RICH RELEVANT LOGICS BASED ON A SIMPLE TYPE OF SEMANTICS

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1. *Aim of this paper*

The logics dealt with in the present paper differ from the well-known Anderson and Belnap systems in two respects. They were arrived at by starting from a simple, clear and intuitive *semantic* idea and they are in a specific sense *richer* than the Anderson and Belnap systems. Both features are interesting. From the first one learns that at least some relevant logics may be characterized by semantic tools which are a lot simpler than the ones devised by Routley and Meyer. Moreover, an interpretation, in the non-technical sense, of the logics is readily available, and the delineation of the contexts in which the logics are adequate is greatly simplified. The second feature is interesting for both technical and philosophical reasons. From a technical point of view, it offers a partial, yet systematic answer to questions suggested on p. 348 of [1] – the semantic systems have indeed quite straightforward connections to Fitch-style systems. From a philosophical point of view, it is quite helpful to counter the objections (and puzzles) raised by people who consider relevant logics too strong in some senses and too weak in others. Apart from all this, the systems discussed below are decidable. For all these reasons they also happen to be useful from a pedagogical point of view.

There is a rich plurality of simple and intuitively clear semantic systems which all lead to relevant logics, not all of them different. It is by no means my intention to give a survey of this plurality. I shall stick to some illustrations of the simplest type, viz. all concerning logics "without" nested implications, i.e. in which nested implications do not have a specific meaning. In such systems it is impossible to derive $(p \to q) \to q$ from p, to derive $p \to q$ from $p \to (p \to q)$, or to derive $(q \to r) \to (p \to r)$ from $p \to q$, although it is possible to derive r from the premises $p \to (q \to r)$, p and q. A small family of this type of systems was first devised by Newton C.A. da Costa and Ayda I. Arruda in [2]; see also the later [3]; do not study these systems from [4], in which they are deformed to (not uninteresting) systems lacking any (object-language) implication at all.

The enrichment I want to illustrate is quite clear and obvious for such simple logics already. Incidentally, the introduction of the enrichment in logics "with" nested implications is a matter of rather straightforward generalization, at least if one sticks to **E**, **R**, and nephews, for interesting alternative roads are easily found. My restricting illustrations to the poorer systems is merely a matter of keeping things as simple as possible with respect to the point to be made; it has nothing to do with preferences.

Since we are at it, I wish to repeat what I wrote several times elsewhere, viz. that I do not believe in some unique and omnipotent logic. The logics of this paper have all the nice features mentioned in the first paragraph, but this is not a reason to consider them better or more adequate in general than other logics or even than other relevant logics. Monologism is as indefensible as any other dogmatism.

This paper is structured as follows. I first offer a brief comment on the logical paradoxes, which is necessary to avoid misunderstandings. Next, I introduce the ideas behind the semantics, and formulate a semantic system for a very poor positive logic **L1+**. A decision method is presented as well as Fitch-style and axiomatic formulations. In the subsequent sections, I consider a variety of extensions of **L1+**, including systems with negation. Finally, I mention some open problems and offer a comment on classical derivability within the systems discussed.

2. *The paradoxes of classical logic*

There are good reasons to distinguish between at least *three* kinds of paradoxes of **PC**, the classical propositional calculus. Let me briefly explain. (2.1) and (2.2) are examples of the first kind, (2.1) a plain and simple one, (2.2) perhaps a somewhat misleading one, as it might be misinterpreted as belonging to the second kind.

(2.1) $p \vdash q \to p$

(2.2) $\sim p \vdash p \to q$

Such examples show that material implication and some of its nephews do not express some suitable connection intended in most if-then statements of natural language. I take (2.1) to mean that from p we can *derive* that q *implies* p. In doing so, I keep implication and derivability neatly separated. (I have nothing against people who want to use both in the same sense, if that is coherently possible; but some don't, and no one should forbid them.) The paradoxes of the second kind derive from the definition of the derivability relation:

(2.3) $q \vdash p \vee \sim p$

(2.4) $q \vdash p \rightarrow p$

From a semantic point of view they are related to the fact that some formulas A are *true in all models*, and hence, for any arbitrary B, that there is no model in which B is true and A is false. The paradoxes of the third kind concern, semantically speaking again, the fact that some formula A is *false in all models*, and hence, for any arbitrary B, there is no model in which A is true and B is false.
Examples are:

(2.5) $p \& \sim p \vdash q$

(2.6) $(p \vee \sim p) \rightarrow (r \& \sim r) \vdash q$

What about remedies? The paradoxes of the first kind are solved by introducing some *relevant* implication. Incidentally, there is nothing wrong with adding such implication to **PC** if all one is worrying about are the paradoxes of the first sort. The paradoxes of the second kind are resolved by allowing for a larger set of models, viz. to the effect that no formula is true in all of them. In other words, one should avoid to have any theorems of logic (respectively, valid formulas) *in the classical sense* – see section 4. In starting from **PC**, to go *paracomplete* constitutes a partial remedy only, as (2.4) shows. Finally, one gets rid of the third kind of paradoxes by, again, allowing for a larger set of models, viz. to the effect that any formula is true in some model. As all connected troubles in **PC** derive from negation, going *paraconsistent* is a choice at hand.

It seems that Anderson and Belnap found another way to solve the paradoxes of the second and third kinds, viz. by changing the definition of derivability; see p. 277 *ff.* of [1]. The situation is rather complex, but as this paper concerns relevant logics, I should at least touch upon it. Anderson and Belnap's view may be rendered – in a fair way, it seems to me – be saying that theorems may be used as means to derive conclusions from premises, but may not be added to the premises in order to derive conclusions. I called the resulting notion 'relevant derivability' (see, *e.g.*, [6], which contains a demonstration that "classical" derivability may be defined from relevant derivability). Personally, I do not like the "identification" of derivability with implication, which lead to the notion of relevant derivability, but the latter is certainly interesting and important. Do Anderson and Belnap go paraconsistent and do they avoid theorems of logic? If the discussion in section 1 of [7] is correct, the answer should be positive, but maybe there is more to it. Also, according to the well-known Routley-Meyer semantics for relevant logics, some worlds (set-ups) are indeed paraconsistent and no formulas are true in all worlds, but maybe a less classical semantic explanation for relevant derivability will be devised in the future.

In the following sections, I shall present systems in which all three kinds of paradoxes are avoided. Still, I shall also consider some systems in which certain (but not all) paradoxes appear, and I think it useful to do so. This entails that I shall stick to the distinction between implication and derivability, and that for the latter I shall distinguish between the classical and the relevant sense. I need to stress this because Routley (alone and with others) has been consistently using in papers on semantics, *e.g.*, in [4] and [5], a way of speech which conflicts with this distinction (and according to which a logic "without" nested implications has no object language implication at all).

3. *The semantic idea*

This was first proposed in [7] and goes as follows. Consider a theory, let us call it *0*, and world, let us call it *1*, to which the theory applies. The fact that the theory applies to the world may be expressed by the (binary) accessibility relation R. Consider the usual definition:

(3.1) $v(A \rightarrow B, 0) = 1$ iff, for all i such that $R0i$, either $v(A, i) = 0$, or $v(B, i) = 1$.

This is known to lead, under usual conditions, to a non-relevant implication. (1) So, let us look for something more interesting about the relation between a theory and a world to which it applies.

If $A \rightarrow B$ is true in some theory, and A is true in a world to which the theory applies, then B is "grounded" in that world. This notion, well-

(1) As appears from section 14, however, the resulting implication is not as uninteresting as it looks.

known from philosophy of science, may be rendered as follows: B is not merely true in the world, but it is true in it *because* some other statement, viz. A, is true in it and the theory applies to this world. ([2]) Instead of 'grounded', one might say 'predictable'.
'Predicted' and 'explained' might also do, but only if some further conditions are fulfilled. Whichever terms are preferred, we are looking for a condition which decently expresses:

(3.2) $v(A \rightarrow B, 0) = 1$ iff, for all i such that $R0i$, either $v(A, i) = 0$, or B is *grounded* in 1.

The easiest way to express the idea in the formal semantics is by introducing a "part" $<1, 0>$ of the world 1, in which are true the statements that are grounded in world 1. From now on I shall use the more idiomatic expression 'aspect-world' to denote such part of a world. Given a theory 0, I shall associate an aspect-world $<i, 0>$ with any world i. If the theory 0 does not apply to world i, it does not matter which statements are true in $<i, 0>$. ([3]) The condition for implication will thus read:

(3.3) $v(A \rightarrow B, 0) = 1$ iff, for all i such that $R0i$, either $v(A, i) = 0$, or $v(B, <i, 0>) = 1$

Whatever is true in aspect-world $<i, 0>$, is grounded, and hence true, in world i:

(3.4) If $v(A, <i, 0>) = 1$, then $v(A, i) = 1$.

In order to formulate a simple (positive) logic, clauses for conjunction and disjunction should be added. Provisionally, I will keep them *classical* in worlds and aspect-worlds, and postpone the decision concerning theories. This treatment of disjunction in aspect-worlds may seem problematic. It is attractive to say, *e,g.*, in some cases in which p is true in the world and $p \rightarrow (q \vee r)$ is true in the theory, that $q \vee r$ is grounded whereas neither q nor r is. However, a somewhat weaker notion of grounding justifies the classical treatment of disjunction and is equally sensible. The weaker notion may be best understood if one refers to "elementary facts"

([2]) 'To be grounded' is usually meant as a pragmatic notion; in the present context I intend the corresponding ontological notion.

([3]) *E.g.*, in some systems one might stipulate $v(A, <i,0>) = 0$ for all A, whenever not $R0i$.

(which are neither "disjunctive" nor "conjunctive"). In te previous example some elementary fact is the case and guarantees the truth of p. The applicability of the theory results in some other elementary fact being the case, which in turn guarantees the truth of $q \vee r$. For this reason, $q \vee r$ is said to be grounded. But the latter elementary fact also guarantees either the truth of q or the truth of r, and hence either of these is grounded. Consequently, a disjunction is grounded (in the weaker sense) if and only if at least one of its disjuncts is grounded (in the same sense). (4)

4. *The logical theory*

Relevant logics require a specific semantic definition of 'valid formula'. No formulas may be true at all set-ups of all models, and no formulas may be true at the privileged set-up of all models (such as *0* in the case of the semantics under construction). Hence, relevant logics do not have any logical truths in the traditional sense.

However, the notion of a logical truth may also be viewed in other ways. I shall follow an approach which has the advantage to lead to an almost direct translation from the semantics to the Fitch-style formulation and vice versa, and to result in an intuitively appealing interpretation.

In standard relevant logics, theorems of logic relate to derivability as follows (where '$\mapsto$' denotes relevant derivability – see section 2):

(4.1) $A_1, \ldots, A_n \mapsto B$ iff $\vdash (A_1 \& \ldots \& A_n) \rightarrow B$.

(4.2) If the members of α are theorems and $\alpha \mapsto A$, then $\vdash A$.

Let us first concentrate on (4.1), and restrict its application in such a way that we do not assign properties to nested implications. Consider the set of implicative "logical truths" as a theory, the "logical theory", to be denoted by $\emptyset$. The properties of $\emptyset$ with respect to grounding are easily found: if i is accessible from $\emptyset$, then

(4.3) $v(A, <i,\emptyset>) =$ iff $v(A,i) = 1$.

In other words, implicative logical truths simply express the logical properties of worlds, the relations between the truth of statements within

(4) To be grounded in the weak sense is an ontological notion, whereas to be grounded in the stronger sense has epistemological components.

worlds. An other, simpler procedure to arrive at the same result consists in saying that implication in $\emptyset$ is characterized by:

(4.4) $v(A \rightarrow B, \emptyset) = 1$ iff, for all i such that $R\emptyset i$, either $v(A,i) = 0$, or $v(B,i) = 1$.

As the logical theory should clearly apply to all worlds, we may simplify this to:

(4.5) $v(A \rightarrow B, \emptyset) = 1$ iff, for all i, either $v(A,i) = 0$, or $v(B,i) = 1$.

Although the latter two formulations are simpler, the combination of (4.3) with (3.3) is more revealing. A theory such as $\emptyset$ has the peculiarity *to ground statements in a specific weak sense*, viz. in function of the meaning of logical terms or, more generally, in function of logical relations between statements. Where "grounded" is meant in this sense, we have, *e.g.*, in view of the classical treatment of disjunction, that $A \vee B$ is grounded in any world in which A is true; in general, a statement is grounded in a world whenever it is a logical consequence of statements which are true in it. In other words, anything true in a world is grounded in it with respect to the logical theory. This is not paradoxical, but just as we want it.

I shall introduce a logical theory $\emptyset$ within each model. As each model contains a set W of worlds which may be incomplete in comparison to the set of all semantically possible worlds, the logical theory $\emptyset$ of some (most) models will contain non-logical truths. For this reason $\emptyset$ is *local* logical theory. The valid formulas are those which are true at the local logical theory of all models.

5. *A first system*

The first system considered will be called **L1+**. It is a very poor positive logic in which (i) nested implications have no meaning, i.e., in which $v(A \rightarrow B, i)$ is arbitrary whenever i is a world, and (ii) non-implicative formulas have no meaning, i.e. $v(A,0)$ and $v(A,\emptyset)$ are arbitrary unless A is of the form $B \rightarrow C$. Here is a semantics for **L1+** :

A model M is a 6-tuple $<0, \emptyset, W, X, R, v>$ in which 0 is a theory, $\emptyset$ is the local logical theory, W is a set of worlds, X is a set of aspect-worlds (such that $<i,0> \in X$ iff $i \in W$), R is a binary relation between $\{0\}$ and W, and v is a valuation fulfilling the following conditions:

(i) $v(A\&B,a) = 1$ iff $v(A,a) = v(B,a) = 1$, for $a \in W \cup X$.

(ii) $v(A \vee B,a) = 1$ iff $v(A,a) = 1$ or $v(B,a) = 1$, for $a \in W \cup X$.

(iii) $v(A \rightarrow B,0) = 1$ iff, for all $i \in W$ such that $R0i$, either $v(A,i) = 0$ or $v(B, <i,0>) = 1$.

(iv) If $v(A, <i,0>) = 1$, then $v(A,i) = 1$.

(v) $v(A \rightarrow B,\emptyset) = 1$ iff, for all $i \in W$, either $v(A,i) = 0$ or $v(B,i) = 1$.

The definition of the semantic consequence relation is exactly as one should expect. The unusual sign is explained immediately.

$A_1, ..., A_n \Mapsto B$ iff, for all M, either $v(A_1,0) = 0$ or ... or $v(A_n,0) = 0$, or else $v(B,0) = 1$.

Finally, I define validity. (5)

$\models A$ (A is valid) iff, for all models M, $v(A,\emptyset) = 1$.

I employ the special sign for the semantic consequence relation in order to indicate that it is *relevant*. The classical consequence relation may be defined from it – see section 14.

It is possible to devise a simpler semantics for **L1+** which lacks an explicit accessibility relation (similar to the simple worlds semantics for **S.5**): all worlds of the model may be considered "accessible" from the theory *0*. This semantics is obtained by dropping all references to the accessibility-relation from the previous semantics. The difference between both semantic systems is as follows. In the second, all that is true at *0* is also true at $\emptyset$. In the first W may contain worlds not accessible from *0*, and hence some formulas true in *0* may be false in $\emptyset$. This difference is immaterial with respect to semantic consequence and validity.

I offer an axiomatization in section 9, but it is useful to state right away that all theorems of **L1+** are implicative and that only implicative formulas matter with respect to the semantic consequence relation. This means that nothing follows from non-implicative formulas, except for these formulas themselves ($A, B, ... \Mapsto A$), and that, *e.g.*, modus ponens and

(5) Alternatively, validity might be defined with respect to the set of "regular models", viz. the models for which $v(A,i) = v(A, <i,0>)$ for all worlds i and formulas A. I shall follow the approach sketched in the text; it has some advantages which will appear later, *e.g.*, with respect to the Fitch-style formulation.

addition do not hold in **L1+**. I shall consider extensions in which these do hold, but at present this extremely poor logic will do to illustrate the enrichment with respect to the corresponding fragments of standard relevant logics.

6. *The net gain*

Each of the following is correct in **L1+** and is not correct in **T**, **E** or **R**:

(6.1) $A \to B,\ (A \& B) \to C \models A \to C$

(6.2) $A \to (B \vee C),\ B \to C \models A \to C$

(6.3) $A \to (B \vee C),\ (A \& B) \to C \models A \to C$

(6.4) $A \to (B \vee C),\ (A \& B) \to D,\ (A \& C) \to E \models A \to (D \vee E)$

(6.5) $A \to (B \vee C),\ C \to D \models A \to (B \vee D)$

(6.6) $A \to B,\ (B \& C) \to D, \models (A \& C) \to D$

The reason why Anderson and Belnap arrived at systems according to which each of these are incorrect will become clear in the section on the Fitch-style formulation. At present I only want to state that there is nothing wrong with any of these from a relevantist point of view. A first argument for this claim is that none of the paradoxes which relevant logicians wanted to avoid obtain in **L1+**. The reader may easily check this by means of the decidability method of section 7. Moreover, **L1+** has extensions which contain nested implications, negation, and the binary connectives inside and outside the scope of implications, which are strictly stronger than the corresponding standard relevant logics, and which avoid all the aforementioned paradoxes (and are all decidable). A second argument for my claim refers to intuitions, and hence is indicative rather than demonstrative in nature. Whether you read the arrow as an entailment, or as a causal implication, or as a relevant nomological implication, (6.1)-(6.6) appear to be sound. I do not claim that they are correct for *all* readings of the arrow. I do indeed think that Anderson and Belnap's systems contain meaningful implications. All I suggest is that, for many uses of implication which occur in natural languages, including scientific language, (6.1)-(6.6) are correct.

Neither of the following obtain in **L1+** :

(6.7) $A \to B \models A \to (A \& B)$

(6.8) $B \to C \models (B \vee C) \to C$

It is well-known that these lead to paradoxes under quite normal conditions on the arrow. Both are avoided in **L1+**; so, *e.g.*, the correctness of (6.1) does *not* depend on (6.7).

7. *Decidability*

The decision method for **L1+** is easily derived from the semantics. I shall use tableaux with two columns, one for truth and one for falsehood. For a tableau i, these columns will be denoted by i_t and i_f respectively. Whenever a tableau i, other than 0 or $\emptyset$, is drawn, one also draws a (aspect) tableau $<i,0>$. Tableaux and tableau-constructions are said to close under the usual conditions for two-sided tableaux (a formula occurs in both columns of some tableau). The rules are as follows (i and j are always different from both 0 and $\emptyset$; i is never an aspect-tableau): (6)

T1 : If $A \rightarrow B$ occurs in $\emptyset_f$, start a new tableau i, write A in i_t and B in i_f.
T2 : If $A \rightarrow B$ occurs in $\emptyset_t$, then, for all i, split i into i' and i'', write A in i'_f and B in i''_t.
T3 : If $A \rightarrow B$ occurs in 0_f, start a new tableau i, stipulate $R0i$, write A in i_t and write B in $<i,0>_f$.
T4 : If $A \rightarrow B$ occurs in 0_t, then, for all tableaux i such that $R0i$, split the tableau into i' and i'', add A to i'_f and add B to $<i'',0>_t$.
T5 : If $A\&B$ occurs in i_t add both A and B to i_t.
T6 : If $A\&B$ occurs in i_f, split i into i' and i'', write A in i'_f and B in i''_f.
T7 : If $A \vee B$ occurs in i_t, split i into i' and i'', write A in i'_t and B in i''_t.
T8 : If $A \vee B$ occurs in i_f, write both A and B in i_f.
T9 : If A occurs in $<i,0>_t$, write A in i_t.
T10: If A occurs in i_f, write A in $<i,0>_f$.

It is provable that:

Theorem. $\models A$ iff the construction started by writing A in $\emptyset_f$ closes.

(6) Splitting tableaux is the common short-hand method for splitting the whole construction. Of course, ineffective moves should not be carried out. *E.g.*, in the case of T2, i should not be split if A occurs in i_f or B occurs in i_t.

Theorem. $A_1, ..., A_n \models B$ iff the construction started by writing A_1 and ... and A_n in 0_t and by writing B in 0_f closes.

The present decision method may easily be extended to all extensions of **L1+** below. The theorems remain provable.

8. *Fitch-style formulation*

There is a direct relation between Kripke-semantics and Fitch-style proofs. (7) The fact that a formula occurs with index $\{i\}$ corresponds to its being true at world i. For usual relevant logics, $\{i, j\}$ corresponds to a world k distinct from i and j (but for which Rijk holds if i precedes j). For **L1+** and its extensions, $\{i,0\}$ corresponds to the aspect-world $<i,0>$ of i.

As derivability and implication should be clearly kept apart in **L1+** – remember that $A \rightarrow (B \& C) \vdash A \rightarrow B$ but that $\nvdash (A \rightarrow (B \& C)) \rightarrow (A \rightarrow B)$ – I need a Fitch-style formulation in which premises, as distinct from hypotheses, may be written. To this end I shall adapt the terminology of [1]. Premises may be written with $\{0\}$ as their set of numerical subscripts and with rank zero. Any number of premises may be written in the proof.

Apart from this, the rules for **L1+** are as those for **E** – see p. 347 of [1] – with the following modifications:

(i) The rule of Hypothesis is restricted to the effect that the maximum rank is 1 (one hypothesis at a time; nested subproofs do not occur).
(ii) One rule is added. It is called RSD (restricted subscript deletion) and reads: from A with the set $\{1,0\}$ of numerical subscripts, to derive A with the set $\{1\}$ of numerical subscripts. From a semantic point of view this corresponds to the fact that grounded statements are true (in the world in which they are grounded).
(iii) $\vee$E is replaced by a stronger rule, called $\vee$E*. This rule is easy to apply but somewhat complex to formulate. I shall give a rather circumstantial description. Let $A \vee B$ occur in a proof, with rank r (in **L1+** always 1) and the set a of subscripts. Copy the proof twice. Add A with rank

(7) Just as each logic has a two valued semantics (as was shown independently by Suszko and by da Costa), each logic characterized by a Fitch-style system has a Kripke-semantic with two-valued valuation functions. Moreover, this semantics is in a direct way derivable from the Fitch-style system and offers a "natural" interpretation for it. For a set of interesting results on the fact that any logic has a two-valued worlds semantics, see [8].

r and set a of subscripts to the first copy and add B with the same rank and set of subscripts to the second. If C is derivable with rank r and any set b of subscripts in *both* copies, then C may be added with the same rank and set of subscripts to the original proof. To see the effect of this rule, show the correctness of (6.2)-(6.5). The rule $\vee$E* indicates that disjunction is extensional: C is derivable from $A \vee B$ in a given context, just in case it is derivable from A as well as from B in the same context.
(iv) &$\vee$ is redundant!
(v) $\rightarrow$I should be restricted: it may not be applied to a subproof in which RSD has been applied and the last formula of which has the same set of subscripts as the hypothesis, viz. 1. By this restriction we make sure that all formulas that occur with the empty set of subscripts in *any* proof are theorems.

Theorem. $\alpha \mapsto A$ iff A occurs with (rank zero and) the set $\{0\}$ of subscripts in a proof in which all premises are members of α.

Theorem. $\models$ A iff A occurs with the empty set of subscripts in some proof in which no premises are introduced (or in any other proof).

Are the rules easy to understand in their application? If one keeps the semantics in mind, they obviously are: premises are given as true in some theory and hypotheses are supposed to be true in some world. In this way the semantics is a guide to the intuitive interpretation of syntactical rules. Fitch-style rules have no independent convincing character. The fact that they are easily fed to undergraduates, usually by misleading tricks, is no argument for their conceptual clarity.

9. *Axiomatization*

I shall not merely consider the delineation of a set of axioms, but also the set of correct inferences (from premises) defined by the axiomatic system. To this effect I shall follow the notation of pp. 277-278 of [1]: for a correct (relevant) deduction it is possible to attach stars to some formulas in such a way that (i) premises are starred, (ii) axioms that are not premises are not starred, (iii) other formulas are starred as indicated below, and (iv) the conclusion is starred as a consequence of (i)-(iii).

Axioms

A1 $A \to A$

A2 $(A \& B) \to A$

A3 $(A \& B) \to B$

A4 $A \to (A \vee B)$

A5 $B \to (A \vee B)$

Rules

R1 $A \to B,\ B \to C\ /\ A \to C$

The conclusion is starred if at least one premise is.

R2 $A \to B,\ A \to C\ /\ A \to (B \& C)$

The conclusion is starred iff both premises are; application of the rule is forbidden in case only one premise is starred.

R3 $A \to C,\ B \to C\ /\ (A \vee B) \to C$

Stars as for R2.

R4 $A \to (B \vee C),\ (A \& B) \to C\ /\ A \to C$

Stars as for R2.

The strength of R4 becomes clear if one realizes that it allows one to demonstrate (6.1), (6.2), and (6.4)-(6.6), as well as the theoremhood of:

(9.1) $(A \& (B \vee C)) \to ((A \& B) \vee C)$

The proofs are straightforward. *E.g.*, for (6.2), $A \to (B \vee C)$ is derivable from $A \to B$, and $A \to C$ is derivable from $A \to (B \vee C)$ and $(A \& B) \to C$.

Theorem. $\alpha \mapsto A$ iff $\alpha \models\!\!\!\to A$.

Theorem. $\vdash A$ iff $\models A$.

10. *Small extensions*

The logic **L1+** does not enable one to derive anything from non-implicative formulas (except for the formulas themselves). Several ways are open to circumvent this restriction. *E.g.*, we may interpret non-implicative formulas true in 0 as true in all worlds, or as true in the real world. The latter turns out to be most appealing.

It seems worth pointing out that the first alternative leads to an interesting logic, the implication of which, however, lacks certain relevance properties usually considered as desirable.

(10.1) $A \rightarrow B, A \models C \rightarrow B$

(10.2) $(A \& B) \rightarrow C, A \models B \rightarrow C$

For (10.1), suppose that both A and $A \rightarrow B$ are true in 0. Then A is true in all worlds, and hence B is grounded in all worlds. But then $C \rightarrow B$ is true in 0 for any C. (10.1) certainly resembles too much (2.1) to keep relevant logicians happy, and (10.2) is considered a source of disaster by most of them. Yet, this logic does have a lot of relevance features; it does not make any of the straightforward paradoxes valid. Typically, it is not one of the usual modal logics. *E.g.*, we have:

(10.3) $A \not\models A \rightarrow A$

(10.4) $A, A \rightarrow B \not\models C \rightarrow C$

The moral is presumably that it does make sense to distinguish between the types of paradoxes discussed in section 2, and that we should care to worry about interpretations, rather than about syntactical properties.

The second alternative consisted in considering the non-implicative formulas true in 0 as true in the real world. Let us single out 1 to denote the latter. Obviously, this corresponds to many uses of the term 'theory'. More specifically, whenever the term is used to denote a set of statements about some domain, one means the non-implicative (and non-modal) statements to be true in the real world, whereas one means implicative statements to be nomologically true, i.e. to be true in a theory in the strict sense. (At least, that is the case if we disregard counterfactuals and similar *locally* true implications.)

The neatest way to deal with the present logic, which I shall call **L2+**, is to define a model as a seven-tuple $<0, \emptyset, 1, W, X, R, v>$, $1 \in W$ denoting the real world, to add to the semantics of **L1+** the clause([8])

(vi) $v(A,0) = v(A,1)$

and to define the semantic consequence relation with respect to 1 instead

([8]) There is no need to stipulate that A be non-implicative.

of *0*. By doing so, we gain a set of correct inferences in comparison to **L1+**. They are easily summarized by noting that an axiomatic version of **L2+** is obtained by adding to that of **L1+** :

R5 $A, A \rightarrow B \ / \ B$

The conclusion is starred iff the minor is; applications in which the major is starred and the minor is not are forbidden.

R6 $A, B \ / \ A \& B$

The conditions on stars are as for R2.

R7 $A \vee B, B \rightarrow C \ / \ A \vee C$

The conditions on stars are as for R5.

R4 and R7 *express the enrichment in comparison to standard relevant logics*. Just as its "implicative counterpart" (6.5), R7 is not objectionable from a relevance point of view and is quite desirable intuitively speaking. (The last phrase in a joke; R7 is desirable on the present interpretation and on some others).

There is an alternative way to formulate **L2+** – if you wish, there is an alternative semantic system which leads to the same set of semantic consequences. It is obtained by collapsing *0* and *1* into *0* as follows. A model is a quintuple as for **L1+**, but with $0 \in W$ and $R00$ (and possibly $v(A, <0,0>) = v(A,0)$ in order to simplify the decision method).

Both formulations are decidable. Both correspond in a straightforward way to Fitch-style systems. However, the latter formulation leads to Fitch-style proofs which are a lot easier to set up and which moreover are closer to those for standard relevant logics.

I briefly touch upon a third way to handle non-implicative premises, viz. by considering them as true in a "part" of the real world, to be denoted by 1^*. Intuitively, this is the part of the real world that the theory pertains to. Formally the matter is handled by defining a model as $<\emptyset, 0, 1, 1^*, W, X, R, v>$ and by requiring:

$v(A,0) = v(A,1^*)$.
If $v(A,1^*) = 1$, then $v(A,1) = 1$.
If $v(A, <1,0>) = 1$, then $v(A,1^*) = 1$.

Let us call the resulting logic **L3+**. It is syntactically identical to **L2+** as it stands, but a striking difference appears for some conditions on negation.

In writing the present section, it was may aim to show that the enrichment this paper is about relies on a semantic interpretation which does not exclude considering non-implicative premises. As a side-effect, I hope to have shown that the present semantic approach to implication opens a variety of perspectives. I pointed to three of them, but it is obvious that more are available. To mention just one example, a non-implicative formula may be interpreted as true in a specific part of any world, viz. the part that the theory pertains to. Again, the resulting logic as it stands will coincide with the first alternative discussed in this section, but it will lead to divergent results for some conditions on negation.

11. *Negation*

Negation too may be handled in a variety of ways. Some of these lead to rather disappointing results. *E.g.*, if one extends the **L1+** semantics by requiring that all worlds, i.e. members of W, are consistent and negation-complete, then one arrives at the result that $(p\& \sim p) \rightarrow q$ is true in the theory, viz. 0, of any model M.(9) I do not have space here to discuss several systems together with their advantages and disadvantages with respect to domains of application. I shall restrict my attention to a single approach to negation and to a single system.

If one starts off in a naive way, and keeps standard relevant logics in mind, one will want to have, *e.g.*, each of the following:

(11.1) $\sim\sim A \mapsto A$ and conversely.

(11.2) $\sim(A\&B) \mapsto \sim A \vee \sim B$ and conversely.

(11.3) $\sim(A \vee B) \mapsto \sim A \& \sim B$ and conversely.

(11.4) $\sim A \rightarrow \sim B \mapsto B \rightarrow A$ and conversely.

Simply adding the corresponding clauses to the semantics might cause people to protest: a semantics should consist of clauses which suggest a sensible interpretation of the meanings of the logical terms. I now present an approach which leads to such an interpretation.

It was said before that B is grounded in a world i to which 0 applies, if A is true in that world and $A \rightarrow B$ is true in theory 0. For reasons that

(9) As, for any M, $(p\& \sim p) \rightarrow q$ would also be true at $\emptyset$, it would be both a non-logical theorem of any theory, and a theorem of logic.

become clear soon, I now reformulate this as: the presence of B in world i is grounded by the presence of A in i. However, it seems desirable to attach a further consequence to $A \rightarrow B$ with respect to grounding: the *absence* of A is grounded by the absence of B. The presence or absence of certain facts may be linked to the outcome of tests. One should keep in mind that, in any somewhat realistic situation, more than one kind of test is available for most predicates, and that it cannot be excluded by an act of faith that something is both present and absent, or that it is neither.

The preceding paragraph suggests that we express the absence of A by $\sim A$. In order to keep things straight, it is helpful to link an absence-world to any world (and aspect-world). Let us denote the absence-world associated to i by i- (not to be confused with the starred worlds of the Routley-Meyer semantics). This results in:

(11.5) $\quad v(\sim A,i) = 1$ iff $v(A,i\text{-}) = 1$

In order to arrive at (11.1)-(11.4) we simply have to request that absence-worlds (and absence-aspect-worlds) are duals to the worlds with which they are associated in that (i) the meanings of conjunction and disjunction are exchanged, the meaning of relevant implication is "reversed", and any world is the absence-world of its absence-world ($i\text{--} = i$, in other words 'i' and 'i-' may be exchanged in (11.5)). The reader may check that this convention is plausible in a large number of situations.

Applying all this to **L2**+, we arrive at the semantics for **L2** by inserting W- and X- in the model, by adding the clauses

$v(A \& B,i\text{-}) = 1$ iff $v(A,i\text{-}) = 1$ or $v(B,i\text{-}) = 1$ for $i \in W \cup X$.
$v(A \vee B,i\text{-}) = 1$ iff $v(A,i\text{-}) = v(B,i\text{-}) = 1$ for $i \in W \cup X$.
$v(\sim A,i) = 1$ iff $v(A,i\text{-}) = 1$ for $i \in W \cup X$.
$v(\sim A,i\text{-}) = 1$ iff $v(A,i) = 1$ for $i \in W \cup X$.

and by modifying the clause for implication as follows:

$v(A \rightarrow B,0) = 1$ iff, for all i such that $R0i$, $v(A,i) = 0$ or $v(B, <i,0>) = 1$, and $v(B,i\text{-}) = 0$ or $v(A, <i,0>\text{-}) = 1$.

and analogously for $v(A \rightarrow B,\emptyset)$. Needless to say, the same additions and modifications may also be applied to **L1**+ and **L3**+, provided one keeps in mind the obvious modifications for the latter.

For the Fitch-style formulation, one adds the following rules:

$\sim\sim$I: from A with the set a of numerical subscripts, to derive $\sim\sim A$ with the same set of subscripts

$\sim\sim$E: from $\sim\sim A$ with the set a of numerical subscripts, to derive A with the same set of subscripts

$\sim\rightarrow$: from $\sim B$ with the set a of numerical subscripts and $A \rightarrow B$ with the set b of subscripts, to derive $\sim A$ with the set $a \cup \{i\}$ of subscripts, provided $b = \{0\}$ or $b = \emptyset$.

These rules lead to the theoremhood of $\sim(A \& B) \rightarrow (\sim A \vee \sim B)$, etc.

The axiomatic formulation is obtained by adding the axioms

$$A \rightarrow \sim\sim A$$
$$\sim\sim A \rightarrow A$$

and the rule

$$A \rightarrow B \;/\; \sim B \rightarrow \sim A$$

the conclusion is starred iff the premise is.

A *richer* system is obtained by adding to the semantic condition for $v(A \rightarrow B,0) = 1$ that $v(\sim A \vee B,0) = 1$ (i.e., $v(\sim A \vee B,I) = 1$ in the first formulation). In this system we obtain, *e.g.*, ([10])

$$A \rightarrow B \models \sim A \vee B$$
$$A \rightarrow B,\ A \rightarrow \sim B \models \sim A$$
$$A \rightarrow (B \& \sim B) \models \sim A$$
$$A \rightarrow \sim A \models \sim A$$
$$A \rightarrow B,\ \sim A \rightarrow B \models B$$
$$(A \vee \sim A) \rightarrow B \models B$$

One may or may not adapt the clause for $v(A \rightarrow B,\emptyset)$ in a similar way. If one does not, the set of theorems is not deductively closed. If one does, one also need to adapt (in a straightforward way) the properties of the other connectives in $\emptyset$ in order not to loose all theorems again. I shall not discuss the matter any further.

([10]) We do *not* have $(A \rightarrow A) \rightarrow B \Rightarrow B$; remember that $v(A \rightarrow A,i)$ is arbitrary whenever i is distinct from 0, and that $v(A \vee \sim A,i)$ may very well be false.

12. *Nested implications*

One may obviously extend the previous logics in such a way that nested implications obtain specific properties. Some such extensions contain one or more standard relevant logics. I cannot work this out here in any detail, but shall merely indicate in which way one may obtain a sensible semantics for such systems.

Let us consider **L2**, viz. the alternative formulation (see section 10) which leads to a technically simpler semantics. Basically, non-implicative formulas are evaluated in a straightforward way within *0*; the evaluation of implicative formulas is guided by the idea that they are true in the theory (in the strict sense) associated to the "real world" *0*. The accessibility relation R expresses that all implicative formulas (those true in the theory in the strict sense) are effective in the accessible worlds. Now, we may associate a theory in the strict sense with the latter, and consider worlds accessible from them as well. Two restrictions might (but need not) be introduced: stipulate that R is transitive and that $< <i,j> ,k> = <i, <j,k> >$. If Rkj and Rji, then both couples denote the aspect-world of i in which is true whatever is grounded in i by the interplay of the implicative statements of j and k; whence one may identify both couples. I cannot offer any proofs, but it is quite obvious that such moves will lead to semantic systems for standard relevant logics enriched by R4, R7 and perhaps some more things.

13. *Open problems*

Important open problems concern the weakest extensions of **T**, **E** and **R** that may be obtained by the present approach.

A technically interesting problem is whether the present logics are in some sense maximal. Consider, *e.g.*, **L1+**. Is it possible to weaken the implication without loosing its relevance properties? I do not have an answer, but I see a possible approach. The non-implicative formulas form a lattice. Implicative theorems are in an obvious way related to this lattice ($A \rightarrow B$ iff there is an upward path from A to B – henceforth denoted by ALB). On the elements of the lattice one may define a transitive binary relation I which corresponds to implicative formulas true in *0*: $v(A \rightarrow B,0) = 1$ iff AIB. I has properties such as: If AIB and BLC, then

AIC. This may be seen as: there is a "contingent" path from A to C if one may reach C from A by a path of which at least one part is contingent. The difference between **L1+** and (the corresponding fragment of) standard relevant logics may be seen as follows. One may reach a point C from a join $A \vee B$ if, whether one "drops down" to A or to B, one is able to reach C. Standard relevant logics require that both paths are of the same type, viz. logical or contingent. **L1+** allows for mixed cases, and merely requires this: one may reach E from D if there is an itinerary such that, in whichever direction you "drop down" at a join, part of your path is contingent. A similar condition applies to meets.

14. *Classical derivability*

According to the spirit of Anderson and Belnap's views – see also [6] – classical derivability is related to relevant derivability as follows, where τ is the set of logical theorems:

$$\alpha \vdash A \text{ iff } \alpha \cup \tau \mapsto \text{A}.$$

It follows that the classical semantic consequence relation (for all systems discussed) may be defined by:

$A_1, ..., A_n \models B$ iff, for all *regular* models, $v(A_1,0) = 0$ or ... or $v(A_n,0) = 0$ or $v(B,0) = 1$.

where

M is regular iff, for all i and for all A, $v(A, <i,0>) = v(A,i)$.

Equivalently:

M is regular iff, for all A, $v(A,0) = v(A,\emptyset)$.

Incidentally, this shows that the kind of approach I considered in the beginning of section 3 is not as far away from relevant logics as one might suppose. If we consider regular models only, one arrives at a system which fits into this approach.

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REFERENCES

[1] Alan Ross Anderson & Nuel D. Belnap, Jr., *Entailment. The Logic of Relevance and Necessity*, vol. 1, Princeton, 1975.

[2] Ayda I. Arruda & Newton C.A. da Costa, "O paradoxo di Curry - Moh-Shaw-Kwei", *Boletim da Socieda Matemática de São Paulo*, 18, 1965, 83-89.

[3] Ayda I. Arruda & Newton C.A. da Costa, *On the relevant systems P and P* and some related systems*, Relatório Interno 174, IMECC-UNICAMP.

[4] Richard Routley & Andréa Loparić, "Semantical analysis of Arruda-da Costa P systems and adjacent non-replacement systems", *Studia Logica*, 37, 1978, 301-320.

[5] Richard Routley & Robert K. Meyer, "Towards a general semantical theory of implication and conditionals. II", *Reports on Mathematical Logic*, 9, 1977, 47-62.

[6] Diderik Batens & Jean Paul Van Bendegem, "Relevant derivability and classical derivability in Fitch-style and axiomatic formulations of relevant logics", *Logique et Analyse*, 109, 1985, 21-31.

[7] Diderik Batens, "Two semantically motivated enrichments of relevant logics", *Proceedings of the 30th Conference on the History of Logic*, Kraków, to appear.

[8] Richard Routley & Robert K. Meyer, "Every sentential logic has a two-valued worlds semantics", *Logique et Analyse*, 71, 1976, 345-364.