# THE REAL WORLD: COMPLETENESS AND INCOMPLETENESS OF A MODAL LOGIC(*)

by Jean PORTE

1. *The real world* – The original model structures for normal modal logics, as expounded by Kripke [9], consisted of a triple <G, K, R> where K was a non-empty set (the "possible worlds"), R a binary relation (the "accessibility relation"), and G a distinguished element of K: the "real world" (or "actual world"). A formula was valid if it was true in the real world for every model (assignment of values) constructed on any model structure.

Later, already in Lemmon [11], It became apparent that the distinguished real world played no role in the normal modal systems (T, S4, etc.). A formula was valid if it was true in every world for every model constructed on any model structure. Then the model structures became the "frames" <K, R>.

In a similar way for the non-normal logics studied by Kripke [10] and Lemmon [12] (S2, E2, etc) the distinguished real world proved useless, after the introduction of certain "non-normal worlds": a formula is valid if it is true in every normal world for every model constructed on any model structure.

Then the distinguished real world all but disappeared from the literature. A notable exception was Zeman's semantics for Sobociński's system S4.4: see Sobociński [21], Zeman [23] and [24] also Zeman [25] (p. 256) – a "real world" is singled out and is accessible only from itself while it has access to all worlds.

S4.4 being a normal logic, it must be complete in a class of frames – perhaps not simple Kripke-style frames, but in a class of Thomason-style generalized frames (or, alternatively, of Makinson-style generalized frames); see Thomason [22], Makinson [13] or [14] Goldblatt [5] and [6]. And indeed S4.4 has been characterized by a class of

(*) This paper is the development of a communication to the "Logic Symposion", Patras, Greece, August 1980. I must thank David Makinson for corrections in the proof of incompleteness.

Kripke-style frames, in Georgacarakos [2], with an accessibility relation which is reflexive, transitive, "convergent", and "remotely symmetrical" (two new fairly complex conditions). Zeman's semantics (with distinguished real world) is noticeably simpler.

Here, I will study a particular non-normal modal logic (previously defined in other papers) and prove that:

(i) It has a very simple semantics, being complete in a certain class of model structure with a distinguished real world.

(ii) It is not complete in any class of frames with non-normal worlds but without real world – the definition of these "non-normal worlds" being allowed to vary along a fairly wide range.

2. *The system $T^+$* – It is axiomatized as follows by five axiom schemas and a rule (the small Latin letters denote wff, the primitive connectives are negation, $\neg$, implication, $\rightarrow$, and necessity, L).

| | | |
|---|---|---|
| A1 | Lt if t is a PC tautology | for all formulas x, y |
| A2 | $L(L(x \rightarrow y) \rightarrow (Lx \rightarrow Ly))$ | |
| A3 | $L(Lx \rightarrow x)$ | |
| A4 | $Lx \rightarrow x$ | |
| A5 | $Lx \rightarrow LLx$ | |
| R1 | $x,\ x \rightarrow y / y$ | |

If A5 is weakened to a rule

R2 $Lx / LLx$

we get an axiomatization of T (the rule of necessitation is admissible). If A5 is strengthened to

A6 $L(Lx \rightarrow LLx)$

we get an axiomatization of S4 (necessitation is admissible). Then, if $T^+$ is considered as a "thetic system" (i.e. identified to the set of its theses) we have at once

$$T \subset T^+ \subset S4$$

And these inclusions are strict: $T \neq T^+$, for A5 is not a schema of theses in T, and $T^+ \neq S4$ for A6 is not a schema of theses in $T^+$ (proof in [16], section 9).

$T^+$ was first defined in [15] and [16], where it was called $\varrho \vee \varrho S_a$. In these papers several modal systems were defined starting from a very weak one, $S_a$, by means of two operations, $\varrho$ and $\vee$. T is $\vee \varrho S_a$ and S4 is $\vee \varrho \vee \varrho S_a = \vee \varrho \vee S_a$. Here I use $T^+$ to denote $\varrho \vee \varrho S_a$ for brevity.

This modal logic is not normal, since, (p being a propositional variable)

$$\vdash Lp \rightarrow LLp$$
$$\nvdash L(Lp \rightarrow LLp)$$

3. *Semantics for $T^+$* – It will be proved that:

*Theorem* – The set of theses of $T^+$ is the set of formulas valid in all Kripke-style model structure $<G, K, R>$, with distinguished real world G, the accessibility of which satisfies the conditions:

(i) every world is accessible to itself;
(ii) G has access to every other world.

When compared to the usual semantics for T and S4, we see that (i) states reflexivity of R, while (ii) implies a kind of "restricted transitivity":

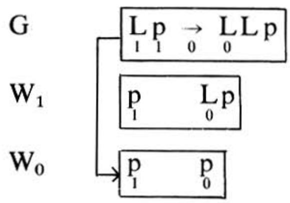
$$\text{if } G\,R\,W_1 \text{ and } W_1\,R\,W_2 \text{ then } G\,R\,W_2.$$

The proof of completeness will be made by modifying the completeness proofs of T and of S4 in Hughes-Cresswell [8], chapters 5 and 6. Familiarity with this book – here denoted by "HC" will be assumed.

It will be sufficient to recall that a HC-diagram is but a compact presentation of a system of Beth tableaux for various worlds in a model-structure. A formula is said to be "assigned 0" (respect. "1") if it is put on the right (respect. left) of a tableau, i.e. if it is given the value false (respect. true) for the given assignment. A rectangle is a world with a tentative assignment of values. An explicitely inconsistent rectangle is one in which the tentative assignment proves impossible, as leading to assign two different values to the same subformula. A formula is valid if every tentative assignment leads to an HC-diagram containing an explicitely inconsistent rectangle.

A formula will be said to be $T^+$-valid if it is valid in every model structure satisfying conditions (i) and (ii) above. An HC-diagram proving that a formula is $T^+$-valid will be called a $T^+$-diagram.

It is obvious that every thesis of $T^+$ is $T^+$-valid and that detachment conserves $T^+$-validity. The $T^+$-validity of A5 is proved by the following diagram




Then every thesis of $T^+$ is $T^+$-valid.

It remains to prove the converse. Similar proofs for T and S4 in HC (pp. 96-102 and 112-115) use two formulas associated with each rectangle $W_j$:

$$w_j' = \beta_1 \rightarrow (\beta_2 \rightarrow \ldots \rightarrow (\beta_k \rightarrow \gamma) \ldots)$$
$$w_j'' = L\beta_1 \rightarrow (L\beta_2 \rightarrow \ldots \rightarrow (L\beta_k \rightarrow \gamma) \ldots)$$

where $\beta_1 \ldots \beta_k$ are the formula which are initially assigned 1 in $W_j$ and $\gamma$ the (unique) formula which is initially assigned 0 in $W_j$ ("initially" meaning: as the starting points of a Beth tableau).

Now if a formula is $T^+$-valid, there is a diagram in which the rectangles are disposed in a descending chain:

$G(= W_{n+1}), W_n, \ldots W_0$ such as

(i) $W_{j+1}$ R $W_j$;
(ii) if $j \neq n$, only $W_{j+1}$ R $W_j$;
(iii) when a subformula, Ly, is assigned 0 in $W_{j+1}$, y is initially assigned 0 in $W_j$;
(iv) $W_0$ is explicitely inconsistent.

We will then consider a new formula associated with each rectangle $W_j$:

$$w_j''' = L^j\beta_1 \rightarrow (L^j\beta_2 \rightarrow \ldots \rightarrow (L^j\beta_2 \rightarrow \gamma) \ldots)$$

with the convention that

$$L^j\beta \text{ is } \underbrace{L\ldots L}_{j \text{ times}}\beta \quad \text{if } j>0$$

$$L^0\beta \text{ is } \beta$$

It results that $w_0''' = w_0'$ and $w_{n+1}''' = w_{n+1}'' = w_{n+1}' = x$ (the formula whose validity is being proved).

Moreover, from A5, A2 and PC, it follows

$$\vdash_{T^+} w_j'' \leftrightarrow w_j''' \quad \text{if} \quad j>0$$

Lemmas 1 and 3 of HC (pp. 97-99), jlemma 7 (p. 113) and the modified form of Lemma 4 (p. 114), hold for $T^+$, i.e. when we read "$\vdash_{T^+}$" instead of "$\vdash_T$" or "$\vdash_{S4}$".

Now we have the:

*Chief Lemma*: In a rectangle of an HC diagram showing the $T^+$-validity of a formula x

$$\vdash_T w_j''' \quad \text{if} \quad 0 \leqslant j \leqslant n$$

If $j = 0$, the Lemma results from $w_0''' = w_0'$ and Lemma 1. If $j>0$ the lemma is proved by constructing a parallel diagram showing the T-validity of $w_j'''$ by initially assigning in $W_j$, $W_{j-1}$,... $W_0$ the same values as in the $T^+$-diagram for x (plus perhaps supplementary assignments for certain subformulas). It follows that if $W_0$ is explicitely inconsistent in the $T^+$-diagram for x, it is as well explicitely inconsistent in the T-diagram for $w_j'''$.

Proof of the Theorem: Let us consider $W_n$ (i.e. the second world, since $W_{n+1} = G$). We get

$$\vdash_{T^+} (L\beta_1 \rightarrow (L\beta_2 \rightarrow \ldots \rightarrow (L\beta_k \rightarrow L\gamma)\ldots)) \rightarrow w_{n+1}'' \qquad (1)$$

as in HC (p. 113). Now, by the chief Lemma

$$\vdash_T w_n'''$$

that is

$$\vdash_T L^n\beta_1 \rightarrow (L^n\beta_2 \rightarrow \ldots \rightarrow (L^n\beta_k \rightarrow \gamma)\ldots)$$

whence, by necessitation, A2 and PC

$$\vdash_T L^{n+1}\beta_1 \rightarrow (L^{n+1}\beta_2 \rightarrow \ldots \rightarrow (L^{n+1}\beta_k \rightarrow L\gamma)\ldots)$$

whence

$$\vdash_{T^+} L^{n+1}\beta_1 \rightarrow (L^{n+1}\beta_2 \rightarrow \ldots \rightarrow (L^{n+1}\beta_k \rightarrow L\gamma)\ldots)$$

and, by A5, A2 and PC

$$\vdash_{T^+} L\beta_1 \to (L\beta_2 \to \ldots \to (L\beta_k \to L\gamma)\ldots) \tag{2}$$

From (1) and (2) by detachment

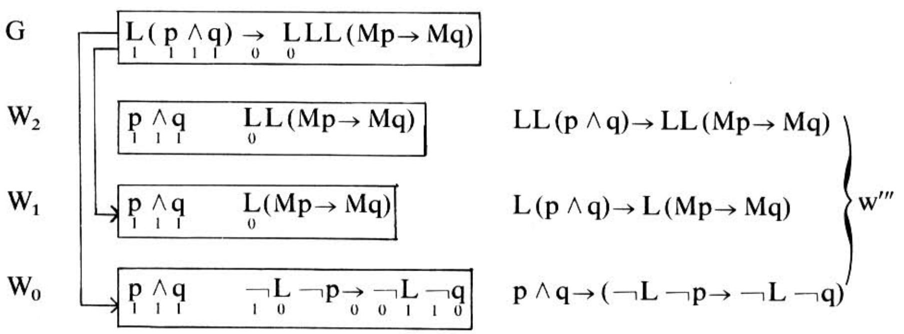
$$\vdash_{T^+} w''_{n+1}$$

that is

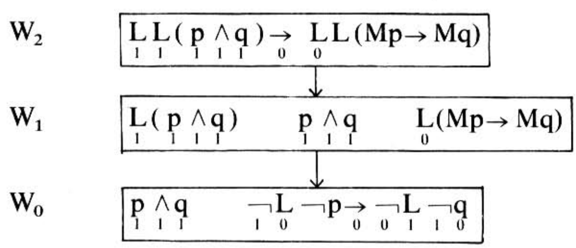
$$\vdash_{T^+} x$$

It remains to deal with the cases when the "sequence" of worlds is actually a tree – i.e. with what HC calls "the †-operators". It will be treated as in HC (p. 114), substituting w‴ for w′ (what is possible because $\vdash_{T+} w_j''' \leftrightarrow w_j''$ for $j > 0$) and $\vdash_{T+}$ for $\vdash_{S4}$ .

As an illustration of the proof of the chief Lemma we may consider the formula $L(p \wedge q) \to LLL(Mp \to Mq)$ – where $p \wedge q$ is an abbreviation for $\neg(p \to \neg q)$ and M an abbreviation for $\neg L \neg$. Its $T^+$ diagram is:




while the T-diagram for $w_2'''$ is as follows



## 4. *"Sound", "Complete", "Determined"*

As the vocabulary is not fixed, I will state explicitly the meaning of the words used in this paper.

Given a frame, M, a formula is *valid* if it is true for all models (i.e. assignment of values in the various worlds belonging to M) constructed on M. Given a class of frames, C, it is valid in C if it is valid in every frame belonging to C. It is (weakly) invalid in C if it is not valid; it is *strongly invalid* in C if it is invalid in every frame belonging to C, i.e. if for every $M \in C$ there is model in which the formula is false.

A *logic* (defined by its set of theses) is *sound* for a class of frames, C, if every thesis is valid. It is (weakly) unsound if it is not sound, i.e. if there is a thesis which is (weakly) invalid in C; it is *strongly unsound* if there is a thesis which is strongly invalid in C. – A logic is *complete* for C if every valid formula is a thesis. – A logic is *determined* by the class of frames C if it is both sound and complete for C.

All the preceding definitions extend to frames containing non-normal worlds (whatever these may be) by the convention that a formula is *valid if, for all models, it is true in every normal worlds*.

## 5. *The non-normal worlds*

It will be proved that $T^+$ cannot be determined by a class of frames with non-normal worlds.

If the non-normal worlds are defined as in Kripke [10], the result is easy. The rule

$$x \,/\, L(Ly \to x)$$

keeps validity in all frames. If we call *eligible* a logic in which this rule is admissible (see Schumm [20]), only eligible logics can be defined by validity in a class of frames. But $T^+$ is not eligible, since

$$\vdash_{T^+} Lp \to LLp$$
$$\nvdash_{T^+} L(L(p \to p) \to (Lp \to LLp))$$

(if the last formula were a $T^+$-thesis $L(Lp \to LLp)$ would also be a thesis, and $T^+$ would be identical with S4).

In that case it is clear that $T^+$ is not complete, and it will be proved presently that it is not sound.

But "non-normal worlds" have been given different (and non-equivalent) definitions, for instance in Cresswell [1], (see also Hughes-Cresswell [8], pp. 286-288), or in Porte [17], section 5.

What can be said about a general concept of non-normal world?

It seems impossible to encompass within a single sentence all the manners in which a possible world may be different from a normal one; see for instance Cresswell [2] or Georgacarakos [4]. But we can analyse the classical (Kripke's) notion of a normal world.

A normal world is characterized by the way an assignment of values in it is related to the same assignment (the same model) in the accessible worlds. In a normal world a formula of the form Lx

(i) takes the value "true", iff x is true in all the accessible worlds;

(ii) takes the value "false", iff x is false in at least one accessible world.

Non-normal worlds will be defined by rejecting those conditions.

(i) *A non-normal world of the first kind (NN1)* is a world in which Lx can be true even though x is false in at least one accessible world.

(ii) *A non-normal world of the second kind (NN2)* is a world in which Lx can be false even though x is true in every accessible worlds.

An NN1-2 is both an NN1 and an NN2.

Care must be taken for the case when no world is accessible from a non-normal world. After the preceding definition, it seems that it could not be an NN1. But that definition says only that in an NN1 Lx true is *compatible* with x false in an accessible world (contrary to what happens in a normal world); there may be other models in which an NN1 behaves as a normal world. Particularly, a world in which formulas of the form Lx may be assigned any value, is an NN1-2.

The non-normal worlds of Kripke [10] are NN2, with the supplementary conditions that no world is accessible from them, and that every formula beginning by L is assigned the value false.

The non-normal worlds of Cresswell [1], for the semantics of SO.5 (sees also Hughes-Cresswell [8], pp. 286-288) are NN1-2, with the supplementary condition that no world is accessible from them. That last condition could be suppressed without altering the completeness proof of SO.5.

The non-normal worlds of Porte [17] (section 5) are NN1. In Porte [18] (section 4) the "non-normal worlds" are NN1-2, while the "semi-normal worlds" are NN1.

It will be proved now that $T^+$ cannot be determined by a class of frames which contains any frame with an "active" NN1 or NN2 world; indeed it is strongly unsound for those classes of frames. An "active" non-normal world is one which is accessible from a normal world; indeed a non active one could be suppressed without changing the validity of the formulas. Then a frame with "active" non-normal worlds must contain a part of the form:

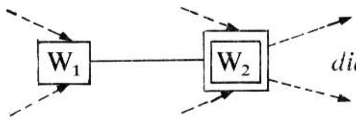


*diagram 1*

where $W_1$ is normal, $W_2$ is either NN1 or NN2, and the dotted arrows may exist or not exist, leading to/from normal or non-normal worlds.

Then here is the proof of unsoundness of $T^+$:

(i) Let $W_2$ be NN2, and let us examine the formula LLt, where t is any classical tautology (for instance $p \to p$). That formula being a $T^+$-thesis, it ought to be valid, and be assigned the value 1 ("true") in $W_1$. But diagram 1 gives the following counter-example:

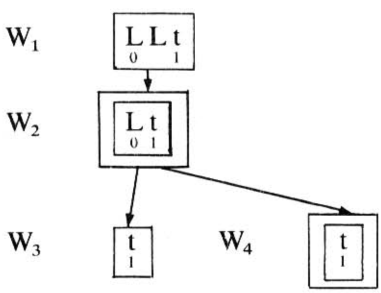
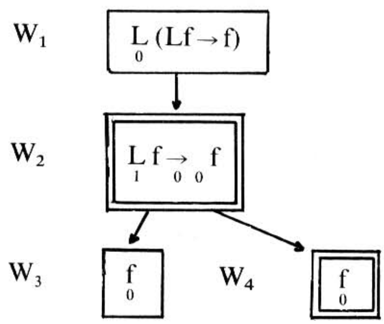



*diagram 2*

(the classical tautologie t is given the value 1 ("true") in every world, normal or not).

Here $W_3$ and/or $W_4$ may not exist; $W_3$ (normal) may be identical with $W_1$, and $W_4$ (non-normal) may be identical with $W_2$ - $W_3$ or $W_4$ wight as well be identical with worlds of the frame not reproduced on diagram 2. That covers all possible cases, and diagram 2 shows that LLt can be assigned the value 0 ("false") in the normal world $W_1$ in a way that respects the definition of normal worlds and of NN2.

(ii) Let $W_2$ be an NN1, and let us examine the formula $L(Lf \to f)$, where f is a classical antilogie (e.g. suppose f is $\neg(p \to p)$). f will be assigned the value 0 in every world (normal or not). And $L(Lf \to f)$, being a $T^+$-thesis, should be valid, and be assigned the value 1 in $W_1$ in every model. But diagram 1 yields the following counter example:




We could easily see, as in the case on NN2, that diagram 3 covers all possible cases, and that it shows that $L(Lf \to f)$ can always be assigned the value 0 ("false") in the normal world $W_1$ in a way which respects the definitions of normal worlds and of NN1.

$T^+$ is then strongly unsound for the classes of frames which use non-normal worlds and then cannot be determined by any of them. – That has been proved only if a non-normal world is "active"; but if the non-normal worlds which may be present in a frame are non-active, they can be suppressed, the result is a classical Kripke frame, and is well known that such frames can determine only normal logics.

*Remark 1* – As far as I know, $T^+$ may be complete for certain classes of frames containing non-normal worlds,...

The only result I have is that it is incomplete when NN1 – respect. NN2 – are strengthen by the convention a formula of the form Lx is always true – respectively: they are always false (Kripke's convention).

The case of strengthened NN2 has been seen above: rule

$$x \,/\, L(Ly \rightarrow x)$$

preserves validity, whence it follows that the non-thesis

$$L(Lt \rightarrow (Lp \rightarrow LLp))$$

is valid.

Similarly, in the case of strengthened NN1, the rule

$x/\, L(\neg Ly \rightarrow x)$ preserves valididy,

whence it follows that the non-thesis

$$L(\neg Lf \rightarrow LLp))$$

is valid.

*Remark 2* – It follows from the result of this section that every non-normal logic where both LLt and $L(Lf \rightarrow f)$ are theses cannot be determined by any class of frames; particularly this is the case for every non-normal logic stronger that T. What is special for $T^+$ is that it is possible to prove its completeness in a class of model structures with distinguished real world.

1 Villa Ornano
75018 PARIS
FRANCE

Jean PORTE

## REFERENCES

[1] Cresswell, M.J. – The completeness of SO.5 – *Logique et Analyse* – vol. 9 (1966), pp. 263-266.

[2] Cresswell, M.J. – Classical intensional logics – *Theoria* – vol. 36 (1970), pp. 347-373.

[3] Georgacarakos, G.N. – Semantics for S4.04, S4.4 and S4.3.2 – *Notre Dame Journal of Formal Logic* – vol. 17 (1976), pp. 297-302.

[4] Georgacarakos, G.N. – Abnormal worlds and the non-Lewis modal systems – *Notre Dame Journal of Formal Logic*, vol. 18 (1977), pp. 95-100.

[5] Goldblatt, R.I. – Metamathematics of modal logic. I – *Reports on Mathematical Logic*, vol. 6 (1976), pp. 41-78.

[6] Goldblatt, R.I. – Metamathematics of modal logic II. – *Reports on Mathematical Logic*, vol. 7 (1976), pp. 21-52.

[7] Hazen, Allen – The eliminability of the actuality operator in propositional modal logic – *Notre Dame Journal of Formal Logic* – vol. 19 (1978), pp. 617-622.

[8] Hughes, G.E. and Cresswell, M.J. – *An Introduction to Modal Logic* – London (Methuen), 1968; reprinted with corrections, 1972.

[9] Kripke, S.A. – Semantical analysis of modal logic. I. – *Zeitschrift für mathematische Logik und Grundlager der Mathematik* – vol. 9 (1963), pp. 67-96.

[10] Kripke, S.A. – Semantical analysis of modal logic, II. *Symposium on the Theory of Models, Berkeley 1963* – Amsterdam (North Holland), 1965 – pp. 206-220.

[11] Lemmon, E.J. – Algebraic semantics for modal logics. I. – *Journal of Symbolic Logic*, vol. 31 (1966), pp. 46-65.

[12] Lemmon, E.J. – Algebraic semantics for modal logics, II – *Journal of Symbolic Logic* – vol. 31 (1966), pp. 191-218.

[13] Makinson, David – A generalization of the concept of a relational model for modal logic – *Theoria* – vol. 36 (1970), pp. 331-335.

[14] Makinson, David – *Aspectos de la lógica modal* – Instituto de Matemática, Universidad Nacional del Sur, Bahía Blanca, 1971 (Notas de Lógica Matemática, nº 28).

[15] Porte, J. – Recherches sur les logiques modales – *Le raisonnement en mathématiques et en sciences expérimentales, Colloque Paris 1955* – Paris (éd. du C.N.R.S.), 1958 – pp. 117-126.

[16] Porte, J. – A research in modal logic – *Logique et Analyse*. vol. 23, nº 89 (Mars 1980), pp. 3-34.

[17] Porte, J. – Congruences in Lemmon's SO.5 – *Notre Dame Journal of Formal Logic* – vol. 21 (1980), pp. 672-678.

[18] Porte, J. – Notes on modal logics – *Logique et Analyse* – vol. 24, nº 95-96 (Sept.-Dec. 1981), pp. 399-406.

[19] Prior, A.N. – Possible worlds – *The Philosophical Quaterly* – vol. 12 (1962), pp. 36-43.

[20] Schumn, F.S. – An incomplete non normal extension of S3 – *Journal of Symbolic Logic* – vol. 43 (1978), pp. 211-212.

[21] Sobociński, Bolesław – Modal system S4.4 – *Notre Dame Journal of Formal Logic* – vol. 5 (1964), pp. 305-312.

[22] Tnomason, S.K. – Semantic analysis of tense logics – *Journal of Symbolic Logic* – vol. 37 (1972), pp. 150-158.

[23] Zeman, J.J. – A study of some systems in the neighborhood of S4.4 – *Notre Dame Journal of Formal Logic* – vol. 12 (1971), pp. 341-357.

[24] Zeman, J.J. – Semantics for S4.3.2. – *Notre Dame Journal of Formal Logic* – vol. 13 (1972), pp. 454-460.

[25] Zeman, J.J. – *Modal logic: The Lewis Modal Systems* – Oxford (Clarendon Press), 1973.