NON - LOGICAL CONSEQUENCES

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0. *Introduction*

The inspiration for this paper came entirely from Richard Sylvan and Newton da Costa's 'Cause as an Implication' [3]. There are some relations between possible states of affairs that are not *logical* relations like entailment. 'that... brings it about that...' is one, or is at least thought to be one by some who believe in casation. (1) Sylvan and da Costa provide an enlightening discussion concerning the many possible ways of axiomatizing causal implication, including helpful review of what other philosophers have written on the topic.

In this paper I shall limit my efforts to sketching out a *very* minimal kind of non-logical consequence relation, which I symbolize as $\rightarrow$.

Since $\rightarrow$ represents a non-logical consequence relation, the rules if $\vdash A \supset B$, then $\vdash \sim(A\rightarrow B)$, and if $\vdash A \supset \sim B$, then $\vdash \sim(A\rightarrow B)$ seem appropriate. Notice that the first rules out $A\rightarrow A$ as a theorem. Indeed, we have $\vdash \sim(A\rightarrow A)$, as well as $\vdash \sim(A\rightarrow T)$ and $\vdash \sim(F\rightarrow A)$, where $\vdash T$ and $\vdash T \equiv \sim F$. And we might as well round things off by having $\vdash \sim(T\rightarrow A)$.

Impossible states of affairs shouldn't be *non*-logical consequences of anything. So we want a rule with the effect of the following: if $\vdash \sim(B_1 \wedge ... \wedge B_n)$, then $\vdash \sim((A\rightarrow B_1) \wedge ... \wedge (A\rightarrow B_n))$.

These axioms and rules have all been concerned with the fact that $\rightarrow$ is non-logical. The following have to do with its being a consequence relation.

It is transitive. We should have $\vdash ((A_1\rightarrow A_2) \wedge ... \wedge (A_{n-1}\rightarrow A_n)) \supset (A_1\rightarrow A_n)$.

$\rightarrow$ should contrapose, i.e., $\vdash (A\rightarrow B) \supset (\sim B\rightarrow \sim A.)$

Finally things that are logically equivalent ought to have the same effects and be the effects of the same things. Thus we have the following

(1) I am guilty here of not discussing whether or not cause *can* be correctly viewed as a propositional connective. [3] contains an excellent treatment of this issue.

pair of rules: if $\vdash A \equiv B$, $\vdash (A \rightarrow C) \supset (B \rightarrow C)$, and if $\vdash A \equiv B$, $\vdash (C \rightarrow A) \supset (C \rightarrow B)$.

Now to summarize the axiomatics of this minimal system.

1. *The Language L*

Let PC be a propositional language with the one-place connective ~ and the two-place connectives ⊃, ∧, and ∨. The language L will be the intersection of all sets S of sentences such that:

(1) if $A \in PC$, $A \in S$,
(2) if $A,B \in S$, $\sim A, (A \supset B), (A \wedge B), (A \vee B) \in S$, and
(3) if $A,B \in PC$, $(A \rightarrow B) \in S$.

We say that a wff A is a *tautology in* L if

(1) → does not occur in A and A is a tautology of PC or
(2) for some $n \geq i$, A is the result of substituting any wffs $B_1, \ldots, B_n$ of L for atomic wffs $p_1, \ldots, p_n$, respectively, at all their occurrences in a wff B which is a tautology of PC.

In the axiom schematization we let T symbolize any tautology of L and F symbolize any wff tautologically equivalent to the negation of a tautology.

The axioms and rules of L are:

(A0) T
(A1) $\sim(T \rightarrow A)$
(A2) $((A_1 \rightarrow A_2) \wedge \ldots \wedge (A_{n-1} \rightarrow A_n)) \supset (A_1 \rightarrow A_n)$
(A3) $(A \rightarrow B) \supset (\sim B \rightarrow \sim A)$

(R1) If $\vdash B \equiv C$, $\vdash (B \rightarrow A) \supset (C \rightarrow A)$ (where → appears in neither B nor C, since if → occurs in, say, B, $B \rightarrow A$ is not well-formed).
(R2) If $\vdash B \equiv C$, $\vdash (A \rightarrow B) \supset (A \rightarrow C)$.
(R3) If $\vdash \sim(B_1 \wedge \ldots \wedge B_n)$, $\vdash \sim((A \rightarrow B_1) \wedge \ldots \wedge (A \rightarrow B_n))$.
(R4) If $\vdash A \supset B$, $\vdash \sim(A \rightarrow B)$.
(R5) If $\vdash A \supset \sim B$, $\vdash \sim(A \rightarrow B)$.

2. *Semantics*

A *model* M for the language L (a L-model) is a triplet <O,At,S>, where O is an actual world, At is the set of atomic sentences true in O, and S is a function that will be described later. (2)

First we have a few definitions involving the language PC, the propositional calculus. We write $\vdash_c$ A when A is a theorem of PC. Let $X \in \mathfrak{P}(PC)$).

(D1) $X \vdash_c A$ if for some $n \geq 1$ there are wffs $A_1,\ldots,A_n \in X$ such that $\vdash_c (A_1 \wedge \ldots \wedge A_n) \supset A$.

(D2) X is *c-inconsistent* if for some wff A, $X \vdash_c A$ and $\vdash_c \sim A$.

(D3) X is *c-consistent* if X is not c-inconsistent.

Given a wff A, where {A} is c-consistent, let $E[A] =_{df} \{B: \vdash_c A \equiv B\}$, $C[A] =_{df} \{B: \vdash_c A \supset B\}$, and $N[A] =_{df} \{B: \sim B \in C[A]\}$.

Now we let $S: \{E[A]: A \in PC\} \rightarrow \mathfrak{P}(PC)$ such that for any wff A of PC the following conditions are satisfied (writing S(A) for S(E[A])):

(1) S(A) is c-consistent,

(2) $S(A) \cap (C[A] \cup N[A]) = \emptyset$,

(3) for all wffs $B \in PC$, $B \in S(A)$ only if $S(B) \subseteq S(A)$,

(4) for all wffs $B,C \in E[D]$ for some wff D, $B \in S(A)$ only if $C \in S(A)$,

(5) $S(T),S(F) = \emptyset$, and

(6) for all wffs $B \in PC$, $B \in S(A)$ only if $\sim A \in S(\sim B)$.

As usual in story semantics, we have the one valuation rule for stories:

(VS) Where $A,B \in PC$, $S(A) \in V(B)$ iff $B \in S(A)$.

This leaves the valuation rules for the actual world:

(VAt) Where A is atomic, $O \in V(A)$ if $A \in At$, and otherwise $O \notin V(A)$.

$(V\sim)$, $(V\supset)$, $(V\wedge)$, and $(V\vee)$ are as expected. Where A is of the form $B \rightarrow C$,

(2) The semantics presented here is a variant of story semantics. For further information about story semantics, see [1] and [2].

(V→) $O \in V(B \to C)$ if $S(B) \in V(C)$, and otherwise $O \notin V(B \to C)$.

We write $<O,At,S> \Vdash A$ when $O \in V(A)$, and $\Vdash A$ when for all models M, $M \Vdash A$.

3. *Soundness*

Theorem. If $\vdash A$, $\Vdash A$.

Proof: *ad* (A1). Let A be a tautology in L. Since (V ~) - (V ⊃) are exactly the valuation rules for classical evaluation substituting 'T' for 'O' and '=' for '∈', A will be sound when the substitutions are reversed.

ad (A2). Suppose $O \notin V(\sim(T \to A))$. By (V ~) $O \in V(T \to A)$. By (V→) $S(T) \in V(A)$. But by (VS) $A \in S(T)$, which contradicts condition (5) on S.

ad (A3). Suppose $O \notin V(\sim(T \to A))$. By (V ~) $0 \in V((A_1 \to A_2) \wedge ... \wedge (A_{n-1} \to A_n))$ and $O \notin V(A_1 \to A_n)$. By (V∧) $O \in V(A_1 \to A_2),...,V(A_{n-1} \to A_n)$. By (V→) $S(A_1) \in V(A_2)$, ...,$S(A_{n-2}) \in V(A_{n-1})$, and $S(A_{n-1}) \in V(A_n)$. By (VS) $A_2 \in S(A_1)$, ...,$A_{n-1} \in S(A_{n-2})$, and $A_n \in S(A_{n-1})$. By condition (3) on S, $S(A_2) \subseteq S(A_1)$, ...,$S(A_{n-2}) \subseteq S(A_{n-1})$. But since $O \notin V(A_1 \to A_n)$, by (V→) $S(A_1) \notin V(A_n)$ and by (VS) $A_n \notin S(A_1)$. Yet since $A_n \in S(A_{n-1}) \subseteq S(A_1)$, we have a contradiction.

ad (A3). Suppose that $O \notin V(A3)$. Then by (V ⊃) $O \in V(A \to B)$ and $O \notin V(\sim B \to \sim A)$. So by (V→) and (VS) $B \in S(A)$ and $\sim A \notin S(\sim B)$, which is impossible by condition (6) on S.

ad (R1). Suppose $\vdash B \equiv C$. Since → occurs in neither B nor C, $\vdash_c B \equiv C$. So $E[B] = E[C]$. Since S is a function, $S(B) = S(C)$. So $A \in S(B)$ iff $A \in S(C)$. By (VS),(V→), and (V ⊃), $O \in V((B \to A) \supset (C \to A))$.

ad (R2). We have as above $E[B] = E[C]$. Suppose $O \notin V((A \to B) \supset (A \to C))$. By (V ⊃) $O \in V(A \to B)$ and $O \notin V(A \to C)$. By (V→) and (VS) $B \in S(A)$ and $C \notin S(A)$, which contradicts condition (4) on S.

ad (R3). Suppose that $O \notin V(\sim((A \to B_1) \wedge ... \wedge (A \to B_n)))$. By (V ~)

$O \in V((A \to B_1) \wedge ... \wedge (A \to B_n))$. By $(V\wedge)$ $O \in V(A \to B_1),...,V(A \to B_n)$. By $(V\to)$ and (VS), $B_1,...,B_n \in S(A)$. Since $\to$ occurs in none of $B_1,...,B_n$, $\vdash_c \sim(B_1 \wedge ... \wedge B_n)$. Thus S(A) is not c-consistent contrary to condition (1) on S.

ad (R4). Suppose that $O \notin V(\sim(A \to B))$. By $(V\sim)$ $O \in V(A \to B)$. By $(V\to)$ and (VS) $B \in S(A)$. Since $\to$ occurs in neither A nor B, $\vdash_c A \supset B$. Thus $B \in C[A]$. So $B \in S(A) \cap C[A]$ contrary to condition (3) on S.

ad (R5). Suppose that $O \notin V(\sim(A \to B))$. By the same reasoning as above $B \in S(A)$, and $\vdash_c A \supset \sim B$. Thus $B \in N[A]$. So $B \in S(A) \cap N[A]$ contrary to condition (3) on S.

This concludes the proof of the *Theorem*.

4. *Completeness*

We reissue definitions (D1)-(D3) for L, substituting $\vdash$ for $\vdash_c$, 'consistent' for 'c-consistent', and 'inconsistent' for 'c-inconsistent'. We also add

(D4) X is *maximal* if for all wffs A of L, $A \in X$ or $\sim A \in X$.

Lemma (4.1). If $A_1,...,A_n \in X$, $X \vdash A_1 \wedge ... \wedge A_n$.

Proof: Immediate from (A0) and (D1).

Given a wff A such that {A} is consistent, a maximal consistent (mc) set Max-A such that $A \in$ Max-A is constructed in the usual way.

Let Max be any mc set.

Lemma (4.2). If $\vdash A$, Max $\vdash A$.

Proof: Suppose $\vdash A$. By (4.1) and (D4) Max $\vdash A$ or Max $\vdash \sim A$. By (A0) $\vdash \sim\sim A$. By (D2) if Max $\vdash \sim A$, Max is inconsistent, which is impossible. So Max $\vdash A$.

Lemma (4.3). If Max $\vdash A$, $A \in$ Max.

Proof: Suppose that Max $\vdash A$ and $A \notin$ Max. By (D4) $\sim A \in$ Max.

By (D1) there are $B_1,\ldots,B_n \in$ Max such that $\vdash (B_1 \wedge \ldots \wedge B_n) \supset A$. By (A0) $\vdash \sim(B_1 \wedge \ldots \wedge B_n)$. By (4.1) $(B_1 \wedge \ldots \wedge B_n) \in$ Max. But then Max is inconsistent, which is impossible.

Lemma (4.4). $A \in$ Max iff Max $\sim$ A.

Proof: Immediate from (4.2) and (4.3).

From any mc set Max we can construct an L-model $<O,At,S>$. We let O = Max, At = {A: A is an atomic sentence and $A \in$ Max}, and let S be that function such that for any wff A $S(A) = \{B: A \rightarrow B \in \text{Max}\}$.

Lemma (4.5). S satisfies conditions (1) - (6) of Section 2 and hence $<\text{Max,At,S}>$ is an L-model.

Proof: *ad* (1). Suppose for some wff A, S(A) is not consistent. Thus for some wff B $S(A) \vdash_c B$ and $\vdash_c \sim B$. By (D1) for $C_1,\ldots,C_n \in S(A) \vdash_c (C_1 \wedge \ldots \wedge C_n) \supset B$. By propositional calculus $\vdash_c \sim(C_1 \wedge \ldots \wedge C_n)$. But then $\vdash \sim(C_1 \wedge \ldots \wedge C_n)$. By (R3) $\vdash \sim((A \rightarrow C_1) \wedge \ldots \wedge (A \rightarrow C_n))$. By construction $A \rightarrow C_1,\ldots,A \rightarrow C_n \in$ Max, since $C_1,\ldots,C_n \in S(A)$. By (A0) $(A \rightarrow C_1) \wedge \ldots \wedge (A \rightarrow C_n) \in$ Max. By (D2) Max is inconsistent, which is impossible.

ad (2). Suppose for some A, $S(A) \cap (C[A] \cup N[A]) \neq \emptyset$. Then for some wff B $B \in S(A) \cap (C[A] \cup N[A])$. By construction $A \rightarrow B \in$ Max, since $B \in S(A)$. Suppose $B \in C[A]$. Then $\vdash_c A \supset B$ in which case $\vdash A \supset B$. By (R4) $\vdash \sim(A \rightarrow B)$, which is impossible since Max is consistent. Suppose $B \in N[A]$. Then $\vdash_c A \supset \sim B$. But then $\vdash A \supset \sim B$. By (R5) $\vdash \sim(A \rightarrow B)$ which is again impossible.

ad (3). Suppose for some B $B \in S(A)$ and $S(B) \not\subseteq S(A)$. By construction $A \rightarrow B \in$ Max, and for some C $B \rightarrow C \in$ Max and $A \rightarrow C \notin$ Max. By construction $\sim(A \rightarrow C) \in$ Max. By (D1) $(A \rightarrow B) \wedge (B \rightarrow C) \in$ Max. By (A2) $((A \rightarrow B) \wedge (B \rightarrow C)) \supset (A \rightarrow C) \in$ Max, and by (A0) $(A \rightarrow C) \in$ Max. By (D1) $(A \rightarrow B) \wedge \sim(A \rightarrow B) \in$ Max, which is impossible.

ad (4). Suppose that $\vdash_c B \equiv C$ and $B \in S(A)$ and $C \notin S(A)$. By construction $A \rightarrow B \in$ Max, $A \rightarrow C \notin$ Max, and $\sim(A \rightarrow C) \in$ Max. Since $\rightarrow$ appears in neither B nor C, $\vdash B \equiv C$. By (R2) $\vdash (A \rightarrow B) \supset (A \rightarrow C)$. By (A0) $A \rightarrow C \in$ Max, which is impossible since Max is consistent.

ad (5). (i) $B \in S(T)$. By construction $T \rightarrow B \in Max$, which is impossible since in view of (A1) Max would then be inconsistent. (ii) Let $B \in S(F)$. By construction $F \rightarrow B \in Max$. $\vdash_c F \supset B$, and so $\vdash F \supset B$. By (R5) $\vdash \sim(F \rightarrow B)$, which is impossible since then Max would be inconsistent.

ad (6). Suppose for some A,B $B \in S(A)$ and $\sim A \notin S(\sim B)$. Then by construction $A \rightarrow B \in Max$ and $\sim B \rightarrow \sim A \notin Max$. By (A0), (A3), and (4.2) $\sim B \rightarrow \sim A \in Max$, wich is impossible.

This completes the proof of the Lemma.

Lemma (4.6). Max $\in V(A)$ iff $A \in Max$.

Proof: By induction on the complexity of A.

Base Case. A is atomic. Max $\in V(A)$ iff, by (VAt), $A \in At$ iff, by construction, $A \in Max$.

Induction Hypothesis (IH). The Lemma holds for all wffs of complexity less than n and A is of complexity n.

A is $\sim B$. Max $\in V(\sim B)$ iff, by ($V\sim$), Max $\notin V(B)$ iff, by IH, $B \notin Max$ iff, by mc, $\sim B \in Max$.

The cases in which A is $B \wedge C$, $B \vee C$, and $B \supset C$ proceed just as straightforwardly.

A is $B \rightarrow C$. Max $\in V(B \rightarrow C)$ iff, by ($V\rightarrow$), $S(B) \in V(C)$ iff, by (VS), $C \in S(B)$ iff, by construction, $B \rightarrow C \in Max$.

This completes the proof of the Lemma.

Theorem (4.7). $\vdash$ A iff $\Vdash$ A.

The proof is an easy consequence of Lemma (4.6).

5. *(A→B) ⊃ (A ⊃ B)*

Let us call a MP-model any L-model in which whenever $O \in V(A \rightarrow B)$, $O \in V(A \supset B)$, which in the presence of (A0) has the same effect as requiring that whenever $O \in V(A)$ and $O \in V(A \rightarrow B)$, $O \in V(B)$. Thus on the set of MP-models the addition to L of the axiom

(A4) $(A \rightarrow B) \supset (A \supset B)$

is sound and complete. These models are of special interest if we view $\rightarrow$ as abstracting from, say, the relation 'that...brings it about that...'. That the burning manuscript is plunged in water brings it about that the fire is extinguished. Now suppose that the burning manuscript is actually plunged in water. We do want it to be the case that the fire is actually extinguished.

The problem is whether models for L+(A4) can be characterized in some more interesting manner.

They can – in a *slightly* more interesting manner. Let $At = \mathrm{At} \cup \{\sim A$: A is atomic and $A \notin \mathrm{At}\}$. Adding the following to the conditions imposed on S in Section 2 will have the desired effect:

(7) for all wffs $B \in PC$, if $B \in S(A)$, then $At \vdash_c A \supset B$.

Clearly for all wffs $A \in PC$, $O \in V(A)$ iff $At \vdash_c A$. That (A7) is sound is easily proved:

ad (A7). Suppose $O \notin V((A \rightarrow B) \supset (A \supset B))$. By $(V \supset)$ $O \in V(A \rightarrow B)$ and $O \notin V(A \supset B)$. By $(V \rightarrow)$ and (VS) $B \in S(A)$. By condition (7) on S, $At \vdash_c A \supset B$, i.e., $O \in V(A \supset B)$, which is impossible.

We can extend the completeness proof in Section 4 by inserting the following case:

ad (7). Suppose for some A,B $B \in S(A)$ and $O \notin V(A \supset B)$. By construction $A \rightarrow B \in \mathrm{Max}$. By (4.2) and (A4) $A \supset B \in \mathrm{Max}$. Clearly $At \vdash_c A \supset B$. So $O \in V(A \supset B)$.

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