IDENTITY CRITERIA

Dr. Stephen POLLARD

Abstract

According to Quine, one reason that properties are to be deplored is that we (allegedly) lack an adequate criterion of property-identity. But Quine's brief remarks on this topic leave us wondering under what conditions an identity criterion is to be considered adequate. I shall offer arguments and examples which suggest that the evaluation of identity criteria is a much more complicated business than one might think and that, at the moment, we have little reason to believe that the extensionality axiom of set theory is preferable in any general sense to property theoretic identity criteria.

According to Quine, one reason that properties are to be deplored is that we (allegedly) lack an adequate criterion of property-identity. But Quine's brief remarks on this topic leave us wondering under what conditions an identity criterion is to be considered adequate. Here are some thoughts which may help to clarify this important matter.

First, a word about the notion of adequacy itself. I understand adequacy to be a quality of identity criteria which is independent of their truth. In what follows, the reader will find examples of true identity criteria which are nonetheless inadequate. And I imagine that an identity criterion could be adequate and yet not be true. In this regard, adequacy is like any one of the standard "virtues" of empirical scientific hypotheses – say, refutability. A proposition can certainly be both refutable and false. And a non-positivist might claim that a proposition could be both true and irrefutable.

Let us say that an *identity criterion* (i.c.) is a sentence of the form

(1) $\forall\alpha\ \forall\beta(\varphi \rightarrow \alpha = \beta)$

where α and β are variables which occur free in φ. We shall also allow a sentence-*scheme* whose instances are i.c. in the above sense to count as an i.c. itself. Of course, one is not interested in just any plausible assertion of the form (1). An *adequate* identity criterion will have some of the qualities of a good definition. (1) itself indicates that an i.c., whether adequate or not, will give a sufficient (though not necessarily a necessary) condition for identity. Furthermore, an adequate i.c. will be "non-circular". That is, if φ would be rejected as a definiens of $\alpha = \beta$ on the ground of circularity, then it is fairly certain that φ cannot serve as the antecedent of an adequate i.c. For example,

(2) $\forall x\, \forall y(x = y \rightarrow x = y)$

or

(3) $\forall x\, \forall y((x = y \lor (P \land \neg P)) \rightarrow x = y)$

(whose "circularity" in the above extended sense is obvious) clearly will not do as identity criteria. For if we are ever in genuine need of help in proving that $a = b$, it would be cold comfort indeed to be told that we need only first prove '$a = b$' or '$(a = b \lor (P \land \neg P))$'. Of course, (2) and (3) are rather outlandish examples. No doubt, one can dream up subtler ones.

To establish the adequacy of an i.c. it is not, however, enough merely to show that it is non-"circular". "Circular" i.c. are to be avoided because they are deductively uninformative: that is, they do not in general help us prove the assertions they are supposed to help us prove (a failing which, I take it, renders them inadequate as i.c.). But i.c. can be deductively uninformative even when they cannot very plausibly be considered "circular". Some remarks by Michael Jubien in an unpublished manuscript bring the following example to mind. Let us steal '$\in$' back from set theory to express the relation of property-possession (i.e. '$a \in b$' is to mean that a has the property b). If a property theory T has

(4) $\forall x\, \exists y\, \forall z(z \in y \leftrightarrow z = x)$

as a theorem (i.e. if T furnishes a "singleton property" for every object), then T supplies us with the i.c.

(5) $\forall x\, \forall y(\forall z(x \in z \rightarrow y \in z) \rightarrow x = y)$

(For just assume the antecedent of (5) and then, by (4), let z be a singleton property of x.) Now (4) is likely to be a theorem of a great number and variety of interesting property theories – theories which will then also contain the i.c. (5). So it is worth asking whether (5) is an adequate i.c. It would be stretching to claim that (5) is "circular" (although our proof of (5), depending as it does on (4), does suggest a kind of circularity). But is (5) deductively informative? Could (5) really help us to make identity judgements? That depends on the sort of theory in which (5) is embedded. (5) could serve as a useful i.c. in a theory which enables us to deduce assertions like

(6) $\forall z(a \in z \rightarrow b \in z)$

in ways which do not themselves depend on our ability to make identity judgements or, in particular, upon our already having established that $a = b$. But there are any number of property theories which lack this characteristic (e.g. the theory whose only axiom is (4)). In these theories, (5) would be no better as an i.c. than (2) or (3).

What have we accomplished so far? Among other things, we have offered a canonical form for i.c. (cf. (1)); we have made the obvious point that an i.c. is to be reckoned as better or worse depending on how much it can help us to make identity judgements; and we have seen that the ability of an i.c. to help us in this way can depend on the make-up of the theory which surrounds it. Actually, this last point seems pretty obvious too; but perhaps some of its consequences are less obvious. Consider the following example. The axiom of extensionality (which says that sets with exactly the same members are identical) is Quine's paradigm of an adequate i.c. It allegedly exalts set theories above their muddy property theoretic rivals. But if the utility of Extensionality can depend on the theory within which it is embedded, then surely it cannot be the case that *every* set theory is superior to *every* property theory with respect to its i.c. For, when placed in an unsupportive environment, Extensionality could well have trouble competing with a property theoretic i.c. from a better neighborhood. It is noteworthy, for example, that our ability to make identity judgements about non-empty sets is, at least insofar as this is a matter of employing Extensionality, no greater than our ability to make identity judgements about the members of those sets. For in order to apply Extensionality to a particular case we must have already determined whether the sets under consideration have identical members. So if Extensionality were put to work in a universe which contained objects which are intractable with respect to questions of identity and if, furthermore, these fuzzy objects were allowed to be members of sets, then our much lauded set theoretic i.c. could easily be feebler than a property theoretic i.c. which is not so encumbered. For the fuzzy objects of the set theory might be fuzzier than the properties of the property theory. Now it could still be the case that no property theoretic i.c. can be reckoned superior (qua i.c.) to Extensionality when the latter is supported by, say, the remaining axioms of Zermelo-Fraenkel set theory and the universe is not infected with fuzzy objects. The moral I want to draw here is simply that sweeping statements about the excellence of Extensionality and the baseness of property theoretic i.c. should be treated with scepticism. For it is misguided to attempt any immediate comparison of isolated i.c. We need instead to weigh the relative merits of i.c. with respect to their roles in given theories. Or, we could say, we need to weigh whole theories, not just the i.c. they contain – a conclusion which a holist like Quine would be hard pressed to deny.

I shall now give an example of just such a comparison. I would like to weigh ZFC (Zermelo-Fraenkel set theory including the axiom of choice) against one of its property theoretic cousins – namely, the theory ZFC_P developed by Norman M. Martin and myself. We shall try to discover what grounds there might be for preferring the extensionality axiom of ZFC to the i.c. of ZFC_P – or vice versa. ZFC_P is formulated in the language L_P which we introduce as follows. Let L be some standard first order language whose only non-logical relation symbol is the binary one '$\in$'. (L is to treat '$=$' as a logical symbol.)

Definition 1 The lexicon of L_P is the same as that of L except that L_P contains the additional logical symbols DF_n and x_n for every natural number n.

Definition 2 Let φ be a finite sequence of symbols of L_P. The *x-rank* of φ is the smallest natural number n such that x_n does not occur in φ.

Definition 3 Let φ be a finite sequence of symbols of L_P of x-rank n. Then $\varphi(x_n)$ is an *x-substitute* for φ if and only if $\varphi(x_n)$ is the result of substituting an occurrence of x_n for at least one occurrence of an individual symbol in φ and no occurrence of x_n in $\varphi(x_n)$ is part of a quantifier.

Definition 4 We define 'formula of L_P' by adding the following clause to the recursive definition of the formulas of L: if φ is a formula of x-rank n and α is an individual symbol, then $DF_n(\varphi(x_n),\alpha)$ is a formula – where $\varphi(x_n)$ is an x-substitute for φ.

For example, '$\forall v_1\, \exists v_2\, Df_0(x_0 = v_1, v_2)$' is a formula of L_P (given an appropriate choice of L). Its intended interpretation is: "Given any object v_1, there is an object v_2 such that v_2 is the property of being identical to v_1." More generally, a Df-clause says that a certain object is the property of being so-and-so. '$Df_0(x_0 \neq x_0, c_1)$' says that the object c_1 is the property of being non-self-identical. (By the way, by attaching natural number sub-scripts to 'Df' and 'x', we allow ourselves to construct nested Df-clauses without ambiguity.)

We form ZFC_P by dropping Extensionality from ZFC and rewriting the existence axioms of ZFC (with the exception of the axiom of choice which is adopted unchanged) using the Df-notation. For example, the pairing axiom becomes:

(7) $$\forall v_1\, \forall v_2\, \exists v_0\, Df_0((x_0 = v_1 \vee x_0 = v_2), v_0)$$

ZFC_P also contains two additional axiom schemes dealing the Df-relation. First, given any x-substitute $\varphi(x_n)$ for a formula of L_P, every closure of

(8) $$((Df_n(\varphi(x_n), v_1) \wedge Df_n(\varphi(x_n), v_2)) \rightarrow v_1 = v_2)$$

is an axiom. Secondly, if $\varphi(x_n)$ is an x-substitute for a formula of L_P and if v_m does not occur in $\varphi(x_n)$, then every closure of

(9) $$(Df_n(\varphi(x_n), v_0) \rightarrow \forall v_m(v_m \in v_0 \leftrightarrow \varphi(v_m)))$$

is an axiom – where $\varphi(v_m)$ is the result of replacing all occurrences of x_n in $\varphi(x_n)$ by occurrences of v_m. (8) is our identity criterion for properties. It merely states that properties with the same definition are identical. A property definition is, for example, an expression of the form 'being φ' in a context of the form 'the property of being φ'. Thus our i.c. amounts to saying that if A is the property of being F and B is also the property of being F, then A = B. (9) just says that the Df-relation determines the extension of a property in the desired way.

That is, A has the property of being F (if there really is a property of being F) if and only if A is F. For example, A has the property of being a denumerable ordinal (if there really is a property of being a denumerable ordinal) if and only if A is a denumerable ordinal.

Now how does ZFC_P stack up against ZFC with regard to its i.c.? An enthusiastic extensionalist might claim right off the bat that (8) is rendered inferior to the extensionality axiom of ZFC by the presence within it of philosophically mysterious terminology – namely the Df-clauses. Since I am convinced that the sets of ZFC are at least as philosophically mysterious as most anything philosophers might trouble themselves over and since "philosophical mysteriousness" is itself a rather muddy notion, I shall pass over this sort of argument. Let us instead try to assess carefully the relative merits of Extensionality and (8) within ZFC and ZFC_P. With regard to at least one very significant branch of mathematics, there seems to be little basis for discrimination. It turns out that ZFC_P is an adequate foundation for classical analysis. And while some classical theorems require more complicated proofs in ZFC_P than in ZFC, other theorems turn out to have simpler proofs in ZFC_P. In any case, the difference between the two theories seems to be small in this domain. Given (8), supported by the rest of ZFC_P, Extensionality would not be much missed.

This is in part because, in developing classical analysis, we happen not to be called upon to prove (the universal closures of) assertions having the form

$$(10) \qquad ((Df_n(\varphi, \alpha) \wedge Df_m(\psi, \beta)) \rightarrow \alpha = \beta)$$

where φ and ψ are formulas differing by more than a change of free variables. The inability of ZFC_P to rule on some such matters might with some justice be viewed as a muddiness in its i.c. But this does not mean that (8) should in some universal sense be considered inadequate (just as Extensionality should not be dismissed on the ground that, within ZFC, it does not allow us to rule on important identity questions such as the continuum hypothesis). It is more reasonable to relativize our notion of adequacy both to a particular *theory* and to a particular *subject matter*. For this allows us to ask questions which we have some hope of answering. (Question: *Within* ZFC_P, is (8) an adequate i.c. *with respect to classical analysis*? Answer: Yes.)

Naturally enough, it turns out that (8) is not an adequate i.c. for ZFC_P with respect to every mathematical theory. For example, the general theory of ordinals seems to be a bit too much for ZFC_P. ZFC does an incomparably better job in this area thanks to some crucial applications of Extensionality. But if we juice up ZFC_P with some additional existence axioms, we can increase the ability of (8) to compete with Extensionality. Or, alternatively, we could achieve the same effect by adopting as axioms (the universal closures of) assertions having the form

$$(11) \qquad (Df_n(\varphi, \alpha) \rightarrow Df_m(\psi, \alpha))$$

where φ and ψ are as in (10). Since this would allow us to use (8) to decide some instances of (10), we could say that the addition of assertions like (11) has clarified our i.c. And enough such clarification could yield a theory for which the ordinals hold no terrors.

The point I wish to emphasize here is that it is reckless to attempt to rule on the adequacy of an i.c. without taking into account its role in particular theories. Indeed, I have suggested that one should specify not only the surrounding theories, but the subject matter at issue as well. For adequacy is clearly (particularly from a Quinean point of view) bound up with *usefulness*. (No matter what the context: an i.c. which is not deductively informative can surely not be considered adequate; and it seems likely that a really convincing case for the inadequacy of a consistent i.c. would have to at least take into consideration the extent to which it is deductively informative.) But the question of usefulness makes little sense unless we specify what we are trying to accomplish and how we propose to go about accomplishing it. Once this has been determined, we can get down to the complicated task of discovering what exactly a given i.c. does and does not do for us. We have some hope of answering, for example, the question of whether (8), supported by the rest of ZFC_P, can determine property-identity in ways which allow for, say, transfinite recursion. Whether (8) "in itself" is muddier that Extensionality "in itself" is, I suppose, a matter to be settled only (if at all) by some sort of induction from the results of our more focused comparisons. Perhaps we could convince ourselves that (8) can deal with huge and complex structures (such as the totality of all ordinals) only when surrounded by a theory which is itself unacceptably complex. And perhaps this could count as evidence that (8) is muddy "in itself". But, as far as I can see, such conclusions are to be reached only after the sort of detailed investigations we have discussed. There is no royal road to the evaluation of identity criteria.

Northeast Missouri State University
Division of Social Science
Kirksville, Missouri 63501
USA

Dr. Stephen POLLARD