, T \sim p$ again says that it is univocally false. $Tp \,\&\, T \sim p$ means that it is both true and false and $\sim Tp \,\&\, \sim T \sim p$ that it is neither true nor false. In the last case one could say that the proposition falls in a *truth-value* *gap*. In the penultimate case one could speak of a *truth-value overlap*.

How many truth-values are there? Shall we say there are two: truth and falsehood? Or count the gaps and overlaps too as truth-values and say there are four in all? As will be seen later, we shall make use of a 4-valued *matrix*. But since all four values are definable in terms of *truth* and *negation*, it would also be possible to say that basically there is only *one* "truth-value", *viz. true*.

Of the 4 possibilities which we listed, Classical Logic (CL) admits only two: univocal truth and univocal falsehood. Propositions neither true nor false are "forbidden" by the Law of Excluded Middle and propositions both true and false by the Law of (Excluded) Contradiction.

There are in all 16 different ways in which one can "permit" or "forbid" some or several of the four cases. (We then include the two extreme cases of permitting all four and permitting none of them respectively.)

These 16 selections answer to 16 different "truth-logics". Not all of them seem to be of interest and some of them, moreover, would seem to be identical with one another. But some of them are worth considering. Four may be singled out for special study, *viz.* classical logic (CL), the logic admitting truth-value gaps (TL), the system admitting truth-value overlaps (T′L) and the logic (T″) which admits both gaps and overlaps. Of the four systems I shall distinguish a stronger and a weaker form, the former but not the latter having *mixed* formulas among its theorems.

Systems which allow truth-value gaps, *i.e.* propositions neither true nor false, may be called *paracomplete*. Systems which allow truth-value overlaps again will be called *paraconsistent*. The terms "paracomplete" and "paraconsistent" are suggestions of Professor Francisco Miró Quesada.

6. The systems which we are going to study have a common "core" which itself is a "truth-logic" of a kind. It has the following axioms:

A0. All formulas which are obtained from tautologies of classical, two-valued, propositional logic (PL) by putting the letter T immediately in front of every variable which occurs in the tautologous formula.

A1. $Tp \leftrightarrow T\sim\sim p$. A proposition is true if, and only if, its negation is false.

A2. $T(p\&q) \leftrightarrow Tp\&Tq$. A conjunction is true if, and only if, all its conjuncts are true.

A3. $T{\sim}(p\&q) \leftrightarrow T{\sim}p \vee T{\sim}q$. A conjunction is false if, and only if, at least one of its conjuncts is false.

The rules of inference or transformation are as follows:

R1. Substitution of formulas for variables. (A variable is also a formula.)
R2. Detachment (*modus ponens*).
R3. The Rule of Truth. If f is a provable formula (in a system of truth-logic), then $Tf\& {\sim}T{\sim}f$ is also provable. In other words: provable formulas are univocally true.

Some theorems of this core system CS will be mentioned. They are valid in all the four systems which we are going then to study in more detail.

$p \vee {\sim}p$ is a tautology of PL. Hence, by A0,

T1. $Tp \vee {\sim}Tp$

is a theorem of truth-logic. It says that every proposition is either true or is not true. This is the form in which the Law of Excluded Middle may be said to hold in all truth-logics. I shall call it the *weak* form of the Law. It should be distinguished from the formula $Tp \vee T{\sim}p$ which says that every proposition is either true or false. This is the *strong* Law of Excluded Middle and is also known as the Law of Bivalence. It holds good in some truth-logics but not in others. The same is true of the formula $T(p \vee {\sim}p)$ which says that the disjunction of a given proposition and its negation is true. It too is one of the several forms of "The Law of Excluded Middle".

The T-operator is distributive, not only over conjunctions (A2) but also over disjunctions. This easily follows from A1 and A3 and the definition of disjunction in the terms of negation and conjunction. Thus we have in all truth-logics the theorem

T2. $T(p \vee q) \leftrightarrow Tp \vee Tq$.

Be it observed in passing that the T-operator is not, in general, distributive over implications.

$Tp \vee {\sim}Tp$ is equivalent with ${\sim}({\sim}Tp \& {\sim}{\sim}Tp)$ by definition, and this again is equivalent with

T3. ${\sim}(Tp \& {\sim}Tp)$

by (A0 and) A1. This is a form of the *weak* Law of Contradiction; it says that no proposition is both true and not true.

$T(p \vee \sim p)$ is equivalent with $T\sim(\sim p \& \sim\sim p)$ which may be transformed into $T\sim(p \& \sim p)$. This is another form of the Law of Contradiction. It says that every contradiction (conjunction of a proposition with its negation) is false. This *strong* version of the law is *not* valid in all truth-logics.

Since $Tp \vee \sim Tp$ is valid in the truth-logics, $T(Tp \vee \sim Tp)$ is also valid by virtue of the Rule of Truth. After distribution we get from this

T4. $TTp \vee T\sim Tp$.

The last formula says that any proposition to the effect that a given proposition is true is itself either true or *false*. The Law of Bivalence thus holds for propositions of the particular form "it is true that –", *i.e.* of the form "T–". This is a noteworthy consequence of the fact that the "core" system of truth-logic includes A0 and R3. It means that, in the truth-logics, expressions of the form "T–" and their molecular compounds, can be "handled" in accordance with the principles of "classical" two-valued propositional logic. Mixed expressions, be it observed, can*not* be unrestrictedly handled in accordance with classical principles.

Since $Tp \vee \sim Tp$ is a theorem, this formula is, by R3, not only true but *univocally* true, *i.e.* also not false. Thus we have, in addition to $TTp \vee T\sim Tp$ also $\sim T\sim(Tp \vee \sim Tp)$. This formula becomes, after distribution and cancellation of double negations, successively $\sim T(Tp \& \sim Tp)$ and $\sim(TTp \& T\sim Tp)$ and $\sim TTp \vee \sim T\sim Tp$.

The last formula can also be written in the form $TTp \rightarrow \sim T\sim Tp$. T4 again can also be written $\sim T\sim Tp \rightarrow TTp$. Joining the two implication formulas we obtain the equivalence $\sim T\sim Tp \leftrightarrow TTp$ and from this, finally,

T5. $T\sim Tp \leftrightarrow \sim TTp$.

It is thus true in all truth-logics (which we are studying here) that the proposition that it is false that a given proposition is true is equivalent with the proposition that it is not true that the given proposition in question is true.

7. We add to the axioms of the core system the following two axioms:

A4. $T\sim Tp \leftrightarrow \sim Tp$
A5. $Tp \rightarrow \sim T\sim p$

A5 says that if a proposition is true then it is not (also) false. The formula can also be written in the form of a 3-termed disjunction $Tp \& \sim Tp \vee T{\sim}p \& \sim Tp \vee \sim Tp \& \sim T{\sim}p$. It says that a(ny) proposition is either univocally true, univocally false or neither true nor false. In other words, it allows for the existence of truth-value gaps, propositions without truth-value. This is the main characteristic of the system of truth-logic which we call TL: that it caters for the possibility of truth-value gaps.

It is a theorem of this logic that

T6. $TTp \leftrightarrow Tp$.

The proof is extremely simple: From T5 we obtain $TTp \leftrightarrow \sim T \sim Tp$ and from A4 $\sim T \sim Tp \leftrightarrow Tp$. Thus by transitivity T6.

Alternatively, we could have replaced A4 by this formula and then proved $T \sim Tp \leftrightarrow \sim Tp$. As can readily be seen this means that in TL every formula which contains an operator T inside another operator T may be reduced to, *i.e.* proved equivalent with, a formula of the first order, *i.e.* one in which no operator occurs in the scope of another.

Another theorem is

T7. $T(p \rightarrow q) \rightarrow (Tp \rightarrow Tq)$.

The T-operator is distributable "one way" over implications. The proof is as follows: $T(p \rightarrow q)$ by definition equals $T(\sim p \vee q)$ which because of the distributivity of the T-operator equals $T{\sim}p \vee Tq$. From the theorem (axiom) $Tp \rightarrow \sim T{\sim}p$ we get by contraposition $T{\sim}p \rightarrow \sim Tp$. (Remember that T-expressions can be handled "classically".) $(p \rightarrow \sim q) \rightarrow ((p \vee r) \rightarrow (\sim q \vee r))$ is a tautology of PL. Hence, by A0, $(Tp \rightarrow \sim Tq) \rightarrow ((Tp \vee Tr) \rightarrow (\sim Tq \vee Tr))$ is a theorem of TL. Now substitute "$\sim p$" for "p", "p" for "q" and "q" for "r". We get $(T{\sim}p \rightarrow \sim Tp) \rightarrow ((T{\sim}p \vee Tq) \rightarrow (\sim Tp \vee Tq))$. Since the first antecedent has already been proved, we can detach $(T{\sim}p \vee Tq) \rightarrow (\sim Tp \vee Tq)$. Here the antecedent is, by T2, equivalent with $T(\sim p \vee q)$ which can also be written $T(p \rightarrow q)$. The consequent again can be written $Tp \rightarrow Tq$. Herewith we have proved T7 above.

Since what is true is not false, the Rule of Truth can, in this particular system, be simplified to

R_1T. If a formula f is provable then also the formula Tf.

One final observation. Since $Tp \rightarrow \sim T{\sim}p$ and, by A4, $\sim T{\sim}p$ is equivalent with $T \sim T{\sim}p$ it follows by transitivity that

T8. $Tp \rightarrow T{\sim}T{\sim}p$.

Thus if a proposition is true it is false that it is false. But the converse is not provable. In this peculiarity, as in the rejection of the Law of Bivalence, TL may be said to resemble intuitionist logic. There are, however, also differences.

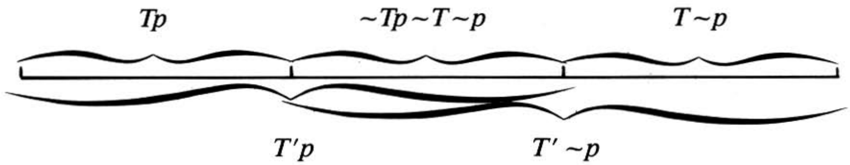
8[1]. Consider a process such as, *e.g.*, rainfall. It goes on for some time and then it stops. Not all of a sudden, let us assume, but gradually. Let p ________________________________ ${\sim}p$ illustrate that, during a certain stretch of time it is first definitely raining (p), later definitely not raining (${\sim}p$), and between these two states in time there is a "zone of transition" when a few drops may be falling – too few to make us say that it is raining then but too many to prevent us from saying that rain has definitely stopped. In this zone the proposition that p is neither true nor false. We can complete the picture as follows:

One could, however, also take the view that as long as some drops of rain are falling then it *is still* raining – but also the view that when there are only a few drops of rain falling, then it *is no longer* raining. When viewing the situation from these points of view one includes the intermediate zone of transition or vagueness both under rain and under not-rain, identifying raining with the state when it is not not-raining, and not-raining with the state when it is not raining. Then instead of saying that it is neither raining nor not raining in the zone, one would say that it is both raining and not raining in this area.

Surely both ways of speaking about the "zone" will strike one as equally natural. They are also, I think, equally common. But it should be observed that a *conceptual shift* has taken place in the notion of truth when we shift from the one way of speaking to the other. It is not the same sense of "true" in which we say that it is neither raining nor not raining *and* say that it is both raining and not r ing in the zone of transition.

(1) This section occurs almost verbatim in my paper "Truth, Negation, and Contradiction", *Synthese 66*, 1986, pp. 3-14.

We could call the former a *strict* sense of "true" and the latter a liberal or *laxer* sense of "true". This laxer notion of truth I shall symbolize by T'. It is defined in terms of strict truth as follows "$T'p$" $=_{df}$ "$\sim T{\sim}p$". We now complete our picture of the process:



To say that here we have two concepts of truth, or are concerned with two senses of "true" need not be a source of "mystification". What it means is simply that the word "true" can be predicated of propositions either in agreement with the rules for "T" or in agreement with the rules for "T′".

9. One can build a logic also for the laxer notion of truth. I shall call this logic T′L.

Since the two notions of truth, the stricter and the laxer, are interdefinable, one can obtain the logic T′L from the logic TL by noting that the concept of truth which figures in T′L equals the negation of falsehood in TL. That is: One can test whether a formula holds good in T′L by testing whether the formula which we get from it by replacing the operator T by the compound $\sim T{\sim}$ holds good in TL.

It follows that if in a formula which is provable in TL we replace all occurrences of T by $\sim T{\sim}$, the formula which we obtain holds good in T′L (but not necessarily in TL). Examples will be given below.

First it should be noted, however, that the core system, *i.e.* the axioms A0-A3 and everything that can be proved from them with the aid of the three rules R1-R3, is not affected by the replacement of T by $\sim T{\sim}$. That is: The formulas resulting from the replacement can, in the core system, be proved equivalent with the formulas in which replacement took place. A few examples will illustrate this:

$p \vee \sim p$ is a tautology of PL. Hence by A0 $Tp \vee \sim Tp$ is a theorem of the core system. We substitute $\sim p$ for p and replace T by $\sim T{\sim}$. We obtain $\sim T{\sim} \sim p \vee \sim \sim T{\sim} \sim p$. A1 in combination with the fact that formulas of the form T– can be handled classically, allows us to simplify this to $Tp \vee \sim Tp$ which is the theorem from which we started.

The reader can easily satisfy himself that the same substitution and replacement leaves, after simplifications, A1-A3 unchanged.

Similarly, R3 remains the same. $Tf \& \sim T{\sim}f$ is equivalent with $\sim T{\sim}f \& \sim \sim T \sim \sim f$. (Cancelling of double negations and commutation of conjuncts.)

It is also easy to see that the axioms and theorems of TL which we mentioned and which have the form of an equivalence are valid in $T'L$. Thus, in particular, the axiom A4 above becomes, after substituting $\sim T \sim$ for T and $\sim p$ for p: $\sim T \sim \sim \sim T \sim \sim p \leftrightarrow \sim \sim T \sim \sim p$. This can then, by A1, be simplified to $\sim T \sim Tp \leftrightarrow Tp$ which, by A0, is equivalent with $T \sim Tp \leftrightarrow \sim Tp$, *i.e.* with the formula (of TL) from which we started.

Theorems of TL, however, which have the form of implications hold in $T'L$ in the reverse direction. Thus, in particular, A5 of TL becomes

A′5 $\sim T{\sim}p \rightarrow Tp$.

It follows that we obtain an axiomatized version of T′L from the axiomatized TL by just reversing the direction of the implication in A5.

In TL it holds that if a proposition is true then it is not false; in T′L again that if a proposition is not false then it is true. The former principle, A5, can be transformed into a disjunction which says that any proposition is either univocally true or univocally false or lacks truth-value. Analogously, A′5 can be transformed into a disjunction which says that any proposition is either univocally true or univocally false or both true and false. Just as TL is a logic which caters for the possibility of truth-value *gaps*, T′L is a logic which caters for the possibility of truth-value *overlaps*. For this reason I shall, following an adopted terminology, call T′L a *paraconsistent* logical system.

TL, we said, resembles intuitionist logic in that, if a proposition is true then it is false that it is false. In T′L the reverse implication holds: If it is false that a proposition is false then it is true:

T′6. $T \sim T{\sim}p \rightarrow Tp$.

In the paraconsistent system the Law of Bivalence holds: $Tp \vee T{\sim}p$. (By virtue of A0, also the principle $Tp \vee \sim Tp$ is true; in TL this is a weaker principle than the Law of Bivalence; in T′L, however, it can be derived from it.)

The form of the Law of Contradiction $\sim(Tp \& \sim Tp)$ is valid in both systems, but the form $\sim(Tp \& T{\sim}p)$ which says that no proposition is both true and false is not valid in T′L. For this reason, also, this version of the principle sometimes called Duns Scotus's Law:

T9. $Tp \& T{\sim}p \rightarrow Tq$

which is valid in TL is not valid in T′L. If by "contradiction" we mean that a proposition is both true and false, then the paraconsistent system is "immune" to trivialization through the occurrence of a contradiction (in a context in which reasoning follows the laws of this logic).

Since ${\sim}(Tp \& {\sim}Tp)$ is a theorem both in TL and in T′L, it follows that if a contradiction of the form $Tp \& {\sim}Tp$, *i.e.* a contradiction which says that one and the same proposition both is and is not true, occurs in a deductive system which accepts either the laws of TL or of T′L, then the system is trivialized. This is a consequence of the fact that T-expressions themselves behave "classically".

Since in T′L, a proposition which is not false is true, it follows that the Rule of Truth R3 can in this system be replaced by

R_2T If a formula f is provable, then the formula ${\sim}T{\sim}f$ is also provable.

10. Next we omit A5 from TL or A′5 from T′L. Then we obtain a new system of truth-logic which I shall call T″. In this system neither the strong Law of Excluded Middle (Law of Bivalence) nor the strong Law of Contradiction is a theorem. The (strong) Law of Double Negation, in this logic, holds neither way. This logic, in other words, caters for the possibility both of truth-value gaps and overlaps.

Since A4 is an axiom of T″L we can also prove the reduction principle $TTp \leftrightarrow Tp$ (T6). (Above p. 10.)

11. We join the two axioms A5 and A′5 respectively to an equivalence $Tp \leftrightarrow {\sim}T{\sim}p$. It can also be written in the form $Tp \& {\sim}T{\sim}p \vee {\sim}Tp \& T{\sim}p$ which says that any proposition is either (univocally) true or (univocally) false. From this follows easily $Tp \vee T{\sim}p$, *i.e.* the Law of Bivalence, and $Tp \leftrightarrow T{\sim}T{\sim}p$, or a strong form of the Law of Double Negation.

If we strengthen either the axiom A5 of TL or A′5 of T′L to the said equivalence we obtain a truth-logic in which every provable formula is reducible to a molecular compound of formulas of the simple form T followed by a variable (p, q, ...). The letter T which in the original systems had the function of distinguishing between ${\sim}T$ or not-truth and $T{\sim}$ or falsehood now loses this function, the two ideas being indistinguishable (from the point of view of truth-value) in this system. The symbol T is now redundant. If we drop it from the axiom A1-A5 these reduce to simple tautologies of classical two-valued propositional logic. A0 just says that all tautologies are theorems (including those to which A1-A5 reduce), and the Rule of Truth likewise is redundant.

But unless we have a rule precisely to the effect that T may be dropped, *e.g.* an axiom $Tp \leftrightarrow p$, all theorems of this truth-logic are still T-expressions, although T-expressions which are provably equivalent with tautologies differing from those of traditional PL only in the fact that all variables have a T in front of them. I shall call this "classical" truth-logic CL.

12. It is a common characteristic of the four systems which we have so far studied that no theorem is a "mixed" formula, *i.e.* a formula composed of T-expressions and PL-expressions (or simply of PL-expressions). This restriction will now be transcended. The resulting systems will be called TLM, T′LM, T″LM, and CLM respectively.

TLM is obtained by adding to the core system the axiom

A6. $Tp \rightarrow p$.

This can be read "if it is true that p then p". A more adequate reading is "either it is not true that p or p", which can also be turned round to "either p or it is not true that p".

It can be proved that, adding A6 to the core system, we can prove the characteristic axioms A4 and A5 of TL. The proof is due, essentially, to Professor Carlos Alchourrón.

From A6 we get by substitution $TTp \rightarrow Tp$. From T5 of the core system we get $\sim T \sim Tp \rightarrow TTp$ and thus by transitivity we have $\sim T \sim Tp \rightarrow Tp$. From this by contraposition we get $\sim Tp \rightarrow T \sim Tp$.

From A6 we also get by substitution $T \sim Tp \rightarrow \sim Tp$. The last two implications jointly give us $T \sim Tp \leftrightarrow \sim Tp$ or A4.

By contraposition we get from $T \sim Tp \rightarrow \sim Tp$ the formula $Tp \rightarrow \sim T \sim Tp$. From T5 we got $\sim T \sim Tp \rightarrow TTp$. Hence by transitivity we have $Tp \rightarrow TTp$. This in conjunction with the first implication above gives the equivalence $TTp \leftrightarrow Tp$ or the reduction formula T6.

From T1 we obtain by substitution $T(p \& \sim p) \vee \sim T(p \& \sim p)$.

The first disjunct can, by virtue of A2 and T6, be transformed to

$TTp \& T{\sim}p$, and the second disjunct, by virtue of A2, PL (de Morgan's Law), and A4, to $T{\sim}Tp \vee T{\sim}T{\sim}p$. Thus the disjunction above becomes $(TTp \& T{\sim}p) \vee (T{\sim}Tp \vee T{\sim}T{\sim}p)$.

From each one of the disjuncts we can "pull out" the outer occurrence of the operator "T", first by an application of A2 and T2, and then by a second application of T2 so as to get $T({\sim}p \& Tp \vee {\sim}Tp \vee {\sim}T{\sim}p)$.

${\sim}p \& Tp$ is, by definition, another form of ${\sim}(Tp \rightarrow p)$ and ${\sim}Tp \vee {\sim}T{\sim}p$ another form of $Tp \rightarrow {\sim}T{\sim}p$. Thus we have $T({\sim}(Tp \rightarrow p) \vee (Tp \rightarrow {\sim}T{\sim}p))$. This again is but another form of $T((Tp \rightarrow p) \rightarrow (Tp \rightarrow {\sim}T{\sim}p))$.

From A6 we obtain $T((Tp \rightarrow p) \rightarrow (Tp \rightarrow {\sim}T{\sim}p)) \rightarrow ((Tp \rightarrow p) \rightarrow (Tp \rightarrow {\sim}T{\sim}p))$. Since we have already proved the antedecent we can detach the consequent (R2). We get $(Tp \rightarrow p) \rightarrow (Tp \rightarrow {\sim}T{\sim}p)$. But since the antecedent of this implication is an axiom (A6) we can again detach the consequent and obtain $Tp \rightarrow {\sim}T{\sim}p$ or the axiom A5 of TL.

The system TLM thus includes TL. The corresponding holds for the other three systems with mixed formula theorems.

13. The "paraconsistent" dual systel T′LM is obtained from the "paracomplete" system TLM by simply replacing the symbol "T" by "${\sim}T{\sim}$" in axiom A6 of TLM. We thus get the characteristic axiom of T′LM

A′6. ${\sim}T{\sim}p \rightarrow p$.

This axiom says that if it is not false that p then p. Or, in a more adequate reading, that either p or it is *false* that p.

Just as one can prove that adding A6 to the core system makes A4 and A5 provable, one can show that adding A′6 makes A4 and A′5 provable.

In T′LM we can, moreover, prove the reverse of A6 or

T′7. $p \rightarrow Tp$.

Proof: By A0 we have ${\sim}Tp \vee Tp$. By A4 $Tp \leftrightarrow {\sim}T{\sim}Tp$. Hence also ${\sim}Tp \vee {\sim}T{\sim}Tp$ which is equivalent with ${\sim}(Tp \& T{\sim}Tp)$. By A2 we get from this ${\sim}T(p \& {\sim}Tp)$ and from this by A1 ${\sim}T{\sim}{\sim}(p \& {\sim}Tp)$. The definition of material implication takes us from this to ${\sim}T{\sim}(p \rightarrow Tp)$ and this together with A′6 gives by R2 T′7. *Q.E.D.*

Thus whereas in TLM we have $Tp \rightarrow p$, we have in T′LM $p \rightarrow Tp$.

14. Next we add to the core system an axiom

A″. $Tp \& {\sim}T{\sim}p \rightarrow p$.

This is weaker than either A6 or A′6. It says that if it is *univocally* true that p then p. A6 only required that the proposition in question be true, leaving open the possibility that it is also false; A′6 required that the proposition be not false, leaving open the possibility that it is not true either; A″ demands of the proposition that it both be true *and* not false.

The resulting system is called T″LM. In it neither the strong Law of Contradiction, nor the strong Law of Excluded Middle (Law of Bivalence), nor the strong Law of Double Negation hold good. It can be proved, however, that A4 and T4 are valid in the system. Thus complete reduction of higher order formulas to first order formulas is possible as in the system T″L. The present system is an extension of T″L so as to include also some theorems which are "mixed".

The proof of the reduction principles is as follows:

Substitute in A″6 Tp for p. We get $TTp \& \sim T \sim Tp \rightarrow Tp$. Since, by T5 of the core system the two conjuncts in the antecedent are equivalent we can simplify the implication either to $TTp \rightarrow Tp$ or $\sim T \sim Tp \rightarrow Tp$. The latter can also be written $\sim Tp \rightarrow T \sim Tp$.

Finally, substitute in A″6 $\sim Tp$ for p. We obtain $T \sim Tp \& \sim TTp \rightarrow \sim Tp$. Because of the equivalence of the two conjuncts we can simplify this to either $T \sim Tp \rightarrow \sim Tp$ or to $\sim TTp \rightarrow \sim Tp$. This last implication can be transformed to $Tp \rightarrow TTp$.

Herewith we have proved the four implications which jointly make up the two equivalences A4 and T4.

15. We add either the axiom A′6 to TLM or the axiom A6 to T′LM. In either case it may be shown (using R3) that the conjunction of the two axioms is equivalent to

A7. $p \rightarrow Tp \& \sim T \sim p.$

or the "reverse" of A″6.

Since the strong Law of Contradiction is valid in TLM but not in T′LM and the strong Law of Excluded Middle in T′LM but not in TLM both laws hold in the new system. The same is true of the strong Law of Double Negation.

Thus also A4, A6, and A′6 are provable. The system is therefore an extended version of CL. In it we can prove the equivalence

$T_{cl}8.$ $Tp \leftrightarrow p.$

With its aid it is easily shown that all theorems of CLM are equivalent with ("reducible to") tautologies of PL.

16. The equivalence $Tp \leftrightarrow p$ is well-known from discussions about the nature of truth. Its meaning is often expressed by saying that the phrase "it is true that", when prefixed to a sentence, is otious or redundant. But *this* is true only if one accepts the laws (of excluded middle and of contradiction) of classical logic. In classical logic the phrase "it is true that" *is* indeed redundant – and this explains why the truth operator is not needed in the object language of the classical calculus. But the classical calculus is only a special, limiting, case of truth-logic. In other truth-logics the truth-operator is *not* redundant.

In TLM we have $Tp \rightarrow p$. We could prove $p \rightarrow Tp$ if we had also $Tp \vee T{\sim}p$ which in fact is equivalent with $T(p \rightarrow Tp)$.

In T′LM we have $p \rightarrow Tp$. We could prove $Tp \rightarrow p$ if we had also ${\sim}(Tp \& T{\sim}p)$ which in fact is equivalent with $T(Tp \rightarrow p)$.

In CLM we have both the required principles and can therefore prove *both* implications. The well-known equivalence $Tp \leftrightarrow p$ is thus based on the assumption that propositions obey the Laws of Excluded Middle and of Contradiction in their strong forms, *i.e.* that propositions are either univocally true or univocally false and never both true and false.

17. The study of truth-logic opens interesting vistas on the antinomies. I shall define an antinomic proposition as follows:

The proposition that p is antinomic if, and only if, assuming that it is either true or false, one can prove that, if it is true it is false, and if false true. This means that the proposition that p is antinomic if, and only if, one can prove a formula $Tp \vee T{\sim}p \rightarrow Tp \& T{\sim}p$.

Now consider what happens to this formula in various "logics".

In classical logic (CLM) the truth-operator is redundant. The formula then "reduces" to $p \vee {\sim}p \rightarrow p \& {\sim}p$. But in classical logic $p \vee {\sim}p$ is valid. Hence, *modo ponente*, we derive $p \& {\sim}p$. In classical logic $p \& {\sim}p \rightarrow q$ is valid. Hence, by a second use of *modus ponens*, we derive q, *i.e.* any arbitrary proposition. Conclusion: If reasoning in a system or other context of propositions obeys the laws of classical logic, the occurrence in that system of an antinomic proposition has the "catastrophic" consequence of "trivializing" the system: anything can now be proved in it.

In TL the Law of Bivalence $Tp \vee T{\sim}p$ is not valid. By A5 (and A1), the conjunction $Tp \& T{\sim}p$ entails the conjunction ${\sim}Tp \& {\sim}T{\sim}p$. Thus, by transitivity, we have $Tp \vee T{\sim}p \rightarrow {\sim}Tp \,\&\, {\sim}T{\sim}p$. The consequent of this implication is the negation of the antecedent. Hence, by the principle which the Schoolmen called *consequentia mirabilis*, the implication is equivalent with its consequent. Thus we have ${\sim}Tp \,\&\, {\sim}T{\sim}p$. Therefore, if reasoning in a system obeys the laws of TL, the appearance of an antinomic proposition in the system means that this proposition lacks truth-value, is neither true nor false.

In T′L the Law of Bivalence holds. Hence if an antinomic proposition appears in a system which obeys the laws of T′L we may legitimately conclude that this proposition is both true and false, $Tp \,\&\, T{\sim}p$. But since Duns Scotus's Law is not valid in T′L (above p. 16) the existence in this system of an antinomic proposition, *i.e.* of a proposition which is both true and false, has no catastrophic consequences.

In a system which obeys the laws of T″, finally, one cannot from the appearance of an antinomic proposition draw the conclusion that this proposition is both true and false, nor the conclusion that it is neither true nor false. The proposition just has no relation at all to truth-values. We can say about it only that if it is either true or false then it is both true and false. We cannot, as in (CL and) T′L, conclude that it must be both true and false, since in T″ the Law of Bivalence does not hold. But nor can we, as in TL, conclude that it is neither true nor false, since the two truth-values are not as in TL exclusive. And since Scotus's Law does not hold in T″ either, the antinomic proposition has no trivializing consequences.

The above observations were designed to show that only in a context of reasoning which proceeds in accordance with the principles ("laws") of classical logic is the occurrence of an antinomy a "catastrophe". From this fact one can draw either one of two possible conclusions. One conclusion is that there is some logical flaw connected with antinomies, that antinomic propositions are not "logically legitimate" constructions. The second conclusion is that classical logic simply is not adequate for dealing with antinomies. The traditional "way out" has been to draw the first conclusion. Then one is faced with the task of showing *what* is "wrong" with antinomies. One way of doing this was the creation of Russell's Theory of Types. There are also other proposals. None of them, however, has seemed entirely satisfactory, free from arbitrariness. Therefore the second "way out" mentioned above may be better, *i.e.* to look for a logic which can cope with antinomies without causing disaster to the reasoning. There are several such ways. One is offered by TL. In a context of reasoning which proceeds in accordance with the principles of TL the occurrence of an antinomic proposition has no trivializing consequences, and the proposition itself can be laid aside as void of truth-value. Another possibility is provided by T′L. It allows us to "embrace" both of two contradictory propositions without catastrophe.

That classical logic cannot, without imposing restrictions on proposition-formation, cope with the antinomies does not show that classical logic is "wrong". Nor does the fact that some non-classical logics can do this show that they are "right". The question which is the "best" way of coping with antinomies is not a question which can be answered *inside logic.*

18. T″LM has a four-valued matrix. The four values are those mentioned in Sect. 5 above: True and false, true but not false (univocally true), false but not true (univocally false), neither true nor false. Let us denote them by "1", "+", "–", and "0" respectively.

In order to show that a truth-logic has a matrix in terms of these values, we need to show that the two truth-functions, negation and conjunction, and the truth-operator have a matrix. The table for negation looks as follows:

p	$\sim p$
1	1
+	–
–	+
0	0

The table can be "computed" from the "meaning" of the four signs 1, +, –, 0 as follows:

That $\sim p$ has the value 1 when p has this value means that $Tp \& T{\sim}p$ in the core system entails $T{\sim}p \& T{\sim}{\sim}p$. Thus if it is both true and false that p it is also both true and false that $\sim p$. Similarly, $Tp \& {\sim}T{\sim}p$ entails $T{\sim}p \& {\sim}T{\sim}{\sim}p$, *i.e.* if it is univocally true that p it is univocally false that $\sim p$. Finally, ${\sim}Tp \& {\sim}T{\sim}p$ entails ${\sim}T{\sim}p \& {\sim}T{\sim}{\sim}p$; if a proposition lacks truth-value its negation also lacks truth-value.

The table for conjunction is this:

p	q	$p\&q$
1	1	1
1	+	1
1	–	–
1	0	–
+	1	1
+	+	+
+	–	–
+	0	0
–	1	–
–	+	–
–	–	–
–	0	–
0	1	–
0	+	0
0	–	–
0	0	0

The principle on which the calculation of the values is based is this: The conjunction has the value 1 if both propositions are true but at least one of them is not univocally true; it has the value + in the sole case when both conjuncts are univocally true; it has the value – when at least one conjunct has the value false, either univocally or not; and it has the value 0 if none of the propositions is false and not both univocally true either.

The table for the truth-operator, finally, is

p	Tp
1	+
+	+
–	–
0	–

The table is based on the entailments between $Tp\&T{\sim}p$ and TTp, $Tp\&{\sim}T{\sim}p$ and TTp, ${\sim}Tp\&T{\sim}p$ and $T{\sim}Tp$, and ${\sim}Tp\&{\sim}T{\sim}p$ and $T{\sim}Tp$. These entailments hold in T″L(M). They thus hold *a fortiori* in the three other systems of truth-logic as well with which we are acquainted (with or without "mixed" theorems).

If from the tables we omit all lines with a "1" we obtain mutilated tables which are valid for TLM. If we omit the lines with a "0" we get tables for T′LM. And if we retain only the lines with + or − we get tables for CLM. As seen, in these last tables the truth-value of *Tp* equals that of *p* and of *T*~*p* that of ~*p*. This means that "*T*" is now otious.

19. If the table for an expression in one of our four truth-logics gets the value + for all distributions of the "allowed" truth-values, *i.e.* 1, +, −, 0 in T″LM; +, −, 0 in TLM; 1, +, − in T′LM, and +, − in CLM, we shall say that the expression in question is a *truth-tautology*.

It may be shown that the class of theorems in a truth-logic is co-extensive with the class of truth-tautologies *in that logic*. This is the sense in which these four logics may be said to be "semantically complete" univocal truth being the criterion of "true theorem" in a logic which admits, beside univocal truth and falsehood, also truth-value gaps and/or overlaps.

By "tautology" one usually understands a formula of the form of a tautology in PL. It may sound strange to have to say that no tautology in this sense is tautologically true (except in PL). But this needs not at all surprise us. If, for example, the proposition expressed by "*p*" lacks truth-value, then it would be absurd to say that "*p*∨~*p*" nevertheless is "true". If again the proposition in question is both true and false, this holds for its negation too, and "*p*∨~*p*" though true is not *univocally* so.

20. Consider the systems without mixed formula theorems. In all of them the non-mixed formulas can be exhibited in the form of molecular compounds of "truth-constituents" *T*– and *T*~– where the place of the dash is taken by a variable, *p*, *q*, ... The formula can then be tested for theoremhood in a truth-table where the only truth-values are + and −. In T″L they can be distributed without restriction over the *T*-constituents. In T′L one has to observe the restriction that the combination −, − is not allowed, in TL the restriction that the combination +, + is not allowed, and in CL that neither −, − nor +, + is allowed. If, and only if, the formula gets the value + for all allowed distributions it is a theorem of T″, T′L, TL, and CL respectively.

For the systems with mixed formula theorems we need matrices in which the initial distributions of truth-values are over the variables *p*, *q*, ... themselves. There are now no restrictions on distributions of truth-values. As we have seen, the matrix for T″ is 4-valued, those for T′LM and TLM 3-valued and that for CLM 2-valued.

21. In TL we have $Tp \& \sim T\sim p \vee \sim Tp \& T\sim p \vee \sim Tp \& \sim T\sim p$: a proposition is either univocally true or false or it lacks truth-value. The two last terms can be contracted to $\sim Tp \& (T\sim p \vee \sim T\sim p)$. The disjunction $T\sim p \vee \sim T\sim p$ is itself a T-tautology and may be dropped. Remains $Tp \& \sim T\sim p \vee \sim Tp$ which can be written $Tp \rightarrow Tp \& \sim T\sim p$ (also obtainable directly from A5). It says that if a proposition is true, then it is univocally so.

If we construct a 4-valued truth table for $Tp \& \sim T\sim p$ we shall find that it has the value $+$ then and then only when p has this value. That p has this value means that it is univocally true and the statement that this is univocally true or $T(Tp \& \sim T\sim p) \& \sim T\sim (Tp \& \sim T\sim p)$ reduces to $Tp \& \sim T\sim p$ itself.

Similarly it may be shown that $T\sim p \& \sim T\sim p$ assumes the value $+$ if, and only if, p takes the value $-$, *i.e.* $\sim p$ the value $+$.

It follows from this that a matrix which will verify $Tp \rightarrow Tp \& \sim T\sim p$ (or simply $Tp \rightarrow \sim T\sim p$) will also verify $Tp \rightarrow p$ and *vice versa*. On this fact is based the possibility of replacing A5 by $Tp \rightarrow p$. After this replacement, as we have seen, A5 becomes redundant and the reduction principles provable.

Similarly a matrix verifying $\sim T\sim p \rightarrow Tp$ will verify $p \rightarrow Tp$ and *vice versa*. Proof: $\sim T\sim p \rightarrow Tp$ is equivalent with $Tp \& T\sim p \vee Tp \& \sim T\sim p \vee \sim Tp \& T\sim p$ which says that any proposition either is univocally true or false or both true and false. Contracting the two first disjuncts and dropping the tautology $T\sim p \vee \sim T\sim p$ we get $Tp \vee \sim Tp \& T\sim p$. Replacing the second disjunct by $\sim p$ we obtain $p \rightarrow Tp$.

In CL $Tp \& \sim T\sim p \vee \sim Tp \& T\sim p$ is a theorem: any proposition is either univocally true or univocally false. Replacing $\sim Tp \& T\sim p$ by $\sim p$ we get from this $p \rightarrow Tp \& \sim T\sim p$.

In T″, finally, all four possibilities are allowed: $Tp \& T\sim p \vee Tp \& \sim T\sim p \vee \sim Tp \& T\sim p \vee \sim Tp \& \sim T\sim p$. Since the disjuncts are mutually exclusive we have $\sim (Tp \& T\sim p \vee \sim Tp \& T\sim p \vee \sim Tp \& \sim T\sim p) \rightarrow Tp \& \sim T\sim p$. We replace the consequent by p alone. The antecedent is easily reduced to $Tp \& \sim T\sim p$ and we get $Tp \& \sim T\sim p \rightarrow p$ or the characteristic axiom of T″LM.

22. It is a common feature of all the truth-logics so far studied by us that propositions to the effect that some other propositions are true, *i.e.* propositions expressed by sentences of the form "*T*–", are bivalent, either true or false. The logic of these propositions, moreover, is classical logic.

This means, for example, that whereas $Tp \vee T{\sim}p$ is not a truth-tautology in TL, $TTp \vee T{\sim}Tp$ is a truth-tautology in TL. This last formula, moreover, is provable in TL. (See above, p. .)

One reason for building a logic which allows truth-value gaps was the following: A *generic* state of affairs such as, *e.g.*, rainfall may be such that criteria for telling whether it obtains or not are not at hand for all occasions in space and time. The occasions which fall in a given spatio-temporal area (span) may then be divided into such in which it is true that the state in question obtains, such in which it is false that it obtains, and such in which it is neither true nor false that it obtains. But what tells us to which of these three categories a given occasion (in the area) belongs? In TL it is just taken for granted that it belongs to one, and only one, of them. But could there not be "gaps" also between any two of the three categories – just as there is a gap between the clear cases when the state in question obtains and the clear cases when it does not obtain? Could there, for example, not be instances of the obtaining state of weather such that, on the available criteria, one cannot tell whether at a certain place and time it is raining or, rather, neither raining nor not-raining? I do not think that this possibility can be dismissed as "unthinkable". If we accept it, we have a 5-partite instead of a tripartite division of logical space: For any given proposition that p there are four mutually exclusive and jointly exhaustive possibilities: TTp, $TT{\sim}p$, $T{\sim}Tp \,\&\, T{\sim}T{\sim}p$, and ${\sim}TTp \,\&\, {\sim}TT{\sim}p \,\&\, {\sim}(T{\sim}Tp \,\&\, T{\sim}T{\sim}p)$. The last, however, can be split in two: ${\sim}TTp \,\&\, {\sim}TT{\sim}p \,\&\, {\sim}T{\sim}Tp$ and ${\sim}TTp \,\&\, {\sim}TT{\sim}p \,\&\, {\sim}T{\sim}T{\sim}p$ which are also mutually exclusive.

In TL the expression ${\sim}TTp \,\&\, {\sim}TT{\sim}p \,\&\, {\sim}(T{\sim}Tp \,\&\, T{\sim}T{\sim}p)$ is refutable. Hence a logic which accepts it as a possibility must differ from TL at least in those features which, in TL, make it possible to refute the expression in question. It is immediately clear that these features have to do with the reducibility of iterated occurrences of the truth-operator to non-iterated or first order occurrences. A way of preventing this reducibility (between the levels here under consideration) is the following:

> We replace the axiom A0 by the following axiom
> B0. Any formula which we obtain from an axiom or theorem of TL when to every occurrence of a variable in the formula we prefix the letter T is a theorem.

The system the logical truths of which can be obtained from theorems of TL by virtue of B0 is a sub-system of TL. Every theorem of the sub-system is a theorem of TL but not *vice versa*. I shall call the new system a truth-logic of *the second order*.

Within every one of the truth-logics of the first order one can isolate such a sub-system of the second order. The true formulas of the sub-system are generated from those of the system by a simple operation of prefixing the letter T to the variables.

In all these second order logics it would hold, however, that the second order T-expressions themselves are bivalent and "behave" according to the laws of classical logic. Thus whereas $TTp \vee T \sim Tp$ would not be a theorem of the second order system obtained from TL, $TTTp \vee T \sim TTp$ would be a theorem.

And the same holds, *mutatis mutandis*, for the third order logics which we get if we denied the "classical" character of second order T-expressions. The levels higher than that of the axioms of a given truth-logic remain "classical". That this should be so is, essentially, a consequence of A0 and R3 in the axiomatic basis of what I have called the "core system". One could also express this by saying that the various truth-logics are *systematized* (studied as logics) *under a classical point of view*. Must classical two-valued logic hold this "privileged position" in a systematic study of all logics? This is a question which I shall not attempt to answer here.

Be it observed in passing that the "isolation" of sub-systems of higher order also applies to the truth-logics with mixed formulas among their theorems. In TLM this would mean, for example, that $TTp \rightarrow Tp$ is a theorem of the second order system, but not $Tp \rightarrow TTp$. In TL'M again it would mean that whereas $Tp \rightarrow TTp$ is a theorem, the reverse, or $TTp \rightarrow Tp$ is not.

23. What is the meaning and purpose of the calculi which I have called truth-logic(s)? To answer the question is to try to say something about the nature of logic.

Logic is concerned with *reasoning*. The study of logic – logic as a "science" – is an attempt to systematize the principles used in correct or sound reasoning. It is, as I see it, a *descriptive*, and not a normative task.

Reasoning takes place in a huge variety of different contexts – in every day life, in mathematics and science, in the courts, and also, by the way, in logic. It cannot be taken for granted that the principles according to which sound reasoning proceeds are the same in all (types of) context. The "laws of logic" are *not* necessarily valid *semper et ubique.*

Most reasoning is with propositions or, which I take to be a different way of saying the same, with sentences which express propositions. The sentences used in reasoning, however, are on the whole not of the form "it is true that –", but are sentences without or stripped of this pre-phrase. Occasionally only, locutions such as "It is true that –" or "Since it is false that –" explicitly enter into reasoning (outside truth-logic itself).

Implicitly, however, all reasoning with propositions is concerned with truth. It is one of the most basic ideas associated with logic that sound logical reasoning is a *truth-preserving* progression of thoughts. When, for example, we argue that if A then B, we maintain that if A *is true* then B is true too. And when by contraposition from this we switch to saying that if not B then not A either what we usually mean is that if B were false A would also be false.

But is this last step justified? In classical logic it is. Classical logic makes no distinction between the denial of truth and the truth of denial, between $\sim T$ and $T\sim$. Both mean falsehood. Not so, however, in our paracomplete system TL. TL distinguishes between "it is not true that" and "it is true that not" and allows us to infer the first from the second, but not *vice versa.* A paraconsistent logic such as our T′L observes the same distinction but allows only the converse inference, *viz.* from "it is not true that" to "it is false that", keeping open a possibility that something false is *also* true.

One cannot ask: Which is the true truth-logic? The question simply makes no sense. But one can ask: According to which logic does reasoning in a given context actually proceed? And, since this question does not always have a univocal answer, one can also ask which truth-logic is best suited for reasoning in a given context. These questions cannot be answered *in logic*, but must be answered so to speak from the platform of the *contexts.* It cannot be taken for granted, for example, that reasoning in all branches of mathematics has to follow the patterns of classical logic. Brouwer produced heavy reasons against the use of classical logic in mathematics and for the use of another type of logic. Whether what he intended was more in line with what has since become known as "intuitionist logic" or with our truth-logic TL is to my mind something of an open question. Be this as it may, it seems to me obvious that for reasoning with *vague* concepts a logic like TL which allows truth-value gaps is better suited than classical logic which does not allow this. And for reasoning about *processes* and the flux of a changing world the use of a paraconsistent type of logic may be more commendable.

There are other possibilities still. But it is perhaps of some interest to note that, if not all, so at least a good many of them can be systematized within a general theory of what I have here called truth-logic(s). In this regard truth-logic represents a *unification* of logical thinking – not in the sense of *one* "true" logic but in the sense of a variety of related alternatives.

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