THE FORMAL STRUCTURE OF THE LIAR PARADOX

Geoffrey HUNT

A distinction is made between absolute and relative truth-values, and using arithmetical analogues for logical operations the formal structure of the Cretan's statement is manifested. It is shown that no contradiction is involved in the statement, and therefore there is no paradox in the sense of a logically unavoidable contradiction. The term 'paradox' is retained however for a certain logical form of appearance and the manner in which this appearance is produced is explained together with an explanation of why this appearance is not produced in the case of the logically identical statement 'This statement is true'.

1 *Absolute and Relative Values*

A recent treatment of the liar paradox (LP) by Herzberger [1982] enables us to regard it as a logical *process*. I extend Herzberger's analysis by using a simple arithmetical device which I believe has the advantage of laying bare once for all the formal structure of the LP.

Herzberger is against an approach which tries to 'enforce a distinction of levels by indexing the truth predicates' and against 'regimentation of what seems to be the intuitive facts either with respect to negation, or with respect to ascription of truth, or with respect to ascent of levels.' (p. 484) What we need, he says, is an 'adequate representation of our naive intuitions.' (p. 485) Thus his 'naive semantics' has a 'tolerance of paradox.' (p. 494) He explicates the LP as a 'valuational pattern.' Thus if we consider the statement 'This statement is false' we do not get a stable true statement nor a stable false statement. What is less obvious we do not get, on Herzberger's view, a statement which is both true and false nor one which is neither true nor false. We get a *pattern* of valuations: 'Its fundamental semantic character is neither a truth-value nor the absence of a truth-value, but is a valuational pattern.' (p. 497)

Now consider the following case of a sequence of valuations of a statement we know to be true:

Snow is white
It is true snow is white
It is true it is true snow is white
etc.

This gives us a pattern of valuations: 1, 1, 1, 1, 1... which has a periodicity of one. Then consider the following case of a sequence of valuations of a statement we know to be false:

Snow is black
It is true snow is black
It is true it is true snow is black
etc.

This gives us the pattern of valuations, in the usual notation: 0, 0, 0, 0, 0... which also has a periodicity of one.

We may regard ordinary language (OL) statements as having two values: true or positive (which I represent by 1) and false or negative (which I represent by -1). These symbols have a crucial advantage over the more usual T and F or 1 and 0, as will soon become clear. To take some examples: granted what we know of relevant empirical states of affairs, the statement 'Snow is white' has a value of 1, and the statement 'Snow is black' has a value of -1. Such statements, which are statements which have as an immediate reference empirical states of affairs I call 'protocol statements' (p-statements). In addition to these there are statements which assign truth or falsity to other statements and I call these truth-assigning statements (a-statements). It should be clear that an a-statement may have as its referent a p-statement or another a-statement.

The statement 'Snow is black' is a p-statement, and the statement '"Snow is black" is true' is an a-statement. The value of the former is −1, given our empirical knowledge. The value of the latter is of two kinds. There is the value it *assigns* (namely 1), which I call its *absolute value*, and there is counterposed to it a value (namely −1), which I call its *relative value*, which is the value relative to the p-statement to which it refers. We may now give inductive definitions of a p-statement and its a-statements, using the usual arithmetical notation for linear and exponential sequences:

Table I.

STATEMENTS	RELATIVE VALUES	DEFINITION
(i.)		
Snow is white		$V_{n+1} = V_n$, for all n
This is true		$V_1 = 1$
'This is true' is true		
etc.	(1), 1, 1, 1...	
(ii.)		
Snow is black		$V_{n+1} = V_n$, for all n
This is true		$V_1 = -1$
'This is true' is true		
etc.	(−1), −1, −1, −1...	
(iii.)		
Snow is white		$V_{n+1} = -V_n$, for all n
This is false		$V_1 = 1$
'This is false' is false		
etc.	(1), −1, 1, −1, 1...	
(iv.)		
Snow is black		$V_{n+1} = -V_n$, for all n
This is false		$V_1 = -1$
'This is false' is false		
etc.	(−1), 1, −1, 1, −1...	

It will be noted that the truth-assignment operation is expressed by multiplication. Also, the p-statement's value is put in brackets in the second column to distinguish it from the relative values which follow. In these examples the absolute values are recurrent. We may give a general formula for this kind of sequence of relative values:

$$V_n = V_p a_1{}^n \qquad (1)$$

V_n signifies the relative value of the nth term/level of a-statements; V_p signifies the value of the p-statement; a_1 signifies the absolute value of the first term/level of a-statements; the n in the expression $a_1{}^n$ is the level index and behaves as an exponent; the product is a sequence of relative values. I call this the general formula for the relative value of any term/level in a recurrent protocol-referential truth-assigning sequence (PRTA rec.). We may write its general instantiations thus:

Table II.

(i.) $V_n = 1$, for all n = 1, 1, 1, 1, 1...
(ii.) $V_n = -1$, for all n = −1, −1, −1, −1, −1...
(iii.) $V_n = (-1)^n$, for all n = −1, 1, −1, 1, −1...
(iv.) $V_n = -(-1)^n$, for all n = 1, −1, 1, −1, 1...

It will be noted that this repeats the forms in Table I. Here, however, the sequence of values on the right omits the p-values.

It is obvious that a sequence of a-statements need not have recurrent absolute values; indeed, in OL, in which such sequences are usually very short (three or four values at most) recurrence is less frequent than non-recurrence. In such cases the general formula would have to be written:

$$V_n = V_p(\prod_{i=1}^{n} a_i) \qquad (2)$$

Here we multiply each absolute value up to the level required, which gives us one product. I call this the general formula for the relative value of any term in a non-recurrent protocol-referential truth-assigning sequence (PRTA non-rec.). We may consider a single example:

(i.) $V_3 = (1) \times -1 \times -1 \times 1 = 1$

It should also be obvious that we may combine (1) and (2) in various ways. For example:

$$V_n = V_p a_1 a_2 a_3{}^n, \text{ where } n \geqslant 3 \qquad (3)$$

This gives us the relative value of any term at or above the third level of a-statements. In this case the recurrence begins at the third level.

2 *The Contradictory Form of Appearance*

I now turn our attention to the LP and shall take up Herzberger's insight that this may be considered as a valuational pattern. We shall examine the simplest form of the LP: 'This statement is false'.

This statement is clearly an a-statement. It would ordinarily be taken to be referring to a p-statement or to a-statements whose relative values are determinable. An ordinary response to 'This statement is false' would be the question 'Which statement?' in the expectation that some other statement would be indicated. The Cretan is no ordinary language-user however. He insists that this a-statement is referring to *itself*. It is self-referential. Let us call it a self-referential truth-assigning statement (SRTA). We have seen that there is nothing problematic about an a-statement referring to another a-statement; but we shall soon see that an a-statement referring to itself has unique consequences which invite the characterisation 'paradoxical'.

As the a-statement is self-referential it gives rise to a sequence of statements with alternating values. We shall see that there is a problem about how to characterise these values. The sequence may provisionally be represented as follows:

Table III.

	(a)	(b)
This statement is false.	−1	
Therefore 'This statement is false' is false.	−1	1
Therefore "'This statement is false' is false" is false.	−1	−1
Therefore '"'This statement is false' is false" is false' is false.	−1	1
etc.	etc.	etc.

Column (a) gives us the sequence of absolute values and column (b) gives us the resultant values. This presentation shows us that we have two counterposed sequences of values. One is a sequence of *negative* values (each value assigned by the statement on each level) and a sequence of resultant *alternating* values with a periodicity of one.

There seems to be no great objection to regarding the sequence of negative values as absolute values, even though they are self-generated. In this respect the LP may be regarded as similar to (1) (iii.) and (iv.). See Tables I and II. Now, can we adequately represent the LP with one of these two inductive definitions? Definition (1) (iv.) seems appropriate as the first assigned value is -1 and the value of the statement to which it is assigned (itself in this case) is -1. Furthermore, the sequence of values in the LP begins with -1 as apparently it does in (1) (iv.) in table I. We might then be tempted to adopt

$$V_{n+1} = -V_n, \text{ for all n, and } V_1 = -1$$

as an adequate inductive definition of the LP. But we would be wrong to do so. In (1) (iv.) the expression $V_1 = -1$ gives the value of the *protocol* statement, an empirically verifiable (and verified) statement. In (1) (iv.) the first *relative* value of an a-statement is 1 not -1. Now, in the LP the first value in the sequence (namely -1) is the value of the a-statement; there is no p-statement. Our initial a-statement is applied to itself, 'another' a-statement. This in turn is applied to 'another' a-statement *ad infinitum*. The inductive definition (1) (iv.) is not appropriate. It does not capture the essential difference between the sequence of statement (iv.) in table I and the Cretan's statement. Another difference between them is that our sequence of absolute values in (1) (iv.) was arbitrarily chosen while in the case of the LP the sequence of absolute values follows from the Cretan's initial statement. This is indicated by the use of 'Therefore' in table III. The sequence is self-determined. How can we capture these points in a general sequential formula? I suggest we take our general formula for PRTA rec. and modify it by leaving out the protocol value, thus:

$$V_n = a_1^n \qquad (4)$$

I call this the general formula for self-referential truth-assigning statements (SRTA). We may write its instantiations exhaustively thus:

Table IV.

(i.) $V_n = -1^n = -1, 1, -1, 1, -1\ldots$
(ii.) $V_n = 1^n = 1, 1, 1, 1, 1\ldots$

In (4) the level index behaves as an exponent. It captures the self-referential character of the Cretan's statement. His statement is nothing more than an a-statement acting upon itself. Let us examine (4) (i.) more closely. Here a recurrent negative truth-assignment $(-1)^n$ gives us the sequence of alternating values of periodicity two which we counterpose to the negative absolute values. We may graphically represent this counterposition thus:

Table V.

(i.)	*absolute values*	*pseudo-relative values*	
	-1		
	-1	1	$\leftarrow$ 'contradiction'
	-1	-1	
	-1	1	$\leftarrow$ 'contradiction'
	-1	-1	
	-1	1	$\leftarrow$ 'contradiction'
	-1	-1	
	etc.	etc.	

(ii.) Alternatively, 'contradiction'

$$\begin{array}{lccccccccc} & & & \downarrow & & & & \downarrow & & \\ \text{Factors:} & -1 & \times & -1 & \times & -1 & \times & -1 & \times & -1\ldots \\ \text{products:} & & = & 1 & = & -1 & = & 1 & = & -1\ldots \end{array}$$

The right hand sequence of values in table V (i.) *appears* as a sequence of relative values, but as there is no p-statement for these to be relative to they cannot be relative values. What, then, are they? They are values relative to the initial a-statement. I call these pseudo-relative values for the simple reason that ordinarily we never take the actual value of an a-statement as the value relative to some other a-statement; that is an implicit rule of OL.

It is, then, this counterposition of 'factors' and 'products' which accounts for the uncanny feeling we have with the Cretan's statement, and this is our reason for calling it a paradox, that a contradiction keeps popping up; a contradiction which we must accept. Indeed, it has usually been assumed that a contradiction is involved in the LP, for example, Wittgenstein assumed this [1967, p. 51]. This has even led some philosophers to reject the law of excluded middle [Rescher & Brandom, 1980].

Our analysis lends further support to Priest's claim [1983] that some semantic paradoxes do not depend on the law of excluded middle and that therefore a strategy of rejecting that law cannot offer a uniform solution of the paradoxes. He mentions Berry's, Richard's and König's, and we can now add the LP. It seems possible that none of the paradoxes, on an analysis like the present one, rests on the law. (Priest does not, of course, rule out the possibility of *any* uniform solution or dissolution of the paradoxes.) It should be observed that I retain the term 'paradox' in this case to refer to the understandably contradictory form of *appearance* of the Cretan's statement.

3 *Double Open-endedness*

In (4) (i.) we began our sequence with -1. This is an absolute value. However, our sequences in (1), (2) and (3) began with a relative value. (In (1) (i.) and (ii.) it just happens that the protocol value and the first relative value are identical in each case.) The subsequent values in (4) (i.) are values relative to that initial absolute value.

Now, are we justified in writing down -1 as our first value? Our justification for writing down -1 as our first value in the inductive definition of (iv.) in table I, for example, is that this is the first *protocol* value. Our justification for writing down 1 as our first value for (iv.) in the general formula for PRTA rec. in table II, for example, is that this is the first *relative* value. But -1 in (4) (i.) is neither a protocol value nor a relative value, but an *absolute* value. In what sense could it be a *first* absolute value? As an absolute value we would ordinarily ask ourselves, in order to identify the relative value, 'What is the condition of the truth of this a-statement?' And we would go on asking this until we struck the relevant p-statement. The difficulty in this case is that we can go on repeating our question *ad infinitum* – we never come to a p-statement. To represent this point we should properly write our sequence for the Cretan's statement in this way:

$$\ldots -1,\ 1,\ -1,\ 1,\ -1 \ldots$$

Unlike the sequences in (1), (2) and (3) this sequence is open at *both* ends. To return to our formula, $V_n = -1^n$, the open-endedness would mean that we do not need to take $n = 1$ as our starting point. That is, instead, $n = -\infty, \ldots, +\infty$. Thus, working backward as well as forward from -1^1 we get the following pattern:

Table VI.

$$\begin{array}{rcr} & & \text{etc.} \\ -1^3 & = & -1 \\ -1^2 & = & 1 \\ -1^1 & = & -1 \\ -1^0 & = & 1 \\ -1^{-1} & = & -1 \\ -1^{-2} & = & 1 \\ -1^{-3} & = & -1 \\ & & \text{etc.} \end{array}$$

This observation also applies to (4) (ii.) the sequence of which should properly be written:

$\ldots 1, 1, 1, 1, 1 \ldots$

It is to this second instantiation of SRTA that I now turn our attention.

4 *The Cretan's Wife's Statement*

I will now examine the little-studied statement made by the Cretan's wife (who was not a Cretan): 'This statement is true'. The fact that this self-referential truth-assigning statement has been almost totally ignored by philosophers over the centuries is intimately linked with the failure to unravel the Cretan's statement. There is a surprising *identity* of structure between the Cretan's statement and that of his wife even though no one has ever claimed the latter to be paradoxical. Its general formula is the same as that of the Cretan's statement, namely $V_n = a_1{}^n$. It is given by (4) (ii.). The difference between (4) (i.) and (4) (ii.) lies merely in the value given to a_1.

The difference in the sequences produced by (4) (i.) and (ii.) is explained by the elementary fact that the first is given by repeated 'multiplications' of a negative value by itself and the second by repeated 'multiplications' of a positive value by itself. A sequence of positive pseudo-relative values and a sequence of alternating pseudo-relative values (periodicity of two) are the only possible results of repeated self-multiplications of 1 and of -1. A sequence of negative pseudo-relative values is impossible.

In the case of the Cretan's wife's statement the sequence of absolute values is identical to the sequence of pseudo-relative values therefore the *appearance* of contradiction does not arise. That, presumably, is why philosophers, to the detriment of progress on this question, have ignored the Cretan's wife's statement. The statement is no ordinary one; its structure is identical with that of the LP; yet it does not present a paradoxical appearance. I call it a *rapadox* to capture these three points. It certainly deserves its own name. There may be cases of rapadox other than that produced by the Cretan's wife's statement.

We may regard the difference between the paradox and the rapadox in this way: while in the former the relative truth-values are underdetermined (or open) and thus indeterminable, in the latter the relative values are overdetermined (or closed) because one of the two values (falsity) is ruled out.

I believe that by employing an arithmetical notation and a sequential analysis we can move from quite ordinary sequences of truth-assigning statements inductively defined, through a series of intermediaries, to extraordinary self-referential sequences. This movement may now be summarised:

Table VII.

PRTA non-rec.	$V_n = V_p(\prod_{i=1}^{n} a_i)$	(2)
PRTA part-rec.	$V_n = V_p a_1 a_2 a_3{}^n$, where $n \geq 3$	(3)
PRTA rec.	$V_n = V_p a_1{}^n$	(1)
SRTA	$V_n = a_1{}^n$	(4)

$V_n = -1^n$	$V_n = 1^n$
(paradox)	(rapadox)

I suggest that the procedure I have employed here has an application more general than that shown in this inquiry which has been concerned only with the Cretan's statement and that of his wife.

5 *Conclusion*

Nothing that has been done here contravenes our expectations as far as *arithmetic* is concerned. There is nothing 'paradoxical' or 'contradictory' about $V_n = -1^n$ as a mathematical formula. But mathematics need not have ontological implications, whereas OL must have. At some point (innumerable points) OL must 'hook' onto the real world, whereas mathematical languages need not (although they can). Thus a simple mathematical device can be used to dissolve the LP, for we can *see* now how the paradoxical appearance arises once our 'hook', namely the protocol statement, is disengaged. But the rules of OL do not allow a disengagement of the hook any more than chess allows one to play in the area off the edge of the board. If they did it would not be *ordinary* language, language which arises from, and orders or constitutes, our everyday practices.

The novelty of employing an analysis in terms of arithmetical sequences should now be evident. The paradoxical consequences of the Cretan's statement do not arise if the rule-breaking procedure embedded in it is expressed *explicitly* in a purely formal structure. The appearance of a contradiction which logically must be accepted lies in the periodic difference between absolute and pseudo-relative values (factors and products) in a self-generating sequence of values. To take a simple and more familiar case: if someone says 'It is true snow is black' the absolute value is positive and the relative value, given empirical states of affairs, is negative. But no appearance of contradiction is presented (and no paradox) for the reason that we can easily and immediately *discriminate* between the p-statement and the a-statement, between the absolute value and the relative value of the statement, and counterpose them. But the Cretan's statement in being self-referential cannot 'contain' any p-statement and therefore we cannot easily and immediately discriminate between absolute and relative values, for what we have are absolute values and *pseudo*-relative values. Thus we tend to counterpose *directly* one positive value against one negative value (i.e. *identify* them) presenting a contradiction. To emphasize, only by clearly discriminating, as we have done by using the arithmetical device, between absolute values and pseudo-relative values can we make the necessary *explicit* counterposition to expose the source of the paradoxical appearance. In the case of the Cretan's wife's statement the counterposition of opposite values cannot arise at all, and therefore the paradoxical appearance does not present itself.

Acknowledgements: I am grateful to David-Hillel Ruben and Olufemi Oguntade for some useful suggestions.

University College Cardiff
Philosophy Department
P.O. Box 78
Cardiff CF1 1XL
Wales
GREAT BRITAIN

Geoffrey HUNT

REFERENCES

Herzberger, H.G. [1982]: 'Naive Semantics and the Liar Paradox', *Journal of Philosophy*, *79*, pp. 479-497.

Priest, G. [1983]: 'The Logical Paradoxes and the Law of Excluded Middle', *Philosophical Quarterly*, *33*, pp. 160-165.

Rescher, N. & Brandom, R. [1980]: *The Logic of Inconsistency*. Rowman and Littlefield.

Wittgenstein, L. [1967]: *Remarks on the Foundations of Mathematics*. Blackwell.