TRANSFORMATIONAL SEMANTICS FOR FIRST ORDER LOGIC

by
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1. *Introduction*

The so-called *possible world semantics* is now pervasive among almost all branches of philosophical logic, owing to the pioneering work of S. Kripke. Now in spite of its great interest, from a philosophical and technical point of view, it seems to be an inadequate tool when treating questions as the cross-world identity of individuals. Now, it is clear what sorts of criteria are usually adopted in everyday life: they turn on the continuity of individuals with respect to possible transformations.

In accordance with these considerations, in this paper we propose a new approach to first order modal semantics. Roughly speaking, if α is a first order formula and d an individual, then $\Diamond\alpha$ is true for d provided it is possible to transform d so that α holds. So, in general, we identify the individual d with the individual d′ provided there exists a transformation f such that f(d) = d′. Such a type of modality is a *contralogical modality* in the sense of Hintikka [5].

2. *Transformational modal structures*

In the sequel $\mathfrak{L}$ denotes a first order language, $\mathfrak{L}^* = \mathfrak{L} \cup \{\Box\}$ its correspondent modal extension, F($\mathfrak{L}$) the set of formulas of $\mathfrak{L}$ and F($\mathfrak{L}^*$) the set of formulas of $\mathfrak{L}^*$. Let W be a class whose elements are called *worlds*, $\mathfrak{M} = (M_w)_{w \in W}$ a family of classical interpretations $M_w = (D_w, I_w)$ of $\mathfrak{L}$ and, for every w, w′ ∈ W, let H(w, w′) be a set of maps from D_w into $D_{w'}$. We assume that the identity map $i: D_w \to D_w$ belongs to H(w,w) for every w ∈ W. Then the pair S = ($\mathfrak{M}$,H) is called *transformational modal structure* (or merely *trasformational structure*) for $\mathfrak{L}^*$ and, if w ∈ W, ($\mathfrak{M}$,H,w) is called *transformational modal model* (in brief, *transformational model*).

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We write $\alpha(x_1,...,x_n)$ to emphasize that the free or bound variables of the formula α are among $x_1,...,x_n$, and we write $t(x_1,...,x_n)$ to emphasize that t is a term whose variables are among $x_1,...,x_n$. If f, R and c are symbols for functions, relations and individual constants, respectively, then we set $f_w = I_w(f)$, $R_w = I_w(R)$ and $c_w = I_w(c)$. If $t(x_1,...,x_n)$ is a term, $w \in W$ and

$a_1,...,a_n \in D_w$ then $t^w[a_1,...,a_n]$ is defined, as usual, by
$t^w(a_1,...,a_n) = a_j$ if t is x_j
$t^w(a_1,...,a_n) = c_w$ if t is the constant c
$t^W(a_1,...,a_n) = f_w(t_1{}^w(a_1,...,a_n),..., t_q{}^w(a_1,...,a_n))$ if t is $f(t_1,...,t_q)$.

If $\alpha(x_1,...,x_n) \in F(\mathfrak{L}^*)$, $w \in W$ and $a_1,...,a_n \in D_w$, the relation $S,w \models \alpha$ $[a_1,...,a_n]$ is defined by recursion on the complexity of α by setting

$S,w \models t_1 = t_2\ [a_1,...,a_n]$ if $t_1{}^w(a_1,...,a_n) = t_2{}^w(a_1,...,a_n)$
$S,w \models R(t_1,...,t_q)\ [a_1,...,a_n]$ if $(t_1{}^w(a_1,...,a_n),...,t_q{}^w(a_1,...,a_n)) \in R_w$
$S,w \models \alpha \wedge \beta\ [a_1,...,a_n]$ if $S,w \models \alpha\ [a_1,...,a_n]$ and $S,w \models \beta\ [a_1,...,a_n]$
$S,w \models \neg\alpha\ [a_1,...,a_n]$ if $S,w \not\models \alpha\ [a_1,...,a_n]$
$S,w \models \exists x_h \alpha\ [a_1,...,a_n]$ if $S,w \models \alpha\ [a_1,...,a,...,a_n]$ for a suitable $a \in D_w$
$S,w \models \Box\alpha\ [a_1,...,a_n]$ if $S,w' \models \alpha\ [f(a_1),...,f(a_n)]$ for every $w' \in W$ and $f \in H(w,w')$.

Moreover, we set

$S,w \models \alpha$ (α is *true* in w) if $S,w \models \alpha\ [a_1,...,a_n]$ for every $a_1,...,a_n \in D_w$
$S \models \alpha$ (α is *valid* in S) if $S,w \models \alpha$ for every $w \in W$.
$\models \alpha$ (α is *valid*) if $S \models \alpha$ for every transformational structure S.

If Σ is a class of transformational structures, we set $\Sigma \models \alpha$, *α is Σ-valid*, provided that $S \models \alpha$ for every $S \in \Sigma$. We call *Σ-logic* the logic corresponding to the Σ-validity, a Σ-logic is *axiomatizable* if the set of Σ-valid formulas is recursively enumerable [8].

The proposed semantics gives an interesting (in my opinion) meaning to the modalities and the iterated modalities. Indeed, $\Box\alpha$ (respectively, $\Box^n\alpha$) holds for $a_1,...,a_n$ if α remains true however one acts on $a_1,...,a_n$ by a transformation (by n successive transformations). Likewise $\Diamond\alpha$ (respectively, $\Diamond^n\alpha$) holds for the individuals $a_1,...,a_n$ if it is possible to realize the property α via a suitable transformation (via n suitable transformations) of $a_1,...,a_n$. Besides, we identify an individual d of D_w with an individual d′ of $D_{w'}$ provided that $d' = f(d)$ for a suitable $f \in H(w,w')$.

3. *Validity and invalidity of some formulas*

It is immediate to verify that

(3.1) $\models \Box(\alpha \rightarrow \beta) \rightarrow (\Box\alpha \rightarrow \Box\beta)$

(3.2) $\models \Box\alpha \rightarrow \alpha$

(3.3) $\models x=y \rightarrow \Box(x=y)$

(3.4) $\models \Box(\forall x_h \alpha) \rightarrow \forall x_h \Box\alpha$ (converse of Barcan formula)

(3.5) $\models \Diamond(\forall x_h \alpha) \rightarrow \forall x_h \Diamond\alpha$

(3.6) $S \models \alpha$ implies $S \models \Box\alpha$ (necessitation rule)

(3.7) $S,w \models \beta \leftrightarrow \beta'$ implies $S,w \models \alpha \leftrightarrow \alpha'$ (substitution of equivalents rule) where α' is obtained from α by substituting β' to some occurrences of β.

If S is a category, i.e., from $f \in H(w,w')$ and $g \in H(w',w'')$ it follows that $gf \in H(w,w'')$, then

(3.8) $S \models \Box\alpha \rightarrow \Box\Box\alpha$ (the characteristic formula of S4).

Finally if, for every $w,w' \in W$ and $f \in H(w,w')$, f is injective, surjective or invertible, then

(3.9) $S \models x \neq y \rightarrow \Box(x \neq y)$

(3.10) $S \models \forall x_h \Box\alpha \rightarrow \Box(\forall x_h \alpha)$ (Barcan formula)

(3.11) $S \models \Diamond\alpha \rightarrow \Box(\Diamond\alpha)$ (the characteristic formula of S5), respectively.

Note that, in spite of the validity of the formula $\forall x \forall y((x=y) \rightarrow \Box(x=y))$, if in $\mathfrak{L}$ there are constants, the semantics we have proposed determines a modal logic with contingent identity. Indeed, the formula $\forall x \forall y((x=y) \rightarrow \Box(x=y))$ expresses only a linguistic convention, i.e. that if actually x and y denote a unique element, then they continue to denote its transformed afterwards. On the other hand, if c is a constant of $\mathfrak{L}$, then the formula $\forall y((c=y) \rightarrow \Box(c=y))$ is not valid. Indeed let $S=(\mathfrak{M}, H)$ be a transformational structure with two worlds w, w' such that $f(d) \neq d$ where $d=I_w(c)$ and f is a suitable element of $H(w,w')$. Then $S,w \models (c=y)$ [d] while $S,w \not\models \Box(c=y)$ [d]. Likewise, one proves that if c and c' are two constants, then $(c=c') \rightarrow \Box(c=c')$ is not valid. It follows also that *the universal specification is not valid in transformational logic and therefore this logic is not an extension of the classical first order logic.*

Indeed, if we assume the validity of the universal specification $\forall x\alpha(x) \rightarrow \alpha(c)$, then from $\forall x\forall y((x=y) \rightarrow \Box(x=y))$ should be possible to derive the two formulas $(c=c') \rightarrow \Box(c=c')$ and $\forall y(c=y) \rightarrow \Box(c=y)$. This happens on account of the different meanings of the variables and the constants in the proposed semantics. The interpretation of the variables is connected with the possible transformations while the interpretation of the constants is connected with the interpretation of the language. The fact that formula $\forall y\exists x\Box(x=y)$ is valid while $\exists x\Box(x=c)$ is not valid, furnishes another example of this difference of meaning. This makes our modal logics very different from the other modal logics and this creates some difficulties to find suitable axiomatizations.

4. *Collapsing transformational structures*

Now we will examine the case in wich the formulas holding in a transformational structure collapses in a non modal system. Recall that a system T of modal formulas *collapses* if $\alpha \leftrightarrow \Box\alpha$ belongs to T for every formula α of $\mathfrak{L}^*$; we says that a transformational structure S *collapses* provided that $\{\alpha \in F(\mathfrak{L}^*)/S \models \alpha\}$ collapses.

PROPOSITION 4.1 A transformational structure S collapses if and only if, for every $w,w' \in W$ and $f \in H(w,w')$, f is an elementary embedding of M_w into $M_{w'}$.

Proof. Let $\alpha(x_1,...,x_n)$ be any formula of $\mathfrak{L}$, $w,w' \in W$ and $f \in H(w,w')$ and assume that S collapses, then $S,w \models \alpha \leftrightarrow \Box\alpha$ and $S,w \models \neg\alpha \leftrightarrow \Box(\neg\alpha)$ for every $w \in W$. Hence, from $M_w \models \alpha\ [a_1,...,a_n]$, i.e. from $M,w \models \alpha\ [a_1,...,a_n]$, it follows that $S,w \models \Box\alpha\ [a_1,...,a_n]$. This implies $S,w' \models \alpha\ [f(a_1),...,f(a_n)]$, and therefore $M_{w'} \models \alpha\ [f(a_1),...,f(a_n)]$. Now, assume that $M_w \models \alpha\ [f(a_1),...,f(a_n)]$ and, for contradiction, that $M_w \models \neg\alpha\ [a_1,...,a_n]$. Since $S,w \models \neg\alpha\ [a_1,...,a_n]$, by hypothesis $S,w \models \Box\neg\alpha\ [a_1,...,a_n]$ and therefore we have that $S,w \models \neg\alpha\ [f(a_1),...,f(a_n)]$, an absurdity. This proves that $M_w \models \alpha\ [a_1,...,a_n]$.
Thus we have

$$M_w \models \alpha\ [a_1,...,a_n] \text{ if and only if } M_{w'} \models \alpha\ [f(a_1),...,f(a_n)]$$

i.e. f is an elementary embedding of M_w into $M_{w'}$.

Conversely, assume that every $f \in H(w,w')$ is an elementary embedding of M_w into $M_{w'}$, and let α be any formula of $\mathfrak{L}^*$; to prove

(4.1) $$S,w \models \alpha \leftrightarrow \Box\alpha \ [a_1,...,a_n]$$

we proceeds by induction on the number n of occurrences of $\Box$ in α. If $n=0$ then (4.1) follows from the hypothesis and the meaning of $\Box$. If $n \neq 0$ then there exists a subformula $\Box\beta$ of α with $\beta \in F(\mathfrak{L})$. Since $S,w \models \beta \leftrightarrow \Box\beta \ [a_1,...,a_n]$, by substitution of equivalents rule it is also $S,w \models \alpha \leftrightarrow \alpha' \ [a_1,...,a_n]$ where α' is obtained by substituting in α the formula $\Box\beta$ by β. Thus, by inductive hypothesis, $S,w \models \alpha' \leftrightarrow \Box\alpha' \ [a_1,...,a_n]$ and therefore $S,w \models \alpha \leftrightarrow \Box\alpha \ [a_1,...,a_n]$.

5. *Logic of the extensions*

An interesting example of Σ-logic is the logic of the extensions in wich *the possibility* means the possibility of extending a given model. Since this logic is extensively examined in [3] and [4], we confine ourselves to sketch it. Let W be a class, $\leq$ a reflexive and transitive relation in W, and $\mathfrak{M}=(M_w)_{w \in W}$ a family of models of $\mathfrak{L}$ such that if $w \leq w'$ then $M_w \subseteq M_{w'}$, i.e. M_w is a submodel of $M_{w'}$. Set

$$H(w,w') = \begin{cases} \{i\} & \text{if } w \leq w' \text{ and } i: M_w \to M_{w'} \text{ is the identical embedding} \\ \emptyset & \text{otherwise.} \end{cases}$$

then $(\mathfrak{M},H)$, in brief $\mathfrak{M}$, is a transformational modal structure. We call *E-structure* and *E-models* the transformational structures and models obtained in this manner and we call *E-logic*, or *logic of the extensions* the logic corresponding to the class of the E-structures. An interesting example of E-structure is associated to every theory T of $\mathfrak{L}$ by considering the class M(T) of the models of T ordered by $\subseteq$, (the submodel relation). In this case $\Diamond\alpha$ means the possibility of extending the given model so that α holds.

In E-logic it is possible to express several important concepts of classical model theory, such as existential completeness and being infinitely generic, that are not expressible in classical logic. We call *model complete* an E-model such that $w \leq w'$ implies that $M_{w'}$ is an elementary extension of

M_w, we call *existentially complete* an E-model $(\mathfrak{M},w)$ such that $w \leq w'$ implies that M_w is existentially complete in $M_{w'}$. Obviously in M(T) this concepts coincide with the corresponding classical ones. From Proposition 4.1 it follows that an E-structure collapses if and only if it is model complete. In [4] we prove that an E-model is existentially complete if and only if it verifies $\Diamond\alpha \leftrightarrow \alpha$ for every existential formula α. In other words the set $\{\Diamond\alpha \leftrightarrow \alpha / \alpha$ is existential$\}$ represents in $\mathfrak{L}^*$ the theory of the existentially complete models. If $a_1,...,a_n$ are elements of a model M_w of $\mathfrak{L}$ and α is a formula of $\mathfrak{L}$, then the relation $\mathfrak{M},w \models \alpha\ [a_1,...,a_n]$, w *infinitely forces* α in $a_1,...,a_n$, is defined inductively in the same way as $\models$ except for what concerns the negation for wich we set $\mathfrak{M},w \models \neg\alpha\ [a_1,...,a_n]$ if and only if $\mathfrak{M},w' \not\models \alpha\ [a_1,...,a_n]$ for every $w' \geq w$.
The E-model $(\mathfrak{M},w)$ is *infinitely generic* if, for every formula α either $(\mathfrak{M},w)$ forces α or $\neg\alpha$. The above two notions coincide with the classical ones in the E-models M(T).

In order to express the infinite forcing, we define a map $t: F(\mathfrak{L}) \rightarrow F(\mathfrak{L}^*)$ by setting $t(\alpha) = \alpha$ if α is atomic and

$$t(\alpha \wedge \beta) = t(\alpha) \wedge t(\beta), \quad t(\alpha \vee \beta) = t(\alpha) \vee t(\beta), \quad t(\exists x_i\, \alpha) = \exists x_i t(\alpha),$$
$$t(\neg\alpha) = \Box(\neg t(\alpha))$$

otherwise. In [4] we prove that a transformational model $(\mathfrak{M},w)$ infinitely forces α if and only if $\mathfrak{M},w \models t(\alpha)$, moreover $(\mathfrak{M},w)$ is infinitely generic if and only if $\mathfrak{M},w \models \alpha \leftrightarrow t(\alpha)$ for every formula α of $\mathfrak{L}$. In other words, the set $\{t(\alpha) \leftrightarrow \alpha / \alpha \in F(\mathfrak{L})\}$ is the theory in $\mathfrak{L}^*$ of the infinitely generic E-models.

In [4] we give some applications of the fact that in E-logic it is possible to express the above quoted concepts of model theory.

With regard to axiomatizability question, in [4] we show that a suitable system of axioms for the E-logic is obtained by adding to first order modal logic QS4 the *rigidity axiom schema* $\alpha \rightarrow \Box\alpha$, where α denotes a *basic* formula, i.e. an atomic or the negation of an atomic formula. This means, in particular, that E-logic is an extension of classical logic.

6. *Logic of invariance*

Let $M = (D,I)$ be a model of $\mathfrak{L}$ and G a group of transformations of D, then it is possible to define a modal structure $S = (\mathfrak{M},H)$ by assuming that $\mathfrak{M}$ coincides with the unique model M and H(M,M) = G. We denote by (M,G) both the transformational structure and the unique transformational model associated to M and G and we write $M,G \vDash \alpha\ [a_1,...,a_n]$ instead of $S,w \vDash \alpha\ [a_1,...,a_n]$. In I-logic it is possible to express the *invariance* of a property represented by the formula $\alpha(x_1,...,x_n)$ via the formula $\forall x_1...\forall x_n(\alpha \leftrightarrow \Box\alpha)$. Indeed $M,G \vDash \alpha \leftrightarrow \Box\alpha\ [a_1,...a_n]$ means that $M \vDash \alpha\ [a_1,...a_n]$ if and only if $M \vDash \alpha\ [f(a_1),...f(a_n)]$ for every $f \in G$. This suggests, for example, the possibility of applying a suitable multimodal version of I-logic to Klein's Erlangen program. It suffices to extend the language $\mathfrak{L}$ of elementary geometry to a multimodal language $\mathfrak{L}^* = \mathfrak{L} \cup \{\Box_m, \Box_a, \Box_p\}$. In this case a modal structure should be of type (M, G_m, G_a, G_p) and the interpretation of the formulas of $\mathfrak{L}^*$ is obtained by an obvious extension of the definitions of Section 2. In particular, we are interested in the case in wich M is a model of the euclidean geometry and G_m, G_a and G_p are the groups of the isometries, the affinities and the projectivities, respectively. In such a logic it is possible to express that a property, represented by the formula $\alpha(x_1,...,w_n)$, is metric, affine or projective, by the formulas

$$\forall x_1...\forall x_n(\alpha \leftrightarrow \Box_m\alpha),\ \forall x_1...\forall x_n(\alpha \leftrightarrow \Box_a\alpha),\ \forall x_1...\forall x_n(\alpha \leftrightarrow \Box_p\alpha)$$

respectively.
Moreover, if ϕ is a formula and $r_1(x_1,...,x_{n(1)}),...,r_p(x_1,...,x_{n(p)})$ are the predicate symbols occurring in ϕ, then the formula

$$\phi \wedge [\forall x_1...\forall x_{n(1)}(r_1 \leftrightarrow \Box_m r_1)] \wedge ... \wedge [\forall x_1...\forall x_{n(p)}(r_p \leftrightarrow \Box_m r_p)]$$

expresses that ϕ is a theorem of metric geometry. Then we can built up a unique multimodal language able to express the metric, affine, projective geometry at the same time.

Another possible application of I-logic is the formalization of those physical theories that are based on the invariance concept.

I-logic has an interesting behaviour with respect to the *de dicto* modalities. Recall that a *de dicto* formula is a formula with no occurrence of subformulas of type $\Box\beta$ in wich β has free variables. A formula is *de re* if is not de dicto. We say that in a modal formula α the modalities are *erasable* if the formula $\alpha \leftrightarrow c(\alpha)$ is I-valid, where $c(\alpha)$ is obtained from α by deleting every occurrence of $\Box$. The following proposition shows that in every de dicto formula the modalities are erasable. Thus, in a sense, de re modalities are the unique modalities of I-logic and the modalities of I-logic are *contralogical* modalities, essentially.

PROPOSITION 6.1 In I-logic the modalities of every de dicto formula are erasable, that is.

(6.1) $\mathrm{I} \vDash \alpha \leftrightarrow c(\alpha)$.

Proof. We proceed by induction on the number n of occurrences of $\Box$ in α. If $n=0$ then (6.1) is obvious. Assume that $n \neq 0$, let $\Box\beta$ a subformula of α where β is a closed formula of $\mathfrak{L}$ and let α^* be obtained by substituting in α the formula $\Box\beta$ with β. By inductive hypothesis we have that $\mathrm{I} \vDash \alpha^* \leftrightarrow c(\alpha^*)$, besides, since β is closed, from the interpretation of $\Box$ we have that $\mathrm{I} \vDash \beta \leftrightarrow \Box\beta$. Thus, by substitution of equivalents rule, $\mathrm{I} \vDash \alpha \leftrightarrow \alpha^*$ and therefore $\mathrm{I} \vDash \alpha \leftrightarrow c(\alpha^*)$. Since $c(\alpha)=c(\alpha^*)$, (6.1) is proved.

Now we will compare the I-models with the Kripke models. To this aim, we assume that in the language $\mathfrak{L}$ there are only relational constants and we associate to every I-structure $S=(M,G)$, with $M=(D,I)$, a Kripke S5-structure K(S) as follows. The class of worlds of K(S) coincides with the group G and to every $f \in G$ a model $M_f=(D,I_f)$ is associated whose domain is D and I_f is defined by

(6.2) $I_f(R) = \{(a_1,...,a_n)/(f(a_1),...,f(a_n)) \in I(R)\}$.

Obviously, if $i: D \to D$ is the identity map then M_i coincides with M and every model M_f is isomorphic to M_g via the map $g^{-1}f$.

PROPOSITION 6.2 Assume that in $\mathfrak{L}$ there are only relational constants, then for every formula $\alpha(x_1,...,x_n)$ and $a_1,...,a_n \in D$

(6.3) $G,M \vDash \alpha\ [a_1,...,a_n]$ if and only if $K(S), i \vDash \alpha\ [a_1,...,a_n]$.

Proof. First we prove that

(6.4) $K(S),i \vDash \alpha\ [f(a_1),...,f(a_n)]$ if and only if $K(S),f \vDash \alpha\ [a_1,...,a_n]$

by induction on the complexity of α. If α is atomic, i.e. of type $R(x_1,...,x_m)$, with $m \leq n$, then

$K(S),i \models R(x_1,...,x_m)\ [f(a_1),...,f(a_n)] \Leftrightarrow (f(a_1),..., f(a_m)) \in I(R) \Leftrightarrow (a_1,...,a_m) \in I_f(R) \Leftrightarrow K(S),f \models R(x_1,...,x_n)\ [a_1,...,a_n]$.

If $\alpha = \Box\beta$, then

$K(S),i \models \Box\beta\ [f(a_1),...,f(a_n)] \Leftrightarrow K(S),g \models \beta\ [f(a_1),...,f(a_n)]$ for every $g \in G \Leftrightarrow K(S),i \models \beta\ [gf(a_1),...,gf(a_n)]$ for every $g \in G \Leftrightarrow K(S),gf \models \beta\ [a_1,...,a_n]$ for every $g \in G \Leftrightarrow K(S),f \models \Box\beta\ [a_1,...,a_n]$.

If $\alpha = \exists x_i\beta$, for example $\alpha = \exists x_1\beta$ then

$K(S),i \models \exists x_1\beta\ [f(a_1),...,f(a_n)] \Leftrightarrow K(S),i \models \beta\ [d, f(a_2),...,f(a_n)]$ for a suitable $d \in D \Leftrightarrow K(S),f \models \beta\ [f^{-1}(d),a_2,...,a_n]$ for a suitable $d \in D \Leftrightarrow K(S),f \models \exists x_i\beta$ $[a_1,...,a_n]$.

The induction steps in the cases $\alpha = \beta \wedge \beta'$ and $\alpha = \neg\beta$ are obvious.

Now, it is possible to prove (6.3) by induction on the complexity of α. If α is atomic (6.3) is immediate, if $\alpha = \Box\beta$ then

$G,M \models \Box\beta\ [a_1,...,a_n] \Leftrightarrow G,M \models \beta\ [f(a_1),...,f(a_n)]$ for every $f \in G \Leftrightarrow K(S),i \models \beta\ [f(a_1),...,f(a_n)]$ for every $f \in G \Leftrightarrow K(S),f \models \beta\ [a_1,...,a_n]$ for every $f \in G \Leftrightarrow K(S),i \models \Box\beta\ [a_1,...,a_n]$.

Since the remaining steps of induction are obvious, (6.3) is completely proved.

From Proposition 6.2 it follows that every S5-valid formula is also I-valid and that I-logic is an extension of S5-logic. In particular, every example of a classically valid formula is I-valid while in Section 3 we have observed that there are examples of universal specification schema invalid in transformational logic. This is a consequence of the absence of constants and functions in $\mathfrak{L}$ rather than a characteristic of I-logic. Indeed it is immediate that if in $\mathfrak{L}$ there is a constant c, then the examples at the end of Section 3 are not I-valid.

We conclude by showing that I-logic is axiomatizable.

PROPOSITION 6.3 I-logic is axiomatizable.

Proof. We proceed by showing that the set V of the I-valid formulas is one-one reducible (see [8]) to a suitable axiomatizable theory of a two sorted first order logic (for example, see [7]). To this aim, let $\mathfrak{L}^s$ be the two sorted language extending $\mathfrak{L}$ whose variables are f_1, f_2,... and x_1, x_2... and with an operation symbol h. Every I-model (G,M) determines a model μ(G,M) for $\mathfrak{L}^s$ in an natural way. Namely, we extend the interpretation in $\mathfrak{L}$ given by M by assuming that the variables f_i and x_i range on G and the domain D of M, respectively. Moreover the interpretation of h is the operation $h^{\#}$ defined by $h^{\#}(f,x)=f(x)$ for every $f \in G$. It is immediate that μ(G,M) verifies the following system S of axioms.

A1 $\exists f_1 \forall x_1 (h(f_1,x_1)=x_1)$ (existence of the identity map)
A2 $\forall f_1 \forall f_2 \exists f_3 \forall x_1(h(f_1,h(f_2,x_1))=h(f_3,x_1)$ (existence of the product)
A3 $\forall f_1 \exists f_2 \forall x_1(h(f_2,h(f_1,x_1))=x_1)$ (existence of the inverse).

Conversely, let $\mathfrak{M}$ be an interpretation of $\mathfrak{L}^s$ verifying the theory $T=\{A_1, A_2, A_3\}$, then $\mathfrak{M}$ individuates a model M=(D,I) of $\mathfrak{L}$, a set Z and a map $h^{\#}: ZxD \rightarrow D$. For every $z \in Z$, let $g_z: D \rightarrow D$ be defined by the equality $g_z(x)=h^{\#}(z,x)$ and set $G=\{g_z/z \in Z\}$. It is matter of routine to prove that

- G is a group of transformations of D and therefore $\sigma(\mathfrak{M})=(G,M)$ is an I-model
- $\mu(\sigma(\mathfrak{M}))$ is isomorphic to $\mathfrak{M}$ for every model $\mathfrak{M}$ of T
- $\sigma(\mu(G,M))$ is isomorphic to (G,M) for every I-model (G,M).

Then it is possible to identify the I-models with the models of T. Now, define the function $t: F(\mathfrak{L}^*) \rightarrow F(\mathfrak{L}^s)$ by setting $t(\alpha)=\alpha$ for every atomic formula α and $t(\alpha \wedge \beta)=t(\alpha) \wedge t(\beta)$, $t(\neg \alpha)= \neg (t(\alpha)$, $t(\exists x_i \alpha)=\exists x_i t(\alpha)$, $t(\Box \alpha)=\forall f_1 [t(\alpha)(h(f_1,x_1),...,h(f_1,x_n))]$ where $t(\alpha)(h(f_1,x_1),...,h(f_1,x_n))$ is the formula obtained from $t(\alpha)$ by substituting every free occurrence of $x_1,...,x_n$ by $h(f_1,x_1),...,h(f_1,x_n)$. It is easy to prove that, for every $\alpha \in F(\mathfrak{L}^*)$ $G,M \models \alpha$ if and only if $\sigma(G,M) \models t(\alpha)$ and therefore that $\alpha \in V$ if and only if $T \vdash t(\alpha)$. Thus, since t is a recursive one-one map, V is one-one reducible to the recursively enumerable subset $\{\alpha \in \mathfrak{L}^s/T \vdash \alpha\}$ and this proves that V is recursively enumerable.

Recall that, since V is recursively enumerable, there is a decidable system S of axioms and a rule of inference able to generate V. Indeed, let α_1, α_2,... be an effective enumeration of the elements of V, set $\beta_1=\alpha_1$, $\beta_{n+1}=\beta_n \wedge \alpha_{n+1}$ and $S=\{\beta_n/n \in N\}$. Since the lengths of the sequence $(\beta_n)_{n \in N}$ are increasing, S is decidable and V is generated by the system of axioms S and the rule $\alpha \wedge \alpha'/\alpha'$. Of course, this is not completely satisfactory. It is an open question to find a more significant system of axioms and inference rules for I-logic.

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