PREDICTION PARADOX REVISITED

Silviu GUIASU

There is no generally accepted solution for the prediction paradox, known also as the surprise examination problem. The numerous previous attempts have tried to solve the paradox by applying tools and methods belonging to classical and modal logic, syntax, set theory, number theory, or epistemology. The object of this paper is to give a solution to the prediction paradox based on a necessary distinction between two different types of surprise.

1. *Introduction*

The prediction paradox, or the surprise examination problem, is well-known: A teacher announces to his class that there will be an examination on exactly one of the five working days of the next week and that the examination will take the students by surprise – the students will not be able to predict with certainty, prior to the day of the examination, on which day it will be held. The students claim that this announcement cannot be fulfilled for if the examination is left until the last day, Friday, the students will be able to anticipate it on Thursday evening; if that day is ruled out, the same reasoning will apply to the next earlier day, and so on until all the days are eliminated. So, according to the students's analysis, a surprise examination is impossible. But in spite of this, the announcement is fulfilled when the teacher gives the examination on one of the days thus catching the students by surprise.

As Scriven [8] pointed out, the flawless reasoning of the student is somehow rudely brought to nothing by the actual occurence of the promised, but apparently impossible, examination; it has "the flavour of logic refuted by the world."

Numerous attempts of solving the prediction paradox have used concepts and techniques borrowed from classical logic and set theory (as in [2], [5]), from modal logic (as in [1]), from syntax and number theory (as in [3]), or from epistemology (as in [6]). The papers just mentioned contain also useful references to other interesting papers dealing with the prediction paradox. Many papers try to find errors in the students's slippery argument. Some other papers introduce some additional, lateral, suppositions on how the students ought to have thought correctly for realizing that a surprize examination is nevertheless possible. Thus, in [7] for instance, the author suggests that the students cannot know whether the teacher will keep his promise or not and that, consequently, they will not know ahead of time whether they are to have the examination or their teacher is to be proved a liar. As Edman says (see [2]), this looks as cutting the Gordian knot. Making suppositions like that, no surprise examination will be possible if, for instance, the teacher becomes ill or if the school closes unexpectedly because of a severe snow storm.

In spite of the numerous attempts, the general opinion is that no simple irrefutable solution to this paradox has yet appeared. The object of this paper is to give a simple solution to the prediction paradox based on a necessary distinction between two different types of surprise. We are dealing with the classical formulation of the prediction paradox, as it is presented in the first paragraph of this section. No other supposition will be allowed. Under the given constraints, is a surprise examination possible? Is the students's argument for rejecting the possibility of a surprise examination correct or not?

2. *Uncertainty*

Surprise, a key concept in the prediction paradox, is directly related to uncertainty. Roughly speaking, surprise has to do with not knowing beforehand and, consequently, it is related to some kind of probabilistic experiment. In the finite case, a probabilistic experiment A is completely characterized by its possible outcomes, called also elementary events, denoted by $a_1, \ldots, a_n$, and their probabilities, denoted by $p_1, \ldots, p_n$. Before performing the experiment, we cannot predict with certainty the future outcome. All we can say is that there is a probability p_i that the outcome a_i will occur.

Dealing with the possible outcomes of a probabilistic experiment, we must distinguish their unity in spite of their diversity. Obviously, the possible outcoumes are different in some respect and this is why we can put different labels on them, but, at the same time, they have something in common, namely, they are elementary events of the same kind, being possible outcomes of the same probabilistic experiment.

Of course, before performing a probabilistic experiment, there is an uncertainty about the future outcome which will occur. Quantitatively, the amount of uncertainty contained by the experiment A is given by Shannon's entropy $H(A) = - \sum_{i=1}^{n} p_i \log_2 p_i$. When all the n outcomes are equally likely, this entropy is equal to $\log_2 n$.

Before performing a probabilistic experiment, the observer is uncertain about the forthcoming outcome, being able to predict only with the probability p_i that the outcome will be a_i, for any $i = 1, \ldots, n$. After performing the probabilistic experiment, the observer is surprised by its outcome. The posterior surprise supplied by the probabilistic experiment is measured by the prior uncertainty about its forthcoming outcome. The larger the prior uncertainty the larger the posterior surprise.

Let us notice that an experiment with only one possible outcome is a particular case of probabilistic experiment whose only elementary event has the occurence probability equal to 1, being in fact the sure event. The corresponding entropy is equal to zero, showing that, in accordance to common sense, there is no uncertainty before performing such an experiment and no surprise after performing it.

The above analysis and the given example alow us to understand only one kind of surprise. When the possible outcomes of a probabilistic experiment are known and surprise comes only from the impossibility to predict with certainty the particular outcome, from this set of known possible outcomes, which will actually occur, we have a so called *type 1 surprise*. Rolling a fair die is a probabilistic experiment alowing only such a type 1-surprise. There is another kind of surprise which is often ignored. If during a probabilistic experiment a new, priorly unknown, outcome occurs, we are facing a *type 2-surprise*.

For realizing the difference between the two types of surprise let us take the following examples:

1°. Suppose that Mr. Brown is waiting at a bus station and he knows that the next bus may be anyone from the set of buses labeled 41, 60B, 117, or GARAGE, passing by. The arrival of such a bus is a type 1-surprise for Mr. Brown.

2°. If one day, at the same station, the next bus proves to be labeled 57 and Mr. Brown is not aware of such a possibility, he will experience a type 2-surprise, unpredictable, of course. Including such a new outcome in the set of previous possible outcomes he will experience again only a type 1-surprise in the following days with respect to the new probabilistic experiment containing one more possible outcome.

Summarizing, the occurence of an outcome, priorly known as possible, produces a type 1-surprise; the occurence of new, unknown, outcomes produces a type 2-surprise.

3. *Two Simple Variants of the Examination Problem*

Before giving a solution to the prediction paradox, let us examine briefly two simple variants of the examination problem. The constraints will be chosen from the following ones:

(A) *Exactly one examination will be given sometime during the next week.*

(B) *The examination will be a surprise, i.e. the student will not be able to predict, with certainty, prior to the day of examination, on which day it will be held.*

(C) *At most one examination will be given next week.*

Of course, (C) is a weaker restriction than (A). We assume that: a) The teacher informs the students about the constraints chosen by him and acts according to the selected constraints; b) The students are informed about the selected constraints and are confident that the professor will act accordingly.

Otherwise a surprise may arise either from teacher's inconsistency or from student's suspicion. Also, we are interested here in the existence of an objective surprise examination. Subjectively the problem is much more delicate. Thus, a good student, prudent and permanently well prepared for a tomorrow examination, will have, subjectively speaking, no surprise at all from an examination on any day of the week.

In what follows, we call surprise examination an examination which cannot be predicted with certainty. Let us denote by E the following experiment: "choose a day for examination".

Variant 1 : Assume only constraint (A). Abbreviating the days of the week by M, Tu, W, Th, F, and denoting by 1 = "examination" and by 0 = "no examination", the possible "histories" allowed by constraint (A) are the five rows of table 1. Obviously, in such a case, any day of the week is a possible outcome of the experiment E. In the evening of each day the remaining possible histories are equally likely. On Sunday evening the experiment E has five possible outcomes (the five days of the next week or, equivalently, the five possible "histories" given in Table 1), the amount of existing uncertainty being $\log_2 5 = 2.3219$. If the examination didn't occur on Monday, in the evening of the same day, the experiment E contains an uncertainty equal to $\log_2 4 = 2$, and so on. Finally, if the examination didn't occur before, on Thursday evening, the experiment E has only one outcome (history 5) and there is no uncertainty on what will happen on the next day. In this variant the examination will be a surprise examination on any day of the week but Friday.

	M	Tu	W	Th	F
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1

Table 1.

Variant 2 : Assume only constraint (C). In such a case there are six possible "histories" (the lines of Table 2), and again any day of the week is a possible outcome of the experiment E. The difference is that in Variant 2 the examination will be a surprise examination on any day of the week, including Friday. If the examination didn't occur before, on Thursday evening, the uncertainty is equal to $\log_2 2 = 1$.

	M	Tu	W	Th	F
1	1	0	0	0	0
2	0	1	0	0	0
3	0	0	1	0	0
4	0	0	0	1	0
5	0	0	0	0	1
6	0	0	0	0	0

Table 2

Therefore, in both variants just presented, a surprise examination is possible without explicitely asking for it! Let us notice also that, in both variants, a type 1-surprise examination is involved.

4. *The Solution of the Prediction Paradox*

Let E be again the experiment: "choose a day for examination". *The constraints of the prediction paradox are* (A) *and* (B). The "histories" allowed by (A) are the rows of table 1 and, as we have seen in Variant 1, any day of the week is an outcome of E *if* only the constraint (A) is imposed. But, according to (B), the outcome of E must be a day on which a surprise examination (not simply an examination!) could occur. Therefore, Friday is not a possible outcome because, according to (A), on Thursday evening there is no uncertainty on what will happen on the next day. Expelling Friday from the set of possible outcomes of E subject to (A) and (B), we see that, if nothing happened during the first days of the week, on Wednesday evening we know that an examination can occur on Thursday but not a surprise examination. Indeed, if nothing happened on the first three days of the week, the last possibility for a surprise examination is Thursday and according to (A) such a surprise examination must necessarily occur on Thursday; therefore the examination is not longer a surprise and Thursday must be ruled out as a possible outcome of the experiment E subject to the constraints (A) and (B). Continuing the same analysis for the other days, we see that no day of the week is a possible outcome of the experiment E under the constraints (A) and (B). This is what the students correctly do in rejecting the possibility of a surprise examination. But in fact the above arguments simply show that the constraints (A) and (B) do not allow a type 1-surprise examination.

At this stage, there is no type 1-uncertainty on what will happen next week. In the evening of any day of the week, as a consequence of the above correct logical argument, the students are in front of a strictly deterministic experiment, with one possible outcome: "no surprise examination tomorrow", which, more accurately must be read: "no type 1-surprise examination tomorrow." But the teacher comes, on any day, and do give an examination, creating a type 2-surprise. If, for instance, the teacher decides to give an examination on Wednesday, he introduces a new possible outcome, namely, "examination on Wednesday", to the students's pattern which contains only one (certain) outcome: "no examination on Wednesday", increasing the uncertainty from 0 to $\log_2 2 = 1$, therefore giving a surprise examination.

Therefore, making a clear distinction between a type 1-surprise and a type 2-surprise we see that the students are right in proving that the constraints (A) and (B) of the prediction paradox do not allow a type 1-surprise examination. But they stop their analysis at this point, unable to predict the possibility of a type 2-surprise examination which is going to be given by the teacher. Now it is clear why the surprise examination problem is called a paradox. Both the students (rejecting the possibility of a type 1-surprise examination) and the teacher (giving a type 2-surprise examination) are right. But a type 2-surprise examination is a surprise anyway!

Remark: Before ending this section let us mention that the constraints (B) and (C) imply the existence of a type 1-surprise examination on any day of the week. Indeed, if the constraint (B) is added to (C), Friday continues to be a possible outcome of the experiment E and nothing changes in Variant 2 analysed in section 3.

5. *Conclusion*

Under the constraints (A) and (C) of the prediction paradox, a type 1-surprise examination is not possible but a type 2-surprise examination is possible. In rejecting the possibility of a type 1-surprise examination the students are right but, unfortunately, they stop their correct analysis here, unable to foresee the possibility of a type 2-surprise examination which is just what the teacher is going to give.

Acknowledgement: The author is indebted to Professor Doris Olin and to the reviewer of this paper for their critical remarks.

York University
Department of Mathematics
Faculty of Arts
4700 Keele Street, North York, M3J 1P3
Toronto, Ontario, Canada

Silviu GUIASU

REFERENCES

[1] Binkley R., The Surprise Examination in Modal Logic. *Journal of Philosophy*, 65 (1968), 127-136
[2] Edman M., The Prediction Paradox. *Theoria*, 40 (1974), 166-175
[3] Fitch F.B., A Goedelized Formulation of the Prediction Paradox. *American Philosophical Quarterly*, 1 (1964), 161-164
[4] Guiasu S., *Information Theory with Applications*. McGraw-Hill, New York-London-Düsseldorf, 1977.
[5] Medlin B., The Unexpected Examination. *American Philosophical Quarterly*, 1 (1964), 66-72
[6] Olin D., The Prediction Paradox Resolved. *Philosophical Studies*, 44 (1983), 225-233
[7] Quine W.V., On a Supposed Paradox. *Mind*, 62 (1953), 65-67
[8] Scriven M., Paradoxical Announcements. *Mind*, 60 (1951), 403-407