THE LOGIC OF NATURALNESS

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Unless we are highly selective about what properties, relations and propositions there are ([1]), we shall have good reason to distinguish between more and less natural properties, relations, and propositions, in the way Lewis does. ([2]) To bring this out it is initially convenient to consider worlds. ([3]) For the sake of definiteness I shall concentrate on propositions, but everything I say applies to properties and relations also. So, with more or less seriousness, we shall treat propositions as sets of worlds. Only some of these will correspond to English sentences, or sentences in any other natural language. Some of them will be quite arbitrary and unconnected sets of worlds. Others, however, and these will include any expressible in a natural language, will be more natural ones.

Naturalness admits of degrees, but for my present purpose I shall simplify matters and operate with a flat natural/unnatural distinction. And I shall assume that the natural propositions are closed under finite conjunction, finite disjunction and negation. ([4]) The intuitive basis for this flat natural/unnatural distinction is that a natural proposition is one which a human mind could have an attitude towards. As an example, consider the operation of infinite conjunction. If Σ is any set of propositions, that is, a set of sets of worlds, then its conjunction $\wedge \Sigma$ is simply the intersection of all the members of Σ. If Σ is infinite this will be an infinite conjunction. Similarly we could define the infinite disjunction

([1]) As an example of a highly selective theory of properties and relations we may note D.M. Armstrong *Nominalism and Realism Vol. II: A Theory of Universals*, Cambridge University Press 1978. Recently Armstrong has come to be less selective, denying that all properties and relations are universals.

([2]) David LEWIS, 'New Work for a Theory of Universals', *Australasian Journal of Philosophy*, 61, Dec. 83, pp. 343-377.

([3]) I say 'worlds' rather than 'possible worlds' because I do not assume they are all possible.

([4]) If we identify natural properties and relations with Armstrongs' universals (as in *A Theory of Universals*) we would not have closure under disjunction and negation. At most this shows that by 'natural' I mean 'fairly natural' not 'the most natural'.

as the union. Now suppose, for the sake of definiteness, that Σ consists of all the propositions expressed by ([5]) English sentences of the form: If x and y are both of negative charge then there is an force repelling them. Here 'x' and 'y' range over all the referring expressions we can form using English. The infinite conjunction of the propositions expressed by those sentences is not expressed by the universal generalisation:

Any two objects of negative charge have a force repelling them.

One reason is that, I submit, it is logically possible that there are objects for which English has not the resources to refer to (perhaps in spatio-temporally detached parts of the universe). And if there were no force attracting them, then the universal generalisation would be false, but the infinite conjunction being considered would be true.

If you grant the above, which I take to be the orthodoxy, then you should question whether the infinite conjunction could ever be the content of any English-speaker's beliefs. And similarly for other languages. The reason for such doubt is the sheer complexity of the infinite conjunction. It is plausible that only an infinite mind could have an attitude towards it, except indirectly by having an attitude to some other proposition, such as the universal generalisation. I shall distinguish between these propositions we humans could not have a direct attitude towards and those we could. The latter I call *natural*, the former *unnatural*. Using this distinction I then conjecture that the weakest natural proposition entailing that expressed by 'If Fx then Gx' for all possible referring expressions 'x' in English, will be a universal generalisation. It will be of the form 'All FHs are Gs' where all the possible referring expressions 'x' are all such that x is H.

Quite generally, if we have characterized some set Σ of propositions, we might be interested in the weakest natural proposition, if there is one, which entails all the members of Σ, and hence entails $\wedge\Sigma$. I shall call that the *natural strengthening* of $\wedge\Sigma$. If, in a formal system, some wff A expresses the proposition $\wedge\Sigma$ then *SA will express the natural strengthening of* $\wedge\Sigma$. S will be an operator, formally like a modal operator. And it is my purpose in this paper to describe a system with this operator.

To continue with the motivation, I note another situation in which we

([5]) What is the connection between a proposition and an sentence? I use the term 'expresses', but this is merely as a place-holder for whatever the relation is.

might consider a natural strengthening. Sometimes there are interesting recursively defined sequences of propositions $\{p_0, p_1 ...\}$ such that p_{n+1} entails p_n. (For example, $\{Q_0, Q_1, ...\}$, where Q_{n+1} is expressed by 'I know that Q_n' for n = 0,1,2,etc.) Then we may be interested in some sort of *limit* of the p_n. In a sense the infinite conjunction of the set of the p_n is itself a limit. But if it is not natural, that is, if no human being could have an attitude towards it, it is not what we are seeking. What we seek is the natural strengthening of the infinite conjunction. Call this p_∞. In the example of the Q_n, Q_∞ might express the sort of totally transparent self-consciousness required in the Cartesian tradition.

In this paper I shall use the worlds semantics for naturalness and for the operation of natural strengthening to find a complete set of axioms for ***S*** and for two ancilliary operators ***N***, and ***T***. This world semantics provides minimal or Scott-Montague models. I start with some simple systems involving operators ***N*** and ***T*** and I work up to the operator of interest ***S***.

One possible motivation for seeking such axioms is scepticism about the worlds. Once we have a complete set of axioms we no longer need the worlds in order to characterise naturalness and the natural strengthening.

I

The account I shall give is within the scope of three simplifications. The first is the assumption that the underlying modal logic is S5. I shall assume that whether or not a proposition is natural is itself non-contingent. Hence if one world is accessible from another they share the same set of natural propositions. Therefore, the interaction between naturalness and modality can be displayed by stipulating, if necessary, that any two worlds connected by a chain of accessibility relations or their converses are themselves mutually accessible. If the modal logic is initially assumed to be as strong as T, or even KD, ([6]) with the stipulated accessibility it is as strong as S5.

The second simplification is that entailment is taken to be a box hook,

([6]) Here I use Chellas' classification. See Brian F. Chellas, *Modal Logic: An Introduction* Cambridge University Press, 1980, p. 131.

that is, 'P entails Q' is explicated as $\Box(P \rightarrow Q)$. The justification for this is simply the need to concentrate on the features introduced by naturalness, without worrying initially about any defects in that account of entailment. (7)

The third simplification is that I shall stipulate bivalence. Intuitively, if A has no natural strengthening because there is no unique proposition which is the weakest natural one entailing the one expressed by A, then SA should be neither true nor false. I simplify matters by taking it to be false.

II

We start then with S5 modal logic, (8) taking the atomic wff to express propositions which need not be natural. Our first step is to consider S5*N*, obtained by adjoining the operator N. We have an extra formation rule telling us that if A is a wff so is NA. We have some extra axiom-schemas to be stated below (*N1* to *N4*), and an extra rule of inference (*REN*). N is an operator standing to $\Box$ as the metalinguistic predicate 'is natural' stands to 'is necessary'. So if, at a given world, A corresponds to a natural proposition, NA is true, at that world. Otherwise NA is false.

Our models will be Kripke models for S5, suitably augmented. Such a model is a triple $<W, R, \| \|>$, where W is the set of worlds, R the relation of accessibility, and $\| \|$ a function assigning to each wff a set of worlds in W, those at which the wff is true (in the model). We augment this to obtain a quadruple $<W, R, \| \|, N>$, which is a model for S5*N*. Here N is a function assigning to each world α in W a set N_α of sets of worlds, that is, a set of propositions. N_α is the set of all propositions which are natural (relative to α). The triple $<W, N, \| \|>$ would then be a Scott-Montague model. We make the usual requirements for Kripke models for S5, namely:

(I) R is transitive, reflexive and symmetrical.

(II) $\|A \wedge B\| = \|A\| \cap \|B\|$, for all wff A, B.

(7) Defects such as those inspiring the Relevant Logic research programme. See, for instance, A.R. Anderson and N.D. Belnap Jr. *Entailment, vol. 1* Princeton University Press, 1975, and Richard R. Routley et al. *Relevant Logics and Their rivals* Ridgeview 1982.

(8) As formulated by Brian F. Chellas, see *Modal Logic: An Introduction.*

(III) $\| \neg A \| = \overline{\| A \|}$ the complement of $\| A \|$ in W, for all wff A.
(IV) $\| \Box A \| = \{\alpha \in W$: For all $\beta \in W$, $\alpha R \beta \rightarrow (\beta \in \| A \|)\}$

In addition, intuitions about naturalness give us further constraints on models for S5N:

(V) $W \in N_\alpha$, for all $\alpha \in W$.
(VI) If $X \in N_\alpha$, then $\bar{X} \in N_\alpha$, for all $\alpha \in W$,
(VII) If X, $Y \in N_\alpha$, then $X \cap Y \in N_\alpha$, for all $\alpha \in W$.
(VIII) If $\alpha R \beta$ then $N_\alpha = N_\beta$, for all α, $\beta \in W$.
(IX) $\| NA \| = \{\alpha \in W$: $\| A \| \in N_\alpha\}$, for all wff A.

Of these requirements, (V) tells us that the trivially true proposition is natural. (VI) and (VII) tell us that the natural propositions are closed under negation and conjunction. (VIII) expresses the non-contingency of naturalness. (IX) expresses the connection between the naturalness operator on wff and the meta-linguistic predicate 'is natural' applicable to propositions.

Guided by the semantics, I adjoin the following axiom-schemas to S5:

$N1$: $\top$
$N2$: $NA \rightarrow \Box NA$
$N3$: $NA \rightarrow N(\neg A)$
$N4$: $N(A) \wedge N(B) \rightarrow N(A \wedge B)$

We have an additional rule of inference:

$$REN: \frac{A \leftrightarrow B}{NA \leftrightarrow NB}$$

The proof of soundness is straightforward. I leave it to the reader. Consistency is obtained from that for the Propositional Calculus by erasing $\Box$ and replacing NA by $\top$ for any wff A.

Suppose we ignore $\Box$, and consider the fragment of S5N consisting of those wff without $\Box$. How does the system then compare with standard modal logics, with N as the operator? $N3$ is incompatible with the analogue of Chellas' schema D, namely:

DN: $NA \rightarrow \neg N \neg A$

Moreover the converse of $N4$, namely Chellas' M, (9) is not a thesis. Indeed neither is the weaker:

(9) Brian F. CHELLAS, *Modal Logic: An Introduction* p. 235.

$$*MWN: A \wedge B \wedge N(A \wedge B) \rightarrow N(A).$$

(See Result Two.) The fragment, then, is rather far from a *normal* modal logic.

To obtain an operator more like a necessity operator we may define T (true and natural) by:

$$Df: TA \leftrightarrow A \wedge NA.$$

We could treat T as the fundamental operator, obtaining a system S5T, an extension of S5 with the axiom-schemas:

$$T1: T\top$$
$$T2: TA \rightarrow \Box(A \leftrightarrow TA)$$
$$T3: TA \wedge TB \rightarrow T(A \wedge B)$$
$$T4: TA \wedge TB \rightarrow T(A \rightarrow \mathrm{B})$$
$$T5: TA \wedge TB \rightarrow T(A \vee B).$$

There is the extra rule of inference:

$$RET: \frac{A \leftrightarrow B}{TA \leftrightarrow TB}$$

We could define N in terms of T by:

$$Df: NA \leftrightarrow TA \vee T\neg A$$

We have:

Result One: S5N and S5T are equivalent systems.

Proof: On the whole this is straightforward. I shall sketch the proofs of $T2$ in S5N, and of $N2$ and $N4$ in S5T.

$T2$: Notice that:

(1) $\vdash NA \rightarrow (A \leftrightarrow TA)$

S5 is monotonic, so:

(2) $\vdash \Box NA \rightarrow \Box(A \leftrightarrow TA)$

So, by $N2$:

(3) $\vdash NA \rightarrow \Box(A \leftrightarrow TA)$

Therefore:

(4) $\vdash TA \rightarrow \Box(A \leftrightarrow TA)$

N2: It is easy to check that:

(1) $\vdash NA \leftrightarrow N\neg A$

From *T2*:

(2) $\vdash TA \rightarrow \Box(A \leftrightarrow A \wedge NA)$

But: (3) $\vdash (A \leftrightarrow A \wedge NA) \leftrightarrow NA$

From (2) and (3):

(4) $\vdash TA \rightarrow \Box(NA)$

Similarly: (5) $\vdash T\neg A \rightarrow \Box(N\neg A)$

From (1) we have:

(6) $\vdash \Box N(\neg A) \leftrightarrow \Box NA$

From (5) and (6):

(7) $\vdash T(\neg A) \rightarrow \Box(NA)$

From (4) and (7):

(8) $\vdash NA \rightarrow \Box(NA)$

N4: the following are instances of *T3*, *T4*, *T4* and *T5*:

(1) $\vdash TA \wedge TB \rightarrow T(A \wedge B)$
(2) $\vdash TA \wedge T\neg B \rightarrow T(A \rightarrow \neg B)$
(3) $\vdash T\neg A \wedge TB \rightarrow T(B \rightarrow \neg A)$
(4) $\vdash T(\neg A) \wedge T(\neg B) \rightarrow T(\neg A \vee \neg B)$

Also, we have by RE*T*:

(5) $\vdash T(A \rightarrow \neg B) \leftrightarrow T(\neg A \vee \neg B)$

and:

(6) $\vdash T(B \rightarrow \neg A) \leftrightarrow T(\neg A \vee \neg B)$.

From (1) to (6):

(7) $\vdash (TA \vee T\neg A) \wedge (TB \vee T\neg B) \rightarrow T(A \wedge B) \vee T(\neg A \vee \neg B)$

That is:

(8) $\vdash NA \wedge NB \rightarrow N(A \wedge B)$.

Even though T is more like a necessity operator than N we still do not have the analogue of schema M, namely:

$*MT$: $T(A \wedge B) \rightarrow T(A) \wedge T(B)$

For this implies:

$*MWN$: $A \wedge B \wedge N(A \wedge B) \rightarrow N(A)$

And we have:

Result Two: Not every instance of $*MWN$ is a thesis of S5N
Proof: Since S5N is sound and consistent it suffices to find a countermodel.
Let W be the natural numbers, and let each be accessible to all the others.
For any n in W, let N_n be the set of all finite and co-finite subsets of W. (A co-finite set is one whose complement is finite). Let the atomic propositions be P_0, P_1 etc.
Define $\| \ \|$ by:

(i) $\| P_0 \| = \{m: m \text{ is prime}\}$
(ii) $\| P_n \| = \{m: n \text{ divides } m\}$

Then $\| \ \|$ is defined recursively using (II), (III), (IV), (IX). It is easy to check that (I) - (IX) hold. So we have a model. But:

$2 \notin \| P_2 \wedge P_0 \wedge N(P_2 \wedge P_0) \rightarrow N(P_2) \|$

So not every instance of *MWN is a thesis.

Finally we have completeness:

Result Three: S5N is complete.

Proof: We use a canonical model $< W^0, R^0, | \ |, N^0 >$ which we characterise as follows.
W^0 is the set of all maximal consistent sets of wff in S5N.
For any wff A, $| A | = \{\alpha \in W^0: A \in \alpha\}$. R^0 is defined,

Kripke-style, by:

$\alpha R^0 \beta$ iff $\{A: \Box A \in \alpha\} \subseteq \beta$.

N^0 is defined by:

$N^0_\alpha = \{ \mid A \mid : NA \in \alpha\}$.

To establish completeness it suffices to show this is indeed a model. ([10]) (I) - (IV) follow as in the proof of the completeness of S5. (V), (VI) and (VII) follow from the fact that all members of W^0 are closed under modus ponens and contain all instances of $N1$, $N3$ and $N4$ respectively. To show (VIII), suppose $\alpha R^0 \beta$ and $X \in N^0_\alpha$. We have to show that $X \in N^0_\beta$. Since $X \in N^0_\alpha$, $X = \mid B \mid$ for some wff B such that $NB \in \alpha$. Since α contains $NB \rightarrow \Box NB$, $\Box NB \in \alpha$ also.
So $\alpha \in \mid \Box NB \mid$. By (IV), $\beta \in \mid NB \mid$. Hence $NB \in \beta$. Therefore $\mid B \mid \in N^0_\beta$. So $X \in N^0_\beta$ as required.
It only remains to show (IX). $N^0_\alpha = \{ \mid A \mid : NA \in \alpha\}$. Therefore $\mid A \mid \in N^0_\alpha$ iff $\mid A \mid = \mid B \mid$ for some wff B such that $NB \in \alpha$. But $\mid A \mid = \mid B \mid$ iff $\vdash A \leftrightarrow B$. And α is closed under *REN*. So if $NB \in \alpha$, then $NA \in \alpha$. Therefore $\mid A \mid \in N^0_\alpha$ iff $NA \in \alpha$. Therefore $\mid NA \mid = \{\alpha \in W: \mid A \mid \in N^0_\alpha\}$ as required.

III

We are interested in *the* natural strengthening, that is the weakest natural strengthening of a given proposition. But first we shall consider more general strengthenings. Suppose S is a strenthening operator. Since SA is a strenthening of A, we have

(X) $\| SA \| \subseteq \| A \|$.

We also require that if Y is any *natural* proposition which entails the proposition $\| A \|$ then Y entails $\| SA \|$. To say that Y entails $\| A \|$ relative

([10]) My procedure differs slightly from Chellas'. Chellas uses a procedure which guarantees we obtain a model. But we have to check that, in the model $\| A \| = \{\alpha \in W: A \in \alpha\}$. I define the "model" in such a way that this holds but have to check it is a model.

to world α is to say that $Y_\alpha \subseteq \|A\|$ where $Y_\alpha = \{\beta \in Y: \alpha R \beta\}$. Hence it is convenient to define the interior of X, Int(X) by:

Int$X$ = $\{\alpha \in X$: There is some $Y \in N_\alpha$ for which $\alpha \in Y$ and $Y_\alpha \subseteq X\}$.

Then the requirement that any natural proposition which is stronger than $\|A\|$ entails $\|SA\|$ can be expressed thus:

(XI) Int $\|A\| \subseteq \|SA\|$

We also require that the strengthening of wff is based on a strengthening of propositions (i.e. sets of worlds). So we require

(XII) If $\|A\| = \|B\|$ then $\|SA\| = \|SB\|$.

I do not yet require that $\|SA\|$ be itself a natural proposition. So we may consider a formal system S5*NS* obtained by adjoining extra wff using the rule that if A is a wff so is SA. A model for S5*NS* is then a model for S5*N* with the extra rules (X), (XI) and (XII). We axiomatise S5*NS* by means of the extra axioms:

S1: $SA \rightarrow A$
S2: $[A \wedge B \wedge NB \wedge \Box(B \rightarrow A)] \rightarrow SA$

We have the extra rule of inference:

$$RES: \quad \frac{A \leftrightarrow B}{SA \leftrightarrow SB}$$

I leave soundness to the reader. Consistency is obtained from that for S5*N* by considering an erasure transform in which S is dropped. We have:

Result Four: *S5NS* is complete.

Proof: We consider canonical models similar to those considered in the proof of the completeness of S5*N*. Because every world is a set of wff closed under modus ponens and containing all instances of S1, (X) is obtained immediately. Again (XII) follows from the requirement that the worlds be closed under *RES*. It only remains to establish (XI). Suppose $\alpha \in$ Int $|A|$. We have to show that $\alpha \in |SA|$. Since $\alpha \in$ Int $|A|$ we have: (i) $\alpha \in |A|$. In addition, there is some $Y \in N^0_\alpha$ such that: (ii) $\alpha \in Y$ and $Y_\alpha \subseteq |A|$. Because $Y \in N^0_\alpha$, for some wff B: (iii) $Y = |B|$ and (iv)

$\alpha \in | NB |$. From (ii) and (iii): (v) $\alpha \in | B |$ and: (vi) $| B |_\alpha \subseteq | A |$. From (i), (iv), (v) and (vi) we have:

(vii) $A \in \alpha$, $NB \in \alpha$, $B \in \alpha$, and $\Box (B \rightarrow A) \in \alpha$.

Since α is closed under conjunction, we have from (vii):

(viii) $[A \wedge B \wedge NB \wedge \Box (B \rightarrow A)] \in \alpha$. Since every instance of *S2* is in α, and α is closed under modus ponens we have from (viii): (ix) $SA \in \alpha$. Therefore: (x) $\alpha \in | SA |$. We have shown that if $\alpha \in \text{Int } | A |$ then $\alpha \in | SA |$, and hence established (XI).

So far we have not assumed that $\| SA \|$, a strengthening of $\| A \|$, is natural. Let us now restrict our attention to a system which is such that for any wff A, $\| SA \|$ is assumed to be natural. That is, our choice of atomic wff is restricted to ensure that for no wff is $\| SA \|$ not natural. Then we have a further semantic rule:

(XIII) If $\alpha \in \| SA \|$ then $\| SA \| \in N_\alpha$.

Let us call the system with this additional rule $S5NS^+$. First we notice that this requirement fixes $\| SA \|$, that is, there is a unique natural strengthening. We have:

Result Five: In the semantics for $S5NS^+$, $\| SA \| = \text{Int } \| A \|$

Proof: By (XI), $\text{Int } \| A \| \subseteq \| SA \|$. We have to show the converse. Suppose $\alpha \in \| A \|$. Then, by (XIII), $\| SA \| \in N_\alpha$. Now $SA \rightarrow A$ is an axiom (S1). So $\| SA \| \subseteq \| A \|$. A fortiori $\| SA \|_\alpha \subseteq \| A \|$. Hence $\| SA \|$ satisfies all the requirements of Y in the definition of $\text{Int } \| A \|$. This establishes that $\alpha \in \text{Int } \| A \|$ as required.

The additional axiom for $S5NS^+$ is:

S3: $SA \rightarrow NSA$.

Once again soundness is straightforward. Consistency is obtained from that for the propositional calculus by erasing $\Box$ and S and replacing NA by $\top$ for any wff A. We have:

Result Six: $S5NS^+$ is complete.

Proof: we use canonical models as before.
$SA \rightarrow NSA$ is now an axiom. So, if $SA \in \alpha$ then $NSA \in \alpha$, for any

maximally consistent set of wff α. Hence if $\alpha \in |SA|$ then $|SA| \in N_{\alpha}$, satisfying (XIII). So the maximally consistent sets do provide a model, which establishes completeness.

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