MANY-DIMENSIONAL TOPOLOGICAL MODAL LOGIC

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0. *Introduction*

In Segerberg (1973) Krister Segerberg develops a semantically complete calculus for two-dimensional modal logic. This apparatus enables us to evaluate formulas at models whose possible worlds are determined by two coordinates. Such formulas are the logical counterparts of ordinary sentences like "It is the case on January 7th 1985 that tomorrow there will be a jazz festival in New Orleans". The truth value of sentences like this one depends on points in a two-dimensional manifold of possible worlds, the dimensions here being space and time. Of course, there are examples with more than two dimensions, in the simplest cases one only takes into account the three dimensions of Euclidean space. There is thus a need for many-dimensional modal logic(1).

Moreover, as Segerberg points out, there is a special need for the evaluation of formulas within two-dimensional manifolds, whose dimensions are both of the same sort, e.g. we might want to determine the truth value of a sentence at two points of time simultaneously. Take for example the statement "It is the case on January 7th 1985 that there will be a jazz festival tomorrow" (S). Since S explicitly refers to a fixed day, it is quite natural to say that its truth value depends on that day. But since S also contains the indefinite time indicator "tomorrow", its truth value depends on the day after its utterance as well, and this day differs from January 7th 1985 if, e.g., S is uttered on January 10th 1985. Thus this statement has to be evaluated at two points in time simultaneously.

(1) Vide also Gabbay (1976), Snyder (1971) and Stuhlmann-Laeisz (1983). Presumably, the concept of a truth system from my (1983) needs some improvement, as U. Nortmann, Göttingen, pointed out to me. But I hope the present argument will do the job as well.
For a somewhat similar topic vide also Nishimura (1981).

Obviously, the logical reconstruction of sentences containing definite or indefinite *positional operators* like "here", "tomorrow", "in New Orleans" and "on January 7th 1985" concerns "topological" logic as well. This type of logic (cf. Rescher and Garson (1968), Garson (1973)) differs from usual modal logic in the following respect: We can express within the object language that a formula is true at some definite point – i.e., in terms of modal logic, in some possible world – whereas whithin the syntax of modal logic we cannot. On the other hand, in topological logic we usually have one-dimensional semantics, i.e. formulas are evaluated at only one point, and this point is determined by one "coordinate" alone.

However, the ideas of these two types of logic can be combined. Within the following paper I shall develop a n-dimensional topological logic (for any natural n) which also contains operators working like Kripke-modalities. – For a special two-dimensional case I shall show then how to define those operators which Segerberg introduces in order to construe sentences which contain indicators like "tomorrow".

The plan of my paper is the following: At first I axiomatically present a calculus for n-dimensional topological logic and define its semantics; this system already contains operators working like Kripke-modalities; secondly I prove completeness. Finally, I shall show how to define the Segerberg-operators within a two-dimensional specialization of many-dimensional topological modal logic.

1. *Axioms and Semantics*

1.1. *Language*

1.1.1. Vocabulary: (i) Here we have as logical constants some functionally complete set of sentential connectives, say $\rightarrow$ and $\neg$; we have the quantifier $\vee$. We further have n operators $\varphi_1, \ldots, \varphi_n$, which refer to the n dimensions of our logic; the job of these operators will be explained with the formation rules, as well as the job of n 1-place and n 2-place predicates (N_i, R_i) will be explained later. Finally, we have as logical constants n weak Kripke modal operators (F_i) and the identity sign $\equiv$.– (ii) Nonlogical constants: Corresponding to the n dimensions we have n enumerable classes $P_1, \ldots, P_n$ of positional constants t, t′, … . These work syntactically like individual constants in the predicate calculus. Further we have an enumerable class of propositional constants p, q, r … . – (iii) Finally, there are n enumerable classes $V_1, \ldots, V_n$ of positional variables $\tau, \tau', \ldots$, which syntactically work like individual variables.

1.1.2. Formulas: (i) As atomic formulas we have the propositional constants and for any positional symbols s, s′ from dimension i (i.e.: $s, s' \in P_i \cup V_i$) the expressions: $N_i(s)$, $s R_i s'$ and $s \equiv s'$. (If $s, s' \in P_i$ are constants and dimension i is that of time, if further s designates 4.30 p.m. on January 7th 1985 and s′ designates 5.30 p.m. on the same day, then N_i (s) might be read as "within the i-th dimension, i.e. that of time, it is now 4.30 p.m. on January 7th 1985", and $s R_i s'$ as "4.30 p.m. on January 7th 1985 is earlier in time than 5.30 p.m. on the same day". – (ii) The set W of formulas of n-dimensional topological modal logic is the smallest set such that: (a) Every atomic formula is in W, and (b) if A and B are members of W, $\tau \in V_i$, $t \in P_i$ then so is any sentential connexion from A,B and further the expressions $\vee \tau A$, $\varphi_i \tau A$, $\varphi_i t A$ and $F_i A$. – (iii) Any formula containing no free variable is a sentence. – If again dimension i is that of time, $\tau \in V_i$ and $t \in P_i$, then if A is a sentence and B contains free τ, $\varphi_i t A$ reads: "At point t of dimension i – i.e. at time t – it is the case that A" and correspondingly $\vee \tau B$: "For some point τ of dimension i – i.e. sometimes – it is the case that B"(2). $F_i A$ would read: "It has been the case that A".

1.2. *Semantics*

1.2.1. Structures: A structure for the language of this logic is a sequence

$U = \langle D_1 \times \ldots \times D_n ; (k_1^0, \ldots, k_n^0) ; m_1, \ldots, m_n ; R^1, \ldots, R^n ; V \rangle$ such that:

(i) D_i is a non-empty class, containing the coordinates of the i-th dimension. A point in the Cartesian product $D_1 \times \ldots \times D_n$ is, from the standpoint of modal logic, a possible world.

(2) Of course – as in ordinary predicat logic – if τ is not free in A, $\vee \tau A$ is equivalent to A. Vide also infra the truth condition for existential statements.

(ii) $k_i^0 \in D_i$. $(k_1^0, \ldots, k_n^0)$ represents (the coordinates of) the actual world (modal logic) or (topological logic) the "here and now and ..." (depending on the number n of dimensions).
(iii) m_i is a function from P_i, the set of positional constants, into the coordinate set D_i.
(iv) R^i is a 2-place relation on D_i. (Earlier than, further behind than,... .).
(v) V is a truth value function on $PC \times D_1 \times \ldots \times D_n$, where PC is the set of propositional constants.

1.2.2. The concept of truth: We now state the conditions under which a sentence A is true for a structure
$U = \langle D_1 \times \ldots \times D_n ; (k_1^0, \ldots, k_n^0) ; m_1, \ldots, m_n ; R^1, \ldots, R^n ; V \rangle$, marking this by $\models_U A$. –
(i) Atomic sentences:

(a) For propositional constants p:

$\models_U p$ iff $V(p; k_1^0, \ldots, k_n^0) = 1$.

(b) For $t, t' \in P_i$:

$\models_U N_i(t)$ iff $m_i(t) = k_i^0$
$\models_U t R_i t'$ iff $m_i(t) R^i m_i(t')$
$\models_U t \equiv t'$ iff $m_i(t) = m_i(t')$.

(ii) The truth conditions for sentential connectives are ordinarily defined.
(iii) The truth condition for the existential quantifier: If $\tau \in V_i$ is a positional variable from the i-th dimension and if t is, with respect to some definite enumeration, the first positional constant from P_i which does not occur in formula A, then

$\models_U \vee\tau A$ iff there is a function m_i'
from P_i to D_i such that $m_i' =_t m_i$
and $\models_{U[m_i'/m_i]} A[t/\tau]$.

$(U[m_i'/m_i]$ is the structure which results from U if on substitutes m_i' for m_i.)

This truth condition for the existential quantifier makes the positional constants and variables also semantically work like individual constants and variables in ordinary predicate calculus.

(iv) Sentences with an operator φ_i. For $t \in P_i$:

$$\models_U \varphi_i\, t\, A \quad \text{iff} \quad \models_{U[m_i(t)/k_i^0]} A.$$

($U[m_i(t)/k_i^0]$ results from U by substituting $m_i(t)$ for k_i^0.)

(v) The Kripke modal operators F_i:

$$\models_U F_i\, A \text{ iff for some } k \in D_i\text{:}$$
$$k\, R^i\, k_i^0 \text{ and } \models_{U[k/k_i^0]} A.$$

A sentence A is valid ($\models A$) if and only if it is true for all structures.

1.3. *Axioms*

The following notation of axioms uses some abbreviations, i.e.: $\wedge$ for $\neg \vee \neg$; F_i for $\neg F_i \neg$ (thus $\wedge$ is the universal quantifier and F_i the strong Kripke modal operator corresponding to the weak F_i); I further write $\varphi^i(t)$ for the sequence $\varphi_i\, t_i \ldots \varphi_1\, t_1$, and I use $\overset{\varphi}{i}(t)$ for $\varphi_n\, t_n \ldots \varphi_i\, t_i$. Thus, $\overset{\varphi}{2}(t)\, A$ denotes the formula $\varphi_n\, t_n \ldots \varphi_2\, t_2\, A$ and the formula $\varphi_3\, t_3\, \varphi_2\, t_2\, \varphi_1\, t_1\, A$ is abbreviated $\varphi^3(t)\, A$.

I define the set of axioms for n-dimensional topological modal logic by induction, i.e. at first I list a class of *ground axioms* and then I state a rule to generate all axioms.

1.3.1. Ground Axioms: Here we have four groups of axioms according to the fact that we have four types of logical constants: sentential connectives, the quantifier(s), the Kripke modal operators F_i (and F_i), and the specific constants of topological logic: the operators φ_i and the predicate constants N_i and R_i. – (i) Since the set of tautologies, i.e. those sentences which are valid due to the truth conditions for the sentential connectives alone, is a decidable set, I take, for sake of simplicity, all tautologies as ground axioms of the first group. – With formulas A, B we then have as further ground-axioms all *sentences* of one of the following types: (ii) Predicate calculus:

(a) $\wedge\tau\, A \rightarrow A[t/\tau] \quad (\tau \in V_i,\ t \in P_i)$.

(b) $A \rightarrow \wedge\tau\, A \quad (\tau \in V_i)$.

(c) $\wedge\tau\, (A \rightarrow B) \rightarrow (\wedge\tau\, A \rightarrow \wedge\tau\, B) \quad (\tau \in V_i)$.

(iii) Kripke modal logic: Here we have one type of axiom, i.e. all sentences

$$F_i(A \to B) \to (F_i A \to F_i B).$$

(iv) Specific axioms for n-dimensional topological logic:

(a) $\varphi_i\, t\, \varphi_j\, t'\, A \leftrightarrow \varphi_j\, t'\, \varphi_i\, t\, A$
$(t \in P_i,\ t' \in P_j,\ i \neq j)$.

(b) $\varphi_i\, t'\, \varphi_i\, t\, A \leftrightarrow \varphi_i\, t\, A \qquad (t, t' \in P_i)$.

(c) $\varphi_i\, t\, A \leftrightarrow \neg \varphi_i\, t\, \neg A \quad (t \in P_i)$.

(d) $\varphi_i\, t\, (A \to B) \to (\varphi_i\, t\, A \to \varphi_i\, t\, B) \quad (t \in P_i)$.

(e) $\wedge \tau\, \varphi_i\, t\, A \leftrightarrow \varphi_i\, t \wedge \tau\, A$
$(t \in P_i,\ \tau \in V_j)$.

(f) $\vee \tau\, N_i(\tau) \qquad (\tau \in V_i)$.

(g) $N_i(t) \wedge N_i(t') \to t \equiv t' \quad (t, t' \in P_i)$.

(h) $\varphi_i\, t\, N_i(t) \qquad (t \in P_i)$.

(i) $N_i(t) \to (A \leftrightarrow \varphi_i\, t\, A)$.

(k) $t' \equiv t'' \leftrightarrow \underset{k}{\varphi}(t)\, t' \equiv t \qquad (t', t'' \in P_i;\ t_k \in P_k,$
$t' \equiv t'' \leftrightarrow \overset{k}{\varphi}(t)\, t' \equiv t'' \qquad k = 1, \ldots, n)$.
$t' \equiv t'' \leftrightarrow \varphi_k\, t\, t' \equiv t''$

(l) $t'\, R_i\, t'' \leftrightarrow \overset{n}{\varphi}(t)\, t'\, R_i\, t'' \qquad (t', t' \in P_i;\ t_k \in P_k,\ k = 1, \ldots, n)$.

$$t'\, R_i\, t'' \leftrightarrow \underset{i+1}{\varphi}(t)\, \overset{i-1}{\varphi}(t)\, t'\, R_i\, t''$$

(Ground axioms (k) and (l) express that statements on the identity of positions as well as statements concerning positional relations hold independently of any position they might refer to: January 7th 1985 is earlier than January 8th 1985 if and only if January 7th 1985 is earlier than January 8th 1985 at any time, at any place, and so on.)

(m) $N_i(t') \leftrightarrow \underset{i+1}{\varphi}(t)\, \overset{i-1}{\varphi}(t)\, N_i(t')$

$$(t' \in P_i;\ t_k \in P_k,\ k = 1, \ldots, i-1, i+1, \ldots, n)$$

(i.e.: Today is January 7th 1985 if and only if at any place (not: at any time!) it is the case that today is January 7th 1985).

(n) $t \equiv t' \to (\varphi_i\, t\, A \leftrightarrow \varphi_i\, t'\, A) \quad (t, t' \in P_i)$.

The last ground axiom finally states a logical relation between positional statements and statements containing a weak Kripke modal operator

(o) $\varphi_i t F_i A \leftrightarrow \vee \tau (\tau R_i t \wedge \varphi_i \tau A)$
$(t \in P_i, \tau \in V_i)$.

(E.g.: It holds at time t that it has been the case that A if and only if for some point τ of time earlier that t it is at τ the case that A.).

1.3.2. Axioms: The set Ax of axioms for many-dimensional topological modal logic is the smallest set such:

(i) Every ground axiom is in Ax.
(ii) If A is a member of Ax, then so are
$F_i A$ and $\varphi_i t A$ $(t \in P_i, i = 1, \ldots, n)$.
(iii) If $A[t/\tau]$ is a member of Ax and if the sentence $\wedge \tau A$ does not contain the constant t, then $\wedge \tau A$ is in Ax $(\tau \in V_i, t \in P_i)$.

As we will see, this class of axioms together with the derivation rule of modus ponens is complete for the set of valid sentences. – Note that – for the sake of simplicity – I did not care for independence within the system of axioms.

1.4. *Soundness*

Before dealing with completeness let us establish validity for some axioms from group (iv). Here I use the notation "A in U" for $\models_U A$ and "$U[m_i(t)]$" for $U[m_i(t)/k_i^0]$ $(t \in P_i, i = 1, \ldots, n)$.

(a) $\varphi_i t \varphi_j t' A$ in U iff A in $U[m_i(t)][m_j(t')]$
$\varphi_j t' \varphi_i t A$ in U iff A in $U[m_j(t')][m_i(t)]$.
But since $i \neq j$, the structure denoted on the right side of the first equivalence is the same as that one denoted on the right side of the second one.
(b) $\varphi_i t' \varphi_i t A$ in U iff $\varphi_i t A$ in $U[m_i(t')]$, iff A in $U[m_i(t)]$, iff $\varphi_i t A$ in U.
(e) I only argue for the more difficult case that $t \in P_i$ and $\tau \in V_i$. Let t' be the first constant from P_i which is not in $\varphi_i t A$. Then $t' \neq t$. $\wedge \tau \varphi_i t A$ in U iff for every m_i'

with $m_i' =_{t'} m_i$: $\varphi_i\, t\, A[t'/\tau]$ in U $[m_i'/m_i]$, iff for every such m_i':
$A[t'/\tau]$ in $U[m_i'(t)/k_i^0 ; m_i'/m_i]$. And:
$\varphi_i\, t \wedge\tau\, A$ in U iff $\wedge\tau\, A$ in $U[m_i(t)]$, iff for every m_i' with
$m_i' =_{t'} m_i$:
$A[t'/\tau]$ in $U[m_i(t)/k_i^0 ; m_i'/m_i]$.
But since $t \neq t'$ we have $m_i'(t) = m_i(t)$,
thus $U[m_i'(t)/k_i^0 ; m_i'/m_i] =$
$U[m_i(t)/k_i^0 ; m_i'/m_i]$.

(h) $\varphi_i\, t\, N_i(t)$ in U iff $N_i(t)$ in $U[m_i(t)]$, iff $m_i(t) = m_i(t)$.

(i) If $N_i(t)$ in U, then $m_i(t) = k_i^0$. Thus:
$\varphi_i\, t\, A$ in U iff A in $U[m_i(t)]$.
But $U[(m_i(t)]$ is U itself.

(o) Let again t' be the first constant from P_i which is not in A. Since $\varphi_i\, t\, F_i\, A$ is a sentence, A doesn't contain any free variable. Hence $A[t'/\tau] = A$.–
$\varphi_i\, t\, F_i\, A$ in U iff $F_i\, A$ in $U[m_i(t)]$, iff for some $k \in D_i$: $k R^i m_i(t)$ and A in $U[k/k_i^0]$ $(U[m_i(t)]\,[k/m_i(t)] = U[k/k_i^0])$.
$\vee\tau(\tau R_i t \wedge \varphi_i \tau A)$ in U iff for some m_i': $m_i' =_{t'} m_i$ and $t' R_i t \wedge \varphi_i t' A$ in $U[m_i'/m_i]$, iff for some such m_i': $m_i'(t') R^i m_i(t)$ and A in $U[m_i'(t')/k_i^0 ; m_i'/m_i]$. Since t' is not in A, this last condition is equivalent to the clause: A in $U[m_i'(t')]$. Hence the truth conditions for both sides of axiom (o) are equivalent: If there is m_i', such that the condition for the right side is fulfilled, choose $k := m_i'(t')$.
Since $m_i'(t) = m_i(t)$ we then have:
$kR^i m_i(t)$ and A in U $[k/k_i^0]$.– If there is $k \in D_i$ such that the condition for the left side is fulfilled, define

$$m_i'(t''):=\begin{cases} k, \text{ if } t''=t', \\ m_i(t''), \text{ otherwise} \end{cases} \qquad (t'' \in P_i).$$

Since $kR^i m_i(t)$ and $t \neq t'$ we then have: $m_i'(t)R^i m_i'(t')$ and A in $U[m_i'(t')]$.

2. *Completeness*

In order to prove that our axiom system with the only derivation rule of modus ponens is complete we need some auxiliary concepts and theorems. Some of these concepts are well known of ordinary predicate logic, others are very similar to concepts used in Henkin proofs for the completeness of Kripke modal logic. – At first, we obviously have the following.

Theorem 1:
If B is derivable from the premises $A_1, \ldots, A_n$, then $\varphi_i tB$ is derivable from $\varphi_i tA_1, \ldots, \varphi_i tA_n$.

This theorem is due to the fact that we have the axioms $\varphi_i t(A \to B) \to (\varphi_i tA \to \varphi_i tB)$ and the rule: If B is an axiom so is $\varphi_i tB$. Here the operators $\varphi_i t$ work like strong Kripke modalities. – Now I define the property of ω-completeness in topological modal logic.

Def. 1:
A set of sentences T is ω-complete iff for every P_i, V_i and every sentence $\vee\tau$ A such that $\tau \in V_i$: if $\vee\tau$ A is in T, then for some $t \in P_i$ the sentence A $[t/\tau]$ is in T.

If we define the concept of a maximal consistent set in the usual way of sentential logic, then many-dimensional topological logic behaves like ordinary predicate logic in the respect that every infinitely extendable (syntactically) consistent set has a Lindenbaum extension. This is said by

Theorem 2:
If T is syntactically consistent and if every set P_i contains infinitely many constants which are not in T, then there is some maximal consistent and ω-complete T_1 such that $T \subseteq T_1$.

Proof: There is an enumeration $A_1, A_2, \ldots$ of all sentences with the following property: For every set of variables V_i: if $\vee\tau$ B is a sentence with $\tau \in V_i$, then there is an A_k such that $A_k = \vee\tau$ B and $A_{k+1} = B[t/\tau]$ for some $t \in P_i$ which is neither in T not in any of the sentences $A_1, \ldots, A_k$. With respect to this enumeration one construes a Lindenbaum extension of T in the usual way:

$L_0 := T$

$$L_{k+1} := \begin{cases} L_k \cup \{A_{k+1}\}, \text{ if this set is consistent,} \\ L_k, \text{ otherwise.} \end{cases}$$

$$L := \bigcup_{k=0}^{\infty} L_k =: T_1.$$

Canonical reasoning shows that L is maximal consistent and ω-complete. –

As next, for every dimension i and every constant $t \in P_i$ I define a function $\Sigma_i\, t$ which associates with every maximal consistent and ω-complete set T another such set.

Def. 2:
For any set T of sentences:
$\Sigma_i\, t\, T := \{A | \varphi_i\, t\, A \in T\}$.

Theorem 3:
If T is maximal consistent and ω-complete then so is $\Sigma_i\, t\, T$.

Proof: (a) If $\Sigma_i\, t\, T$ is not consistent then there are $A_1, \ldots, A_m$, $B \in \Sigma_i\, t\, T$ such that $\neg B$ is derivable from the A_k. Hence by theorem 1 and axiom (iv c): $\neg \varphi_i\, t\, B$ is derivable from $\varphi_i\, t\, A_1, \ldots, \varphi_i\, t\, A_m$. But the $\varphi_i\, t\, A_k$ and $\varphi_i\, t\, B$ are in T. Thus T would not be consistent.
(b) That $\Sigma_i\, t\, T$ is maximal consistent is a simple consequence of axiom (iv c).
(c) ω-completeness: Let $\vee \tau\, A$ be in $\Sigma_i\, t\, T$ for some $\tau \in V_j$. If there is no $t' \in P_j$ such that $A[t'/\tau]$ is in $\Sigma_i\, t\, T$, then by maximal consistency for every $t' \in P_j$:
$\neg A[t'/\tau] \in \Sigma_i\, t\, T$, hence $\varphi_i\, t \neg A[t'/\tau] \in T$. But T is ω-complete, hence (as in ordinary predicate logic) $\wedge \tau\, \varphi_i\, t \neg A \in T$ (with $\tau \in V_j$). As a consequence of axiom (iv e) and by maximal consistency of T:
$\varphi_i\, t \wedge \tau \neg A \in T$, hence $\wedge \tau \neg A$ is in $\Sigma_i\, t\, T$, which contradicts the hypothesis.

Q.E.D.

In the following Henkin-proof for completeness we will associate with every syntactically consistent set T a structure U, which is a model of T (i.e.: all the sentences in T are true for U). As usual, this structure will be built up by linguistic entities, and – L being a Lindenbaum extension of T – the sets $\Sigma_i\, t\, L$ will work as elements of

D_i, i.e. as coordinates of the i-th dimension. It will turn out that a sentence A is true at a point $(\Sigma_1 t_1 L, \ldots, \Sigma_n t_n L)$ iff A is a member of the set $\Sigma_1 t_1 \ldots \Sigma_n t_n L$. – But we still need some theorems. – Due to axiom (iv a) we obviously have the

Theorem 4:
If $i \neq j$, $t \in P_i$, $t' \in P_j$:
$\Sigma_i t \Sigma_j t' T = \Sigma_j t' \Sigma_i t T$.

Corollary:
If $i_1, \ldots, i_k \in \{1, \ldots, n\}$ are mutually distinct and if $t_{i_\tau} \in P_{i_\tau}$ ($\tau = 1, \ldots, k$), then for any permutation $j_1, \ldots, j_k$ of the $i_1, \ldots, i_k$:
$\Sigma_{i_1} t_{i_1} \ldots \Sigma_{i_k} t_{i_k} T = \Sigma_{j_1} t_{j_1} \ldots \Sigma_{j_k} t_{j_k} T$.
In what follows T always denotes a maximal and ω-complete set of sentences.

Theorem 5:
For any $t, t' \in P_i$:
$\Sigma_i t T = \Sigma_i t' T$ iff $t \equiv t' \in T$.

Proof: (a) Sufficiency. The axiom (iv n)
$\varphi_i t N_i(t)$ is in T, hence
$N_i(t)$ is in $\Sigma_i t T$, thus by hypothesis:
$N_i(t) \in \Sigma_i t' T$. But $N_i(t')$ is in $\Sigma_i t' T$ as well. Hence by axiom (iv g): $t \equiv t'$ is in $\Sigma_i t' T$, hence $\varphi_i t' t \equiv t'$ is in T, thus (by axiom (iv k)) $t \equiv t' \in T$.
(b) Necessity. If $t \equiv t' \in T$, then by maximal consistency and axiom (iv n):
for any A: $\varphi_i t A \in T$ iff $\varphi_i t' A \in T$. Hence by definition:
$\Sigma_i t T = \Sigma_i t' T$.

Q.E.D.

Theorem 6:
$\Sigma_i t T = \Sigma_i t' T$ iff $N_i(t) \in \Sigma_i t' T$.

Proof: (a) Since $N_i(t)$ is in $\Sigma_i t T$ we at once have sufficiency.
(b) Necessity. We have $N_i(t')$ in $\Sigma_i t' T$. Therefore: If also $N_i(t)$ is in

$\Sigma_i\, t'\, T$, we have (by axioms (iv g), (iv k) and definition): $t \equiv t' \in T$. Hence by theorem 5: $\Sigma_i\, t\, T = \Sigma_i\, t'\, T$.

Q.E.D.

In what follows I shall use two more abbreviations, i.e.: I write $\overset{i}{\Sigma}(t)$ for the sequence $\Sigma_1 t_1 \ldots \Sigma_i t_i$, and I denote the sequence $\Sigma_i t_i \ldots \Sigma_n t_n$ by $\underset{i}{\Sigma}(t)$. (These sequences, of course, are functions as the $\underset{i}{\Sigma}\, t$ are.)

Theorem 7:
For $t \in P_i$: If $N_i(t) \in T$, then $\Sigma_i\, t\, T = T$.

Proof: If $N_i(t) \in T$, then by axiom (iv i) for all A: $A \leftrightarrow \varphi_i\, t\, A$ is in T. Hence by def of $\Sigma_i\, t$:
$A \in \Sigma_i\, t\, T$ iff $\varphi_i\, t\, A \in T$, iff $A \in T$.
Thus $\Sigma_i\, t\, T = T$.

Q.E.D.

Corollary:
If $N_i(t_i) \in T$ $(i = 1, \ldots, n;\ t_i \in P_i)$ then $\overset{n}{\Sigma}(t)\, T = T$.

Theorem 8:
If for $i = 1, \ldots, n$: $\Sigma_i\, t_i\, T = \Sigma_i\, t_i'\, T$ $(t_i, t_i' \in P_i)$, then $\overset{n}{\Sigma}(t)\, T = \overset{n}{\Sigma}(t')\, T$.

Proof (by induction):
(i) $\Sigma_n\, t_n\, T = \Sigma_n\, t_n'\, T$ (by hypothesis).
(ii) Let be $\underset{k+1}{\Sigma}(t)\, T = \underset{k+1}{\Sigma}(t')\, T$.

Since $\Sigma_k\, t_k\, T = \Sigma_k\, t_k'\, T$ we have by theorem 5: $t_k \equiv t_k' \in T$. Hence by axiom (iv k): $\underset{k+1}{\varphi}(t)\, t_k \equiv t_k'$ is in T. Hence $t_k \equiv t_k'$ is in $\underset{k+1}{\Sigma}(t)\, T$.

Therefore: $\varphi_k\, t_k\, A$ is in $\underset{k+1}{\Sigma}(t)\, T$ iff $\varphi_k\, t_k'\, A$ is in $\underset{k+1}{\Sigma}(t)\, T$ (by axiom (iv n)), hence by def of
$\Sigma_k\, t_k$: $\Sigma_k\, t_k\, \underset{k+1}{\Sigma}(t)\, T = \Sigma_k\, t_k'\, \underset{k+1}{\Sigma}(t)\, T$

which, by induction hypothesis, yields $\underset{k}{\Sigma}(t)\, T = \underset{k}{\Sigma}(t')\, T$.

Q.E.D.

Let now Γ be any syntactically consistent set of sentences such that for every P_i there are infinitely many constants $t \in P_i$ which are not in Γ. Let L be a maximal consistent and ω-complete set containing Γ. We construe a structure U to the effect, that every sentence $A \in L$ is true for U, hence U is a model for Γ. As is well known, this procedure proves completeness. – Construction of U:

(1) The coordinate sets D_i. Define
$D_i := \{\Sigma_i\, t\, L | t \in P_i\}$.

(2) $k_i^0 := \Sigma_i\, t^0\, L$ iff $N_i(t^0) \in L$.
k_i^0 is well-defined: If $N_i(t^1) \in L$, then $t^0 \equiv t^1 \in L$, hence by theorem 5:
$\Sigma_i\, t^0\, L = \Sigma_i\, t^1\, L$. – Furthermore by axiom (iv f) and theorem 7 : $k_i^0 = L$.

(3) The interpretation functions for positional constants $t \in P_i$ are the following ones:

$$\omega_i(t) := \Sigma_i\, t\, L.$$

This definition makes all coordinates k_i in D_i, i.e. all the $\Sigma_i\, t\, L$ with $t \in P_i$, be designated by some constant in P_i ($\Sigma_i\, t\, L$ is designated by $t \in P_i$). This property of the structure U to be construed here allows substitutional interpretation for quantified statements, i.e. the following equivalence holds (for $\tau \in V_i$):

$$\models_U \vee \tau\, A \text{ iff for some } t \in P_i : \models_U A[t/\tau].$$

I shall use this equivalence tacitly.

(4) The relation R^i on D_i is defined as follows:

$$\Sigma_i\, t\, L\, R^i\, \Sigma_i\, t'\, L \quad \text{iff} \quad t\, R_i\, t' \in L.$$

We have to show that R^i is well-defined. Let be $\Sigma_i\, t\, L = \Sigma_i\, t_1\, L$ and $\Sigma_i\, t'\, L = \Sigma_i\, t_1'\, L$.
Then by theorem 5: $t \equiv t_1 \in L$ and $t' \equiv t_1' \in L$. Hence by axiom (iv n): $t\, R_i\, t' \in L$ iff $t_1\, R_i\, t_1' \in L$. This shows well-definedness.

(5) The truth value function V. With $t_i \in P_i$ $(i = 1, \ldots, n)$
$(\Sigma_1\, t_1\, L, \ldots, \Sigma_n\, t_n\, L)$ is a point in $D_1 \times \ldots \times D_n$.

Define:

$$V(p; \Sigma_1 t_1 L, \ldots, \Sigma_n t_n L) = 1 \text{ iff } p \in \overset{n}{\Sigma} (t) L.$$

By theorem 8, V is well-defined.

I now prove the main theorem.

Theorem 9:
For every sentence A and every point $(\Sigma_1 t_1 L, \ldots, \Sigma_n t_n L)$ in $(D_1 \times \ldots D_n)$
A is true for $U[\Sigma_1 t_1 L, \ldots, \Sigma_n t_n L]$ iff $A \in \overset{n}{\Sigma} (t) L$ $(:= \Sigma_1 t_1 \ldots \Sigma_n t L)$.
Here $U[\Sigma_1 t_1 L, \ldots, \Sigma_n t_n L]$ results from U by substituting $\Sigma_i t_i L$ for k_i^0. Let us abbreviate: $U[\Sigma t L] := U[\Sigma_1 t_1 L, \ldots, \Sigma_n t_n L]$.

The theorem is proved by induction. Again I write "A in $U[\Sigma t L]$" instead of $\models_{U[\Sigma t L]} A$. – (a) I start with atomic sentences.

(i) The theorem holds by definition for propositional constants.
(ii) $N_i(t')$ in $U[\Sigma t L]$ iff $m_i(t') = \Sigma_i t_i L$
(by def of the concept of truth), iff $\Sigma_i t' L = \Sigma_i t_i L$
(by def of m_i), iff $N_i(t') \in \Sigma_i t_i L$
(by theorem 6), iff $\underset{i+1}{\varphi}(t) \overset{i-1}{\varphi}(t) N_i(t') \in \Sigma_i t_i L$
(by axiom (iv m), iff $\varphi_i t_i \underset{i+1}{\varphi}(t) \overset{i-1}{\varphi}(t) N_i(t') \in L$
(by def of $\Sigma_i t_i$), iff $N_i(t') \in \overset{n}{\Sigma}(t) L$
(by the corollary of theorem 4).
(iii) Let be $t', t'' \in P_i$. $t' \equiv t''$ in $U[\Sigma t L]$ iff $m_i(t') = m_i(t'')$, iff $\Sigma_i t' L = \Sigma_i t'' L$, iff $t' \equiv t'' \in L$ (by theorem 5), iff $\overset{n}{\varphi}(t)\, t' \equiv t'' \in L$ (by axiom (iv k)), iff $t' \equiv t'' \in \overset{n}{\Sigma}(t) L$.
(iv) $t' R_i t''$ in $U[\Sigma t L]$ iff $m_i(t') R^i m_i(t'')$, iff $t' R_i t'' \in L$ (by def of m_i, R^i), iff $\overset{n}{\varphi}(t)\, t' R_i t'' \in L$ (by axiom (iv l)), iff $t' R_i t'' \in \overset{n}{\Sigma}(t) L$.
(b) For sentential connectives the theorem is shown by canonical reasoning.
(c) Existential quantification. Let be $\tau \in V_i$ and let the theorem hold for all sentences $A[t/\tau]$ with $t \in P_i$. $\vee \tau A$ in $U[\Sigma t L]$ iff for some $t_1 \in P_i$
$A[t_1/\tau]$ in $U[\Sigma t L]$, iff for some $t_1 \in P_i$: $A[t_1/\tau] \in \overset{n}{\Sigma}(t) L$ (by

induction hypothesis), iff $\vee\tau$ A $\in \overset{n}{\Sigma}$ (t) L (by maximal consistency and ω-completeness of $\overset{n}{\Sigma}$ (t) $\wedge$ L).

(d) I now prove the theorem for sentences containing an operator φ_i. Let the theorem hold for A and let be $t' \in P_i$. φ_i t' A in U[Σ t L] iff A in U[$\Sigma_1 t_1 L, \ldots, \Sigma_{i-1} t_{i-1} L, m_i(t'), \Sigma_{i+1} t_{i+1} L, \ldots, \Sigma_n t_n L$] (by the definition of truth), iff

$A \in \overset{i-1}{\Sigma}(t)\, \Sigma_i t'\ \Sigma\ (t)$ L (by induction hypothesis), iff

$A \in \Sigma_i t'\ \overset{i-1}{\Sigma}(t) \underset{i+1}{\Sigma}(t)$ L (by the corollary of theorem 4), iff

$\varphi_i t' A \in \overset{i-1}{\Sigma}(t) \underset{i+1}{\Sigma}(t)$ L, iff

$\varphi_i t_i \varphi_i t' A \in \overset{i-1}{\Sigma}(t) \underset{i+1}{\Sigma}(t)$ L (by axiom (iv b) and since $t_i \in P_i$),

iff $\varphi_i t' A \in \overset{n}{\Sigma}(t)$ L.

(e) Sentences containing a weak Kripke modal operator F_i: Let the theorem be true for A. Then it is as well true for any φ_i t' A – as I have just proved. Furthermore, the theorem is true for all atomic sentences. Thus we get: F_i A in U[Σ t L] iff for some $t' \in P_i$: $\Sigma_i t' L R^i \Sigma_i t_i L$ and:
A in U[$\Sigma_1 t_1 L, \ldots, \Sigma_{i-1} t_{i-1} L, \Sigma_i t' L, \Sigma_{i+1} t_{i+1} L, \ldots, \Sigma_n t_n L$] (by the def of truth and by def of the coordinate set D_i), iff for some $t' \in P_i : t' R_i t_i \in L$ and

$A \in \overset{i-1}{\Sigma}(t)\, \Sigma_i t' \underset{i+1}{\Sigma}(t)$ L (by induction hypothesis and by the definition of R^i), iff for some $t' \in P_i$:

$t' R_i t_i \in \overset{i-1}{\Sigma}(t) \underset{i+1}{\Sigma}(t)$ L and

$\varphi_i t' A \in \overset{i-1}{\Sigma}(t) \underset{i+1}{\Sigma}(t)$ L (by axiom (iv l) and the corollary of theorem 4), iff (with $\tau \in V_i$):

$\vee\tau (\tau R_i t_i \wedge \varphi_i \tau A) \in \overset{i-1}{\Sigma}(t) \underset{i+1}{\Sigma}(t)$ L,

iff $\varphi_i t_i F_i A \in \overset{i-1}{\Sigma}(t) \underset{i+1}{\Sigma}(t)$ L (by axiom (iv o)), iff $F_i A \in \overset{n}{\Sigma}(t)$ L.

– This finishes the proof.

Q.E.D.

We then have

Corollary 1:
For every sentence A:
$\models_U$ A iff $A \in L$.

Proof: By definition:
$U = U[\Sigma_1 t_1^0 L, \ldots, \Sigma_n t_n^0 L]$, where $N_i(t_i^0) \in L$ ($i = 1, \ldots, n$). Hence by the theorem: $\models_U$ A iff $A \in \overset{n}{\Sigma}(t_i^0) L$. But by the corollary of theorem 7: $\overset{n}{\Sigma}(t_i^0) L = L$

Q.E.D.

Now since $\Gamma \subseteq L$ we have

Corollary 2:
For every $A \in \Gamma$: $\models_U$ A.

Thus we have proved that every syntactically consistent set which is infinitely extendable has a model. Canonical reasoning shows that *every* syntactically consistent set has a model. Thus we have (strong) completeness for many-dimensional topological modal logic:

> If A is a semantical consequence from a set Γ of premises, then A is syntactically derivable from Γ.

3. *The Segerberg-operators*

We now turn to the definition of the operators which Segerberg introduces in order to solve the problem of sentences which contain indicators like "tomorrow". To do this, we regard a special case of n-dimensional topological logic, i.e. the case that $n = 2$ and $D_1 = D_2 =: D$. Thus our class of possible worlds – to adopt the language of modal logic – is a two-dimensional space, but we have only one set of coordinates, as we have, e.g., in analytical geometry of the two-dimensional plane, where the set of coordinates is the field $\mathbb{R}$ of real numbers and the plane is represented by the Cartesian "square" $\mathbb{R}^2$. In two-dimensional modal logic of the Segerberg type, the class of possible worlds is represented by the set of ordered pairs (k_1, k_2) with $k_1, k_2 \in D$. If (k_1, k_2) is some fixed point within D^2, it makes sense to talk – with respect to (k_1, k_2) – of "all points on this longitude", i.e. all points (k_1, k) with $k \in D$, or "all pojnts on this latitude", i.e. all points (k, k_2) with $k \in D$; it makes sense to talk of "the diagonal point on this longitude", i.e. the point (k_1, k_1), as well as to talk of the "mirror point", i.e. (k_2, k_1). The Segerberg-operators use these expressions, here they are: ([3])

□ A – everywhere, A.
◫ A – everywhere on this longitude, A.
⊟ A – everywhere on this latitude, A.
⦶ A – at the diagonal point on this longitude, A.
⊖ A – at the diagonal point on this latitude, A.
⊗ A – at the mirror point, A.

Informally, the truth conditions for these operators are the following ones: Let again $(k_1, k_2) \in D^2$ be some fixed point, then

□ A is true at (k_1, k_2) iff for
all $(k, k') \in D^2$: A is true at (k, k');

◫ A is true at (k_1, k_2) iff for
all $(k_1, k) \in D^2$: A is true at (k_1, k);
and so on, e.g.

⊗ A is true at (k_1, k_2) iff A is true at (k_2, k_1).

I shall now show how to define these operators within the apparatus developped so far. – Since $n = 2$ and $D_1 = D_2 =: D$ we need within syntax only one class of positional constants and variables, respectively, hence let us assume that $P_1 = P_2 =: P$ and $V_1 = V_2 =: V$. Furthermore we add to the concept of a structure the conditions $m_1 = m_2 =: m$, and $R^1 = R^2 =: R$ (though we don't talk about that relation yet). – Now I define the operators by identifying the formulas □ A, ◫ A, etc. with sentences expressed in topological logic which have the intended truth conditions.

([3]) Vide Segerberg (1973), p. 81.

(1) $\Box$ A: $= \bigwedge\tau \bigwedge\tau' \varphi_1\tau \varphi_2\tau' A$.
(2) $\boxminus$ A: $= \bigvee\tau \bigwedge\tau' (N_1(\tau) \wedge \varphi_1\tau \varphi_2\tau' A)$.
(3) $\boxminus$ A: $= \bigvee\tau \bigwedge\tau' (N_2(\tau) \wedge \varphi_1\tau' \varphi_2\tau' A)$.
(4) $\varobar$ A: $= \bigvee\tau(N_1(\tau) \wedge \varphi_1\tau \varphi_2\tau A)$.
(5) $\ominus$ A: $= \bigvee\tau(N_2(\tau) \wedge \varphi_1\tau \varphi_2\tau A)$.
(6) $\otimes$ A: $= \bigvee\tau_1 \bigvee\tau_2 (N_1(\tau_1) \wedge N_2(\tau_2) \wedge \varphi_1\tau_2 \varphi_2\tau_1 A)$.

Let us see now whether the sentences on the right side have the intended truth conditions listed above. Let U = $<D^2$; (k_1^0, k_2^0); m; R; V$>$ be a structure. Then (k_1^0, k_2^0) is the point to which our sentences $\otimes$ A etc. refer. Denote by $U[k_1, k_2]$ the structure which results from U by substituting k_i for k_i^0 (i = 1, 2). We have to prove then e.g.: $\otimes$ A in U iff A in $U[k_2^0, k_1^0]$ (using once more this notation for $\models_{U[k_2^0, k_1^0]} A$) and

$\boxminus$ A in U iff for all $k \in D$: A in $U[k_1^0, k]$.

Remember for the following proofs that A is a sentence, thus A does not contain any free variable, hence $A[t/\tau]$ is always the same as A. Now let t and t' be the first and the second constant from P which do not occur in A.

(1) $\Box$ A in U iff for all $k_1, k_2 \in D$: A in U $[k_1, k_2]$

Proof: Let be $\Box$ A in U, then by def: for all functions m' such that $m' =_{t,t'} m$: $\varphi_1 t\, \varphi_2 t'$ A in $U[m'/m]$. Now let be $k_1, k_2 \in D$. The function m' defined by

$$m'(t''):= \begin{cases} k_1, \text{ if } t'' = t, \\ k_2, \text{ if } t'' = t', \\ m(t''), \text{ otherwise} \end{cases}$$

fulfills the condition. Thus $\varphi_1 t \varphi_2 t'$ A in $U[m'/m]$, hence A in $U[k_1, k_2; m'/m]$, therefore, since t, t' are not in A and $m' =_{t,t'} m$: A in $U[k_1, k_2]$. – Let us now have A in $U[k_1, k_2]$ for all $k_1, k_2 \in D$ and let be $m' =_{t,t'} m$ any function. By hypothesis: A in $U[m'(t), m'(t')]$. But since t, t' are not in A, this is the same as A in $U[m'(t), m'(t'); m'/m]$, which is, by def, equivalent to $\varphi_1 t \varphi_2 t'$ A in $U[m'/m]$. Thus: $\Box$ A in U.

Q.E.D.

(2) $\boxminus$ A in U iff for all $k \in D$: A in U $[k_1^0, k]$.

Proof: Let be ◫ A in U. Then by def: There is a function m′ such that: $m' =_t m$ and for all m″: if $m'' =_{t'} m'$ then $N_1(t) \wedge \varphi_1 t \varphi_2 t'$ A in U[m″/m], which is the same as: $m''(t) = k_1^0$ and A in $U[m''(t)/k_1^0, m''(t')/k_2^0; m''/m]$, hence for all m″: if $m'' =_{t,t'} m$, then A in $U[k_1^0, m''(t'); m''/m]$. Now let be $k \in D$. The function m″ defined

$$m''(t'') := \begin{cases} k_1^0, \text{ if } t'' = t, \\ k, \text{ if } t'' = t', \\ m(t''), \text{ otherwise} \end{cases}$$

fulfills the condition. Hence: A in $U[k_1^0, k; m''/m]$. But since t, t′ are not in A, this the same as A in $U[k_1^0, k]$. – Let us now have A in $U[k_1^0, k]$ for all $k \in D$. Define

$$m'(t'') := \begin{cases} k_1^0, \text{ if } t'' = t, \\ m(t''), \text{ otherwise.} \end{cases}$$

Let be $m'' =_{t'} m'$, hence $m''(t) = k_1^0$, hence $N_1(t)$ in U[m″/m] and A in U[m″(t), m″(t′); m″/m], hence $N_1(t) \wedge \varphi_1 t \varphi_2 t'$ A in U[m″/m], hence $\vee\tau \wedge\tau' (N_1(\tau) \wedge \varphi_1 \tau \varphi_2 \tau'$ A) in U.

Q.E.D.

(3) ⊟ A in U iff for all $k \in D$: A in $U[k, k_2^0]$.

This can be proved likewise as (2).

(4) ⦶ A in U iff A in $U[k_1^0, k_1^0]$.

Proof: Let be ⦶ A in U. Then by def for some function m′ such that $m' =_t m$

$N_1(t) \wedge \varphi_1 t \varphi_2 t$ A in U[m′/m]. But this implies $m'(t) = k_1^0$ and A in $U[k_1^0, k_1^0]$. – If, on the other hand, we have A in $U[k_1^0, k_1^0]$, define

$$m'(t'') := \begin{cases} k_1^0, \text{ if } t'' = t, \\ m(t''), \text{ otherwise.} \end{cases}$$

Then we have $N_1(t) \wedge \varphi_1 t \varphi_2 t$ A in U[m′/m]. Hence $\vee\tau (N_1(\tau) \wedge \varphi_1 \tau \varphi_2 \tau$ A) in U.

Q.E.D.

Likewise one proves

(5) ⊖ A in U iff A in $U[k_2^0, k_2^0]$.

Finally I show

(6) $\otimes$ A in U iff A in $U[k_2^0, k_1^0]$.

Proof: If $\otimes$ A in U, then by def: for some function m″ such that $m'' =_{t,t'} m$ we have $N_1(t) \wedge N_2(t') \wedge \varphi_1 t' \varphi_2 t$ A in $U[m''/m]$. Hence: $m''(t) = k_1^0$ and $m''(t') = k_2^0$ and A in $U[m''(t'), m''(t); m''/m]$, which is the same as A in $U[k_2^0, k_1^0]$. –

If, on the other hand, we have A in $U[k_2^0, k_1^0]$ define:

$$m''(t'') := \begin{cases} k_1^0, \text{ if } t'' = t, \\ k_2^0, \text{ if } t'' = t', \\ m(t''), \text{ otherwise.} \end{cases}$$

Then we have: $N_1(t) \wedge N_2(t') \wedge \varphi_1 t' \varphi_2 t A$ in $U[m''/m]$, thus $\vee \tau_1 \vee \tau_2 (N_1(\tau_1) \wedge N_2(\tau_2) \wedge \varphi_1 \tau_2 \varphi_2 \tau_1 A)$ in U.

Q.E.D.

This finishes the introduction of Segerberg-operators into many-dimensional topological modal logic.

Further semantical reasoning shows that all of Segerberg's axioms[4] for two-dimensional modal logic turn out to be valid formulas in topological logic if they are translated by the operator definitions (1) - (6). Thus, Segerberg's logical system is a specialization of many-dimensional topological modal logic. – However, we must once more look at completeness: Since by the assumptions $D_1 = D_2$, $m_1 = m_2$ and $R_1 = R_2$ we have tightened the concept of a structure, the model which we construct within the course of the completeness proof has to satisfy these additional assumptions. The model construed does not, however: Even if $P_1 = P_2 =: P$ the set $D_1 := \{\Sigma_1 t L \mid t \in P\}$ is different from $D_2 := \{\Sigma_2 t L \mid t \in P\}$.

Can we get rid of this problem? Indeed we can – even in the n-dimensional case. This is shown by some arguments very similar to those we already know.

Using the abbreviation

$$\Sigma t \Gamma := \Sigma_1 t \ldots \Sigma_n t \Gamma \text{ (for } t \in P)$$

we have the following theorems (Γ being some maximal consistent and ω-complete set):

(4) Vide ibid., p. 82.

Theorem 10:
For all t, t' $\in$ P:

$\Sigma\, t\, T = \Sigma\, t'\, T$ iff $t \equiv t' \in T$.

Theorem 11:
If for i = 1, ..., n: $\Sigma\, t_i\, T = \Sigma\, t'_i\, T$
then $\overset{n}{\Sigma}(t)\, T = \overset{n}{\Sigma}(t')\, T$ (for any t_i, $t'_i \in P$).

These theorems resemble theorems 5 and 8, and so do their proofs:

(10) Sufficiency: Let be $\Sigma\, t\, T = \Sigma\, t'\, T$.
From axioms (iv h,m) we have $\varphi_{n\,t} \ldots \varphi_1\, t\, N_1(t) \in T$, hence $N_1(t) \in \Sigma\, t\, T$ and $N_1(t) \in \Sigma\, t'\, T$.
But $N_1(t') \in \Sigma\, t'\, T$ as well. Hence (axiom (iv g)) $t \equiv t' \in \Sigma\, t'\, T$ hence (by axiom (iv k)) $t \equiv t' \in T$. – Necessity: If $t \equiv t' \in T$, then by theorem 5: $\Sigma_n\, t\, T = \Sigma_n\, t'\, T$. Now let be $\Sigma_k\, t \ldots \Sigma_n\, t\, T = \Sigma_k\, t' \ldots \Sigma_n\, t'\, T$. Since $\varphi_n\, t \ldots \varphi_k\, t\, t \equiv t'$ is in T, $t \equiv t'$ is in $\Sigma_k\, t \ldots \Sigma_n\, t\, T$. Hence $\Sigma_{k-1}\, t \Sigma_k\, t \ldots \Sigma_n\, t\, T = \Sigma_{k-1}\, t' \Sigma_k t \ldots \Sigma_n\, t\, T$.
By induction hypothesis: $\Sigma_{k-1}\, t \Sigma_k\, t \ldots \Sigma_n\, t\, T =$
$\Sigma_{k-1}\, t'\, \Sigma_k\, t' \ldots \Sigma_n\, t'\, T$.

Q.E.D.

(11) Let be $\Sigma\, t_i\, T = \Sigma\, t_i'\, T$ (i = 1,..., n).
Then by theorem 10: $t_i \equiv t_i' \in T$
(i = 1,..., n), hence by theorem 8:
$\overset{n}{\Sigma}(t)\, T = \overset{n}{\Sigma}(t')\, T$.

Q.E.D.

Now I define a model U:

(1) D: = $\{\Sigma\, t\, L \mid t \in P\}$.
(2) $k_i^0 := \Sigma\, t_i^0$ iff $N_i(t_i^0) \in L$.
k_i^0 is well-defined: if $N_i(t_i^1) \in L$, then $t_i^0 \equiv t_i^1 \in L$, hence by theorem 10: $\Sigma\, t_i^0\, L = \Sigma\, t_i^1\, L$.
By axiom (iv f): $N_i(t_i^0) \in L$ for some $t_i^0 \in P$.
(3) $m(t) := \Sigma\, t\, L$.
(4) $\Sigma\, t\, L\, \bar{R}\, \Sigma\, t'\, L$ iff $t R t' \in L$.
$\bar{R}$ is well-defined by axiom (iv n) and theorem 10.
(5) $V(p; \Sigma\, t_1\, L, \ldots \Sigma\, t_n\, L) = 1$ iff $p \in \overset{n}{\Sigma}(t)\, L$.
V is well-defined by theorem 11.

If $t_1,\ldots, t_n \in P$ let U (Σ t L) denote the structure
$U[\Sigma\, t_1\, L/k_1^0,\ldots, \Sigma\, t_n\, L/k_n^0]$.

We then have

Theorem 12:
For all $t_1,\ldots, t_n \in P$ and all sentences A:
A in U (Σ t L) iff $A \in \overset{n}{\Sigma}(t)\, L$.

The proof is left to the reader. It results from the proof for theorem 9 by substituting U (Σ t L) for U[Σ t L], by dropping the indices at the appropriate places and by using the new theorems 10 and 11.

Corollary:
For all A: $\models_U$ A iff $A \in L$.

Proof: By definition:
$U = U[\Sigma\, t_1^0,\ldots, \Sigma\, t_n^0]$, where $N_i(t_i^0) \in L$. Hence by the corollary of theorem 7:
$\overset{n}{\Sigma}(t^0)L = L$, and by theorem 12:
A in U iff $A \in \overset{n}{\Sigma}(t^0)\, L$.

Q.E.D.

Thus we have completeness for many-dimensional topological modal logic also in the case where there is only one set D of coordinates and D^n is our class of possible worlds.

Let us have one final look at the indicators "here" and "tomorrow" mentioned in the introduction. (Definite operators like "in New Orleans" and "on January 7th 1985" are simply construed by $\varphi_1 t_1$, $\varphi_2 t_2$, respectively, t_1 being interpreted "in New Orleans" by the function m_1, t_2, via m_2, designating the 7th of January 1985.) Well, if $D_1 = D_2$ is the dimension of space and $U[k_1, k_2]$ is a structure with $k_i \in D$ ($i = 1,2$), let us interprete the relation *A in* $U[k_1, k_2]$ to mean "The sentence A, regarded as being uttered at k_2, is true at k_1". But if in "Here, A" the indicator "Here" is meant to refer to the point in space where the sentence A is uttered, this yields:

Here, *A* in $U[k_1, k_2]$ iff *A* in $U[k_2, k_2]$ ([5]).

Thus "Here" is the operator $\ominus$ defined above.

In order to construe the indicator "tomorrow" we use the relation-component R of a structure U and the additional assumption that R is a successor relation, i.e. R has the properties:

(i) For every $k \in D$ there is some $k' \in D$
such that for all $k'' \in D$:
$k \overline{R} k'$, and: if $k \overline{R} k''$, then $k'' = k'$.

(This condition makes R a function on D.)

(ii) For every $k \in D$: not $k R k$.

In order to save the completeness of our logic, we have to add some further ground axioms. Canonical reasoning shows, that the following set will do the job (since $R_1 = .. = R_n$, I drop the index):

(i) $\wedge\tau \vee\tau' \wedge\tau'' (\tau R \tau' \wedge (\tau R \tau'' \rightarrow \tau' \equiv \tau''))$
(ii) $\wedge\tau \neg\tau R \tau$

Let us now define "tomorrow". Since the relation *Tomorrow, A in U* $[k_1, k_2]$ is understood to mean "The sentence *Tomorrow, A*, regarded as being uttered at k_2, is true at k_1" and since "Tomorrow" refers to the day after the sentence's utterance, the truth condition will be

> *Tomorrow, A* in $U[k_1, k_2]$ iff *A* in $U[k, k]$, where k is the unique element from D such that $k_2 R k$.

Now, by the conditions on R, this obviously amounts to:

> *Tomorrow, A* in $U[k_1, k_2]$ iff *A* in $U[k, k]$ for some $k \in D$ such that $k_2 R k$.

([5]) This makes the second component in (k_1, k_2) be the point of a sentence's utterance. Since this component is referred to by the operator $\varphi_2 t_2$, the sentence $\varphi_1 t_1 \varphi_2 t_2$ *Here, A* is understood to mean: The sentence *Here, A*, regarded as being uttered at t_2, is true at t_1. Thus the truth condition of $\varphi_1 t_1 \varphi_2 t_2$ *Here, A* ought to be the same as that one of $\varphi_1 t_2 \varphi_2 t_2 A$. – Well, this is the case:
$\varphi_1 t_1 \varphi_2 t_2$ *Here, A* in $U[k_1, k_2]$ iff
Here, A in $U[m(t_1), m(t_2)]$, iff
A in $U[m(t_2), m(t_2)]$, iff $\varphi_1 t_2 \varphi_2 t_2 A$ in $U[k_1, k_2]$.

I now define:

Tomorrow, A : = $\vee\tau \vee\tau'$
$(N_2(\tau) \wedge \tau R\tau' \wedge \varphi_1 \tau' \varphi_1 \tau' \varphi_2 \tau' A)$.

Tomorrow, A has the intended truth condition.

Proof: Let t, t′ be the first and second constant from P which are not in A. Let $U = <D^2; (k_1^0, k_2^0); m; R; V>$ be a structure.
If we have *Tomorrow, A* in U, then by def for some function $m' =_{t,t'} m$:
$N_2(t) \wedge tRt' \wedge \varphi_1 t' \varphi_2 t'$ A in $U[m'/m]$.
Hence: $m'(t) = k_2^0$, $m'(t) R m'(t')$ and A in $U[m'(t'), m'(t'); m'/m]$.
Hence (t, t′ not being in A): For $k := m'(t')$ we have $k_2^0 R k$ and A in $U[k, k]$. – Now let be A in $U[k, k]$ for some $k \in D$ such that $k_2^0 R k$.
Define

$$m'(t'') := \begin{cases} k_2^0, \text{ if } t'' = t, \\ k, \text{ if } t'' = t', \\ m(t''), \text{ otherwise.} \end{cases}$$

Since t, t′ are not in A we then have by our assumption and def of m′:
$N_2(t) \wedge tRt'$ in $U[m'/m]$ and A in $U[m'(t'), m'(t'); m'/m]$. Hence:
$N_2(t) \wedge tRt' \wedge \varphi_1 t' \varphi_2 t'$ A in $U[m'/m]$.
Thus: *Tomorrow, A* in U.

Q.E.D.

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