PARTIALITY AND NONMONOTONICITY IN CLASSICAL LOGIC

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Recent developments in semantics have broken with what are generally perceived to be two major presuppositions in classical logic: *complete* information, and *cumulative* inference. In this report, we want to show that the matter is more complex. Lack of completeness and failure of cumulation do occur in classical modal logic and, in the last analysis, even in ordinary classical logic itself. Although the locus of these phenomena becomes less definite in this way, the classical analogy may also have some positive heuristic virtues.

The following discussion has been restricted to propositional languages, for reasons of expedience rather than principle. Three results obtained appear to be new: a semantic tableau analysis of "strong consequence" (Sec. 1), a modal reduction of "data logic" (Sec. 2), and an axiomatization of nonmonotonic classical logic (Sec. 4).

Partiality

Recent semantic theories have advocated a "partial" perspective, often as regards our information about semantic objects, sometimes also as regards these objects themselves. For instance, in tense logic, people have studied partially interpreted languages over mathematically definite points in time. But also, extended "periods" have been investigated, as a kind of temporal item which itself represents a range of eventual mathematically precise locations. (Cf. van Benthem, 1982.) Similar moves have been proposed in possible world semantics for modalities: cf. Humberstone, 1981.

Actually, the term "partial" may be inappropriate in these and similar cases – as it suggests cutting off parts from some complete given entity. But eventually, one may come to regard the partial cases as the fundamental ones, viewing the former semantic objects as some kind of ideal extrapolation out of the latter. Instead of musing about these issues in a general fashion, we shall now take a look at some concrete examples, pointing at more general features as they arise.

1. *Strong Consequence in Situation Semantics*

One of the fundamental observations in "situation semantics" (cf. Barwise and Perry, 1983) has been that an adequate account of the propositional attitudes needs to employ a finer grid of identifying propositions than that provided by classical logic. In fact, "harmless" classical equivalences such as $p \leftrightarrow (p \,\&\, (q \vee \sim q))$ turn out to be unacceptable in an analysis of direct seeing. The proposed solution has been that (partial) situations can "support" p without supporting $p \,\&\, (q \vee \sim q)$, when lacking information about q. In general a situation can support a statement or its negation, and then the matter is decided henceforth ("persistence of information"); but, the statement can also be undecided, becoming supported only in a larger situation. Only those statements will be identified now into one single proposition which are indistinguishable in terms of supporting situations.

Kamp's Analysis

Kamp (1983) proposes an interesting analysis of the above ideas, of which the following fragment is relevant here. Situations may be regarded as partial valuations V, assigning truth values 0,1 to some (perhaps not all) proposition letters. A simultaneous recursion now defines

$$V \models \varphi \ (V \text{ supports } \varphi), \quad V \dashv \varphi \ (V \text{ rejects } \varphi),$$

for formulas φ in a propositional language with $\sim$ (negation), & (conjunction), and $\vee$ (disjunction):

(i) $V \models p$	if $V(p) = 1$,		$V \dashv p$	if $V(p) = 0$
(ii) $V \models \varphi \,\&\, \psi$	if $V \models \varphi$ *and* $V \models \psi$,		$V \dashv \varphi \,\&\, \psi$	if $V \dashv \varphi$ *or* $V \dashv \psi$
(iii) $V \models \varphi \vee \psi$	if $V \models \varphi$ *or* $V \models \psi$,		$V \dashv \varphi \vee \psi$	if $V \dashv \varphi$ *and* $V \dashv \psi$

and, simple, but subtle,

(iv) $V \models \sim\varphi$ if $V \dashv \varphi$, $\qquad V \dashv \sim\varphi$ if $V \models \varphi$.

Alternatively, these clauses could be formulated using three-valued truth tables.

In the restricted area of propositional logic, but also in the much more complex field of predicate logic with an added operator for direct perception statements (treated by Kamp), this semantics displays the earlier-mentioned features. We do not have $V \models \varphi$ or $V \dashv \varphi$ for all φ (i.e., bivalence fails); hence, for instance, $q \vee \sim q$ is not always supported, which blocks the strong inference from p to $p \& (q \vee \sim q)$. On the other hand, persistence does hold: if $V \models \varphi$ ($V \dashv \varphi$), then $V' \models \varphi$ ($V' \dashv \varphi$) for all valuations V' extending V.

Kamp gives a completeness argument deriving the full logic of this scheme. Basically, all those classical equivalences remain valid which involve the same vocabulary on both sides: including most constructively controversial principles, such as the De Morgan law $\sim(p \& q) \leftrightarrow \sim p \vee \sim q$, or just Double Negation $\sim\sim p \leftrightarrow p$. Thus, perhaps surprisingly, the grid of strong equivalence is not that much finer than that of classical equivalence after all.

Semantic Tableaus

Once presented, the Kamp semantics suggests an alternative, completely orthodox view upon the matter. In so-called "Beth tableaux" for classical logic (cf. Smullyan, 1968), a counter-example is sought for a top sequent $P_1,\ldots,P_n \cdot Q_1,\ldots,Q_m$ representing an inference from P_1 and ... and P_n to the disjunctive conclusion Q_1 or ... or Q_m. That is, some valuation V is to be found verifying all of $P_1,\ldots,P_n$, while falsifying all of $Q_1,\ldots,Q_m$. Connective rules break this problem down into (alternate) sets of less complex requirements, down to an operator-free level (if need be). Thus, through a systematic application of these rules to all complex formulas, a tree is created, each of whose branches represents a purported counter-example. Formulas to be made true occur on the left-hand side, those to be made false to the right-hand side of the branch. Such a branch may "close", when the same formula appears on both sides (then, the corresponding counter-example turns out to be spurious); otherwise, it remains "open".

Example: a closed tree

$$\sim p \vee (q \& r), \sim q \cdot \sim p$$
$$\sim p \vee (q \& r) \cdot \sim p, q$$
$$\sim p \vee (q \& r), p \cdot q$$

$q \& r, p \cdot q$	$\sim p, p \cdot q$
$q, r, p \cdot q$	$p \cdot q, p$
closure	closure

There are no counter-examples:

$$\sim p \vee (q \,\&\, r), \sim q \quad \text{imply} \quad \sim p.$$

Example: an open tree

$$\sim p \vee (q \vee r), \sim q \cdot \sim p$$
$$\sim p \vee (q \vee r) \cdot \sim p, q$$
$$\sim p \vee (q \vee r), p \cdot q$$

$q \vee r, p \cdot q$		$\sim p, p \cdot q$
$q, p \cdot q$	$r, p \cdot q$	$p \cdot q, p$
closure	open	closure

The valuation V with $V(r) = V(p) = 1$ and $V(q) = 0$, corresponding to the open branch, is a counter-example to the top sequent.

Now, one interesting thing about the Beth refutation method is this. Counter-examples occur when a branch remains "open", that is, no statement occurs both on the left and the right side of the branch. But, there is no presumption of completeness: not every proposition letter need be decided on the branch.

Example: In refuting $\sim p \vee q \cdot \sim q$, the tableau may leave p undecided:

$$\sim p \vee q \cdot \sim q$$
$$\sim p \vee q, q \cdot$$

$\sim p, q \cdot$	$q, q \cdot$
$q \cdot p$	

Left counter-example: $V(q) = 1$, $V(p) = 0$, and right counter-example: $V(q) = 1$. Notice that the former is a special case of the latter.

Significantly, this indeterminacy shows in the form of the induction establishing the desired behavior of the valuation V associated with an open branch:

if φ occurs on the 1-side, then $V \models \varphi$
if φ occurs on the 0-side, then $V \models \sim \varphi$.

Usually, one extends V to some arbitrary total $V^+ \supseteq V$. But, evidently, the above partial valuations would also be quite suitable, with $V(\varphi) = 1$ if and only if $V \models \varphi$ and $V(\varphi) = 0$ if and only if $V \dashv \varphi$. Thus, Beth tableaus already induce the above partial perspective.

But then the question arises how the outcome of such a partial perspective can still be classical logic. The answer lies in the definition of consequence. In a partial valuation framework, there are at least two options for defining consequence, which were still co-extensive in the bivalent case (a latitude well-known from many-valued logic). One possibility is to set $\varphi \models \psi$ if

for all V, if $V(\varphi) = 1$, then $V(\psi) = 1$.

This is "strong consequence" in the above sense. Another possibility is to have

for all V, if $V(\varphi) = 1$, then $V(\psi) \neq 0$.

It is the latter notion which is being tested in Beth tableaus, and hence it is the latter which gives rise to classical consequence.

Still, using the tableau perspective, it is now also relatively easy to give a complete description of strong consequence. The idea is simply to reinterpret the two labels 1 and 0 on the two sides as 1 and $\neq 1$ (i.e., 0 or undefined). All rules of decomposition for &, $\vee$, and $\sim$ except one work out in entirely the same way. (For instance, $p \mathbin{\&} q \neq 1$ if and only if $p \neq 1$ or $q \neq 1$.) The only exception are negations with the $\neq 1$-label. If $V(\sim\varphi) \neq 1$, then not necessarily $V(\varphi) = 1$; it might be undefined. Thus, reading off *Gentzen sequent rules* from tableau rules in the usual way (recall the and/or-reading mentioned above), we obtain the following valid inferences:

(1) the usual basic sequents, the usual (introduction) rules for & and $\vee$, as well as the left $\sim$-rule: $A \Rightarrow C, D$ implies $A, \sim C \Rightarrow D$.

Notice that the missing right $\sim$-rule would be involved in deriving the invalid sequent $\Rightarrow p, \sim p$ (i.e., *tertium non datur*). The left $\sim$-rule, however, gives us $p, \sim p \Rightarrow$ (i.e., *ex falso sequitur quodlibet*). Even the latter would have failed us, if we had also admitted "over-defined" valuations V, assigning both 0 and 1 in some cases – that is, if a four-valued semantics had been adopted.

Nevertheless, there remain quite a few principles governing negations in the conclusion (and premise) set, mirroring the given evaluation rules. Basically, these allow negations to move inward:

(2) $\sim\sim A$ is equivalent to A, $\sim(A \mathbin{\&} B)$ to $\sim A \vee \sim B$, and $\sim(A \vee B)$ to $\sim A \mathbin{\&} \sim B$.

These observations yield a simple proof for a result first obtained by Kamp:

Theorem: The complete logic of strong consequence, expressed in Gentzen sequents, is given by the principles (1) and (2).

Proof: The above principles are all valid in the intended sense. Conversely, a nonderivable sequent may be brought into an equivalent form having negations only in front of atomic formulas. To the latter, the ordinary tableau rules for & and $\vee$ may be applied, yielding some open branch, which induces a counter-example as follows. The valuation V defined by setting $V(p) = 1$ if p occurs on the 1-side of the branch, $V(p) = 0$ if $\sim p$ so occurs, will give value 1 to all formulas on the 1-side and a different value to those on the other side. (The relevant induction step is guaranteed by the above reinterpretation of the tableau rules.) Q.E.D.

Thus, a simple validity test for classical consequence can be made to work for strong consequence as well.

Actually, the Beth test is one of a family of classical methods sharing similar partial traits. Another relative would be Hintikka's well-known method of "model sets," being partial specifications of a model under construction.

Representation

There are other possible connections between the above partial semantics and its total ancestor. A partial valuation V can be extended in several ways to a total one and the idea lies at hand to attempt a "reduction" of truth at the former to truth at the latter. More specifically, the following equivalence seems plausible:

$$V(\varphi) = 1 \quad \text{iff} \quad V^+(\varphi) = 1 \text{ for all total valuations } V^+ \text{ extending } V.$$

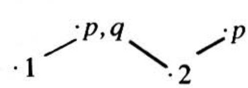
When taken as a definition, this amounts to the "supervaluation" approach to partial truth. But as a statement the equivalence fails. For example, $p \vee \sim p$ is true in all total V^+, but not necessarily in all partial V. What can be shown, however, is the earlier-mentioned persistence: V-values once assigned remain the same for extensions of V. (Actually, not even this much will be guaranteed in later sections, when absence of information may validate possibility statements, which need not be persistent.) Thus, on the whole, the above connection fails. This does not preclude more sophisticated formulations – which certainly exist – but they are beyond the scope of this report.

2. *Strong Consequence in Data Semantics*

An extension of the above propositional semantics to the case of modal notions has been proposed in Veltman (1981). In "data semantics", ordered sets of information states, provided with partial valuations, are used to model possible growth of information. This setting is needed to describe the semantics of implication, as well as the natural language modalities MAY and MUST, which refer to some set of possibilities, not all of them (necessarily) actual. We shall give an outline of this theory, which has a lingering orthodox modal flavor. The latter feeling is then validated by presenting an embedding of data semantics into the more traditional possible worlds semantics for the modal logic *S4.1*.

Data models $M = \langle I, \sqsubseteq, V\rangle$ have a partially ordered set $\langle I, \sqsubseteq\rangle$ of information states, each of whose maximal chains ends in a greatest element. (The idea is that every search for complete information is eventually succesful. Compare the similar condition in Scott's "domain semantics," where all omega-chains are required to have limits.) To each i in I, the function V assigns some partial valuation V_i for the proposition letters, subject to a monotonicity constraint: if $i \sqsubseteq j$, then V_j extends V_i. Moreover, V_i ends up being total in the maximal elements of $\langle I, \sqsubseteq\rangle$.

Notice that not all possible partial valuations need occur in a model. This means that certain constraints may be encoded in the pattern of available continuations for a state i in I with valuation V_i. Conversely, certain partial valuations may recur at different states. This allows us to model distinctions such as those displayed in the following pattern:



In state 1, p and q are not yet true, but, whenever p becomes true, so does q. In node 2, p and q are not yet true either, p will become true, but it may or it may not entail q. This kind of observation is also standard in modal logic: mere state descriptions cannot replace possible worlds (except in the semantics of such a simple logic as *S5*), since verification of higher modal operator statements may require the occurrence of identical state descriptions at different locations in the pattern of alternative worlds.

The truth definition is as before for the propositional part of the language ($\sim$, &, $\vee$). In addition, Veltman presents the following operators, with their truth clauses:

$M \models \varphi \rightarrow \psi [i]$	if	*for all* $j \sqsupseteq i$, $M \models \varphi [j]$ *only if* $M \models \psi [j]$
$M \dashv \varphi \rightarrow \psi [i]$	if	*for some* $j \sqsupseteq i$, $M \models \varphi [j]$ *and* $M \dashv \psi [j]$
$M \models \text{MAY}\, \varphi [i]$	if	*for some* $j \sqsupseteq i$, $M \models \varphi [j]$
$M \dashv \text{MAY}\, \varphi [i]$	if	*for all* $j \sqsupseteq i$, *not* $M \models \varphi [j]$
$M \models \text{MUST}\, \varphi [i]$	if	*for all maximal* $j \sqsupseteq i$, $M \models \varphi [j]$
$M \dashv \text{MUST}\, \varphi [i]$	if	*for some maximal* $j \sqsupseteq i$, $M \dashv \varphi [j]$

Evidently, different views of these operators could be (and have been) implemented as well.

As was indicated above, this time, persistence is not guaranteed: growth of information may mean loss of possibilities, and hence true MAY-statements, or, say, false $\rightarrow$-statements, can change their truth value. Veltman charts this phenomenon systematically.

With the definition of strong consequence as before, this semantics generates a "data logic" extending the earlier one of Section 1. (A complete axiomatization is obtained in Veltman 1985.) For instance, the implication $\rightarrow$ has an interesting logic, with features reminiscent of classical, intuitionistic, and even counterfactual conditionals. Also, MUST and MAY behave in an attractive way. Like their counterparts in traditional modal logic ("necessarily" and "possibly"), they satisfy some reasonable reduction postulates, such as MUST MUST $\varphi \models \text{MUST}\,\varphi$ or $\sim \text{MAY} \sim \varphi \models \text{MUST}\,\varphi$. On the other hand, they avoid the principle $\text{MUST}\,\varphi \rightarrow \varphi$, which seems too strong for ordinary language. The semantic point here is that MUST φ may hold, in the sense that φ is true at all maximal information states extending the present one, while φ is still not true. More informally, the implicit evidence may make φ inevitable, while we still lack direct evidence to assert φ. This intuitive distinction, hard to draw in classical logic, is reminiscent of the difference between $\sim\sim\varphi$ and φ in intuitionistic logic. (This observation, which will be clarified below, is even valid in a precise technical sense.)

A Modal Embedding

Even though the presentation of data semantics involves partial information, the above account evokes analogies with classical modal logic. And indeed, a reduction is possible, as sketched in the following five steps. Presumably, this type of argument can be made to work for a variety of partial semantics in the above vein.

First, we determine a modal logic with enough "locomotive power" to drive the reduction.

A. The modal logic of all partial orders with greatest elements for maximal chains is S4.1

Here, *S4.1* is the logic arising from the modal logic *S4* by adding the so-called McKinsey Axiom $\Box\Diamond\varphi \rightarrow \Diamond\Box\varphi$.

The soundness side of this assertion follows by a simple inspection: all *S4.1* axioms are valid on data structures, when the modal operators are interpreted in the usual way over the "alternative" relation $\sqsubseteq$. In particular, the McKinsey Axiom holds thanks to the presence of maximal elements.

Completeness is less immediate. *S4.1* is known to be complete with respect to those possible worlds frames that are pre-orders in which every node has a successor that is an end-point. This will not quite do here: such orders may still lack maximal elements for all maximal chains. But, a closer look at the actual proof techniques employed in Segerberg (1971) provides more information. As is shown there, *S4.1* nontheorems may be refuted in finite reflexive transitive frames, in which each upward path eventually ends in some reflexive end-point. Now, an application of modal "bulldozing" methods will turn such a counter-example into a partial order all of whose maximal chains have maximal elements.

Other structural stipulations on data models might have produced the even stronger modal logic *S4Grz*, or perhaps rather the weaker *S4* itself. Interestingly, in this perspective the modal logic of the earlier-mentioned Scott domains remains just *S4*. (The reason is, essentially, that one can use ascending omega-1-chains to model infinite upward alternations $\Box\Diamond p \,\&\, \Box\Diamond{\sim}p$ $(= {\sim}\Diamond\Box p)$, while obeying the letter of the omega (i.e., omega-0) completeness requirement.)

B. A translation from data formulas to modal formulas

With the understanding that $\Box$ will mean "*in all* $j \sqsupseteq i$," and $\Diamond$ "*in some* $j \sqsupseteq i$" (as is usual in modal semantics), we define a translation by simultaneous recursion:

$$
\begin{array}{llll}
p^{+} & = \Box p & p^{-} & = \Box{\sim}p \\
({\sim}\varphi)^{+} & = \varphi^{-} & ({\sim}\varphi)^{-} & = \varphi^{+} \\
(\varphi \,\&\, \psi)^{+} & = \varphi^{+} \,\&\, \psi^{+} & (\varphi \,\&\, \psi)^{-} & = \varphi^{-} \vee \psi^{-} \\
(\varphi \vee \psi)^{+} & = \varphi^{+} \vee \psi^{+} & (\varphi \vee \psi)^{-} & = \varphi^{-} \,\&\, \psi^{-} \\
(\varphi \to \psi)^{+} & = \Box(\varphi^{+} \to \psi^{+}) & (\varphi \to \psi)^{-} & = \Diamond(\varphi^{+} \,\&\, \psi^{-}) \\
(\mathrm{MAY}\,\varphi)^{+} & = \Diamond\varphi^{+} & (\mathrm{MAY}\,\varphi)^{-} & = \Box{\sim}\varphi^{+} \\
(\mathrm{MUST}\,\varphi)^{+} & = \Box\Diamond\varphi^{+} & (\mathrm{MUST}\,\varphi)^{-} & = \Diamond\Box\varphi^{-}.
\end{array}
$$

Actually, the specific clauses are irrelevant here. The main idea resides in the double recursion (an unusual feature in logical reductions), and the modal expressibility of the earlier truth conditions.

Now it remains to set up correspondences between the models for these two languages.

C. From data models to modal models

Given a data model $M = \langle I, \sqsubseteq, V\rangle$, set $M^* = \langle I, \sqsubseteq, V^*\rangle$, with $V^*(p) =$ (def) $\{i \text{ in } I \mid \text{either } V_i(p) = 1, \text{ or } V_i(p) \neq 0 \text{ and for some } j \sqsupseteq i : V_j(p) = 0\}$.

Claim: $M \models \varphi[i]$ iff $M^* \models \varphi^{+}[i]$
$M \mathrel{=\!\!\!|} \varphi[i]$ iff $M^* \models \varphi^{-}[i]$.

There is a story to this particular choice of V^*. It would seem more natural to have $V^*(p)$ simply be equal to $\{i \text{ in } I \mid V_i(p) = 1\}$. But then M^* could validate $\Box{\sim}p$ at some point i merely because of a lack of p-successors, something which is not sufficient for $V_i(p) = 0$ in data semantics.

The claim is proved by induction on φ. Since the translation has been designed exactly to mirror evaluation in data semantics (with one minor modification in the MUST-clause), the only nonevident case is that of the atomic formulas. The relevant reasoning goes as follows.

If $M \models p\,[i]$, then $V_i(p) = 1$, and hence $V_j(p) = 1$ for all $j \sqsupseteq i$. Therefore, $M^* \models \Box p\,[i]$. Conversely, if $M^* \models \Box p\,[i]$, then *a fortiori* $M^* \models p\,[i]$. That is, $V_i(p) = 1$ (and we are done), or $V_i(p) \neq 0$ and some $j \sqsupseteq i$ has $V_j(p) = 0$. But, the latter contradicts $M^* \models p\,[j]$ (implying $V_j(p) \neq 0$).

If $M \dashv p\,[i]$, then $V_i(p) = 0$ and $V_j(p) = 0$ for all $j \sqsupseteq i$. Therefore, $M^* \models \Box \sim p\,[i]$. Conversely, let $M^* \models \Box \sim p\,[i]$. Consider any maximal $j \sqsupseteq i$. Since p fails at j, and V_j was total, $V_j(p) = 0$. Now, suppose $V_i(p) \neq 0$. Then i would qualify as a member of $V^*(p)$ after all: *quod non*. Therefore, $V_i(p) = 0$.

A check for the MUST-case clinches the argument. Q.E.D.

Evidently, the existence of maximal points in data models plays an important role in the above argument, even in the atomic case. It would be of interest to extend the present reduction to the case where no such existence is postulated, something which ought to produce an embedding into modal *S4*.

D. *From modal models to data models*

Given an *S4.1* model $N = \langle I, \sqsubseteq, V\rangle$ of the kind obtained in step A, set $N' = \langle I, \sqsubseteq, V'\rangle$, with $V'_i(p) = 1$ if $N \models \Box p\,[i]$, and $V'_i(p) = 0$ if $N \models \Box \sim p\,[i]$. Again, an obvious inductive proof, this time also with an obvious base step, establishes the following:

Claim: $N' \models \varphi\,[i]$ iff $N \models \varphi^+\,[i]$
$N' \dashv \varphi\,[i]$ iff $N \models \varphi^-\,[i]$.

It now remains to reduce the corresponding logics.

E. *Strong consequence in data semantics reduced*

For a set $\varphi_1,\ldots,\varphi_n$ of premises and a conclusion ψ in data semantics, we have the following:

Theorem: ψ follows from $\varphi_1,\ldots,\varphi_n$ in data semantics if and only if ψ^+ follows from $\varphi_1^+,\ldots,\varphi_n^+$ in *S4.1*.

Proof: Counter-examples to inferences can be transformed both ways, as has been shown in steps *C* and *D*. Q.E.D.

This result does not diminish the interest of the above partial approaches as such. What it does provide is some direct information about their meta-theory. For instance, *S4.1* is known to be decidable, and hence so is strong consequence. Thus, traditional modal proof techniques and results remain useful in the new setting. But, the above reduction also shows that the traditional possible worlds semantics, despite the frequent accompanying "totalitarian" ideology, can model partiality after all. (Actually, a similar use of the modal system *S4* has been around for quite a while – witness the next sections.)

Much more could be said on the topic of "partial" versus "total" interpretation of classical modal frameworks. Suffice it to note here that, at least, Carnap's original use of finite state descriptions as possible worlds has a partial flavor. Philosophically, hard-headed reification of complete possible worlds has only been one current in the development of the subject.

3. *Possible-worlds Models as Information Models*

That the possible worlds set-up can also be interpreted partially has been evident since the sixties, when Kripke presented his modal semantics for intuitionistic logic, inspired by Gödel's translation of the latter into modal *S4*. (Cf. Fitting, 1969, for a comprehensive exposition.) Briefly models $\langle I, \sqsubseteq, V\rangle$ are now thought of as partially ordered "growth patterns" $\langle I, \sqsubseteq\rangle$, with pieces of information ("forcing conditions") attached to every mode i in I. Unlike the earlier approach, however, there is just one relation of truth ("forcing"), without the above "refutation." In a sense, Kripke's account is more radical, as the aspect of information growth now shows, even in the basic clause for negation:

$M \models \varphi \,\&\, \psi[i]$ if $M \models \varphi[i]$ *and* $M \models \psi[i]$
$M \models \varphi \vee \psi[i]$ if $M \models \varphi[i]$ *or* $M \models \psi[i]$
$M \models {\sim}\varphi[i]$ if *for all* $j \sqsupseteq i$, *not* $M \models \varphi[j]$ ("never").

In addition, implication is treated "modally":

$M \models \varphi \rightarrow \psi[i]$ if *for all* $j \sqsupseteq i$, $M \models \varphi[j]$ *only if* $M \models \psi[j]$.

What this semantics shares with the earlier Kamp set-up is Persistence (or "monotonicity"):

if a statement holds at i, it will continue to hold at all extensions $j \sqsupseteq i$.

This behavior will have to be stipulated for atomic formulas, of course, and it then extends to complex formulas. (Notice that the presence of additional modal operators may invalidate persistence in general, as is the case in data semantics.) An additional important feature of this approach is that traditional explications for logical constants are reassessed, and sometimes have to be modified, in a partial setting. (Eventually, one may also come to modify the explications given for & and $\vee$, employing further $\sqsubseteq$-extensions. Cf. van Benthem, 1984a.)

There is nothing intrinsically intuitionistic about this approach to partiality, however:

Possible-worlds Semantics for Classical Logic

A plausible "partial" criticism of current completeness proofs (Henkin style) in standard logic objects to the use of an arbitrary "maximally consistent" extension of a given consistent set, in order to produce a model for the latter. (For an exposition, compare any standard introductory text in mathematical logic.) But, in practice, this additional information is never used, and hence one would prefer a cleaner model construction, without this arbitrary indeterminacy. (Cf. van Benthem, 1981.)

Now, the usual answer here is that merely consistent sets of formulas do not "decompose" their statements in the recursive way required of a potential "truth set," whereas maximally consistent sets are precisely the ones which do. But, this response loses force once one considers the natural environment in a partial perspective, viz. the complete universe of all consistent theories, ordered by inclusion. For, there, the following reductions do apply for provability:

$S \vdash \varphi \,\&\, \psi$ iff $S \vdash \varphi$ *and* $S \vdash \psi$
$S \vdash \sim\varphi$ iff *for all* $S' \supseteq S$, *not* $S' \vdash \varphi$
$S \vdash \varphi \to \psi$ iff *for all* $S' \supseteq S$, $S' \vdash \varphi$ *only if* $S' \vdash \psi$.

The reduction for disjunction is somewhat more complicated (eventual, as opposed to immediate choice):

$S \vdash \varphi \vee \psi$ iff *for all* $S' \supseteq S$, *there exists* $S'' \supseteq S'$ with *either* $S'' \vdash \varphi$ *or* $S'' \vdash \psi$.

(Compare the equivalence $\varphi \vee \psi \leftrightarrow \sim(\sim\varphi \& \sim\psi)$.) Classical logic does not enforce on the spot decisions at gun-point, as happened in the earlier intuitionistic clause for disjunction.

Thus, the Henkin model of all consistent theories itself forms one unique canonical model verifying consistent theories at their own appropriate stages, when reinterpreted as a possible worlds model M in the obvious way. (More precisely, we will have $M \models \varphi[S]$ if and only if $S \vdash \varphi$, for all φ and S.) A little care is needed to make the right abstraction from here, however. One wants models $\langle I, \sqsubseteq, V\rangle$ with a partial order $\langle I, \sqsubseteq\rangle$ upon which evaluation takes place according to the above scheme. But, to obtain classical logic, two constraints are to be enforced (starting with the assignment V.) One condition is *Persistence*, as so often before. The other is *Stability* (or "cofinality"):

if for all $j \sqsupseteq i$ there exists some $k \sqsupseteq j$ where φ holds, then φ holds already at i itself;

or, equivalently,

if φ does not hold at i, then $\sim\varphi$ holds at some $j \sqsupseteq i$.

In other words, lacking the information that φ, there must still be the possibility of obtaining further information excluding φ altogether. This principle has been proposed by various advocates of partial semantics (cf. Humberstone, 1981).

Simple though it is, this Henkin model forces one to reflect upon certain semantic options in modelling partiality. For instance, the consistent theories may explicitly embody both atomic information ("facts") and higher information ("generalizations"). In general, this seems to be quite a realistic view of what information people possess, both "factual" and "higher-order." But, if desired, one could also allow direct access only to the facts – all higher information remaining encoded, so to speak, in the pattern of possible extensions.

Another relevant issue is that of direct and indirect evidence. Instead of deductively closed consistent theories, one might consider all consistent sets of sentences; doing so creates a distinction between "direct" evidence – $\varphi \in S$ – and "indirect" ("derived") evidence – $S \vdash \varphi$. The latter notion can be rewritten as

for all $S' \supseteq S$, there exists $S'' \supseteq S'$ with $\varphi \in S''$.

Notice that "growth" now involves both additional information and fresh consequences of old information. If one defines a valuation on this Henkin model, setting $V(p) = \{S \mid p \text{ in } S\}$, while defining the logical constants as above, there will be no direct equivalence between provability and truth at S, but rather

$$M \models \sim\sim\varphi[S] \quad \text{iff} \quad S \vdash \varphi.$$

A Modal Henkin Model

Many of the above clauses have a modal ring. So, let us introduce two operators $\Box$ ("in all extensions") and $\Diamond$ ("in some extension"), to make this modality explicit. Thus, our language can now also express consistency of its statements. Moreover, Persistence may be formulated as the validity of the implication from φ to $\Box\varphi$, while Stability amounts to the transition from $\Box\Diamond\varphi$ (i.e., $\sim\sim\varphi$) to φ.

This simple addition makes the logic of the model much more complex. In general, the truth relation $S \models \varphi$ can no longer be recursively enumerable; as, for example $\emptyset \models \Diamond\varphi$ if and only if φ is consistent, for predicate-logical formulas φ. (For the case of propositional logic, the complexity may remain manageable.) Still, one can study the modal logic of this scheme, consisting of the inferences valid at each point. There are some interesting phenomena here. For the operator $\Box$, the logic becomes at least *S4*, with, for example, $\Box(\varphi \& \psi) \leftrightarrow \Box\varphi \& \Box\psi$, $\Box\varphi \rightarrow \varphi$, and $\Box\varphi \rightarrow \Box\Box\varphi$. But, $\Diamond$ is no longer its dual: $\Diamond\varphi \leftrightarrow \sim\Box\sim\varphi$ is not a valid principle; and indeed, $\Diamond$ fails to satisfy even the basic minimal modal law $\Diamond(\varphi \vee \psi) \leftrightarrow \Diamond\varphi \vee \Diamond\psi$. (The reason is that, on the above classical account of disjunction, $\vee$ is upward preserved, whereas $\Diamond(\varphi \vee \psi)$ need not be.) Actually, this difficulty will disappear once we take $\vee$ to be the earlier intuitionistic disjunction. (Cf. the corresponding move in Gabbay, 1982.)

Evidently, there are also attractive structural variations on the above theme. For instance, the above observation about nonrecursive enumerability of truth at stages seems a bit "extraneous." It would be interesting to devise some kind of relative computability result, separating traditional complexity problems from those (if any) induced by the Henkin structure itself. More radically, changes in the latter structure might be studied, once the underlying system of deduction is allowed to vary from classical logic to weaker variants. With these speculations, we leave the above Henkin model as an object for further study.

Finally, if a connection is to be established with the traditional Henkin approach employing maximally consistent sets, one may study the maximal chains through the Henkin model, and associate models with these. (Cf. the corresponding idea of a maximal search in data semantics.) Again, the relevant questions of total representation for partial stages will not be addressed here.

Nonmonotonicity

Classical logic seems to obey the maxim of "what we have, we hold". Likewise, situation semantics stresses the persistence of information once obtained. This close connection between classical inference and monotonicity is studied in van Benthem (1984b), where (in a suitable generalized quantifier setting) classical entailment is shown to be the only notion of inference which allows strengthening of antecedents with weakening of consequents.

On the other hand, classical model theory does not hold that statements true in a model would hold in all extensions of that model. Neither does classical modal *S4* imply that if $\Diamond\varphi$, then $\Box\Diamond\varphi$ for all statements φ. In other words, nonmonotonicity is a term with various senses, and hence some distinctions are needed.

4. *Varieties of Nonmonotonicity*

In a logical semantics, three levels of monotonicity may be distinguished. First, at the indices of evaluation, one may have

I. $i \models \varphi, i \sqsubseteq j$ only if $j \models \varphi$.

Then, in the object-language, an implication connective may satisfy the law

II. $\varphi \to \psi \models \varphi \,\&\, \chi \to \psi$ (strengthening of antecedents).

Finally, in the meta-language, a notion of consequence may obey addition of premises:

III. $\varphi \models \psi$ only if $\varphi, \chi \models \psi$.

For example, the nonmodal fragment of the above data semantics exhibits all three forms of monotonicity. However, the modal logic *S4*, with its usual notions of (object-level) entailment and (meta-level) semantic consequence, has only II and III. And other types of behavior have occurred too. The counterfactual implication of Lewis' possible worlds semantics fails to obey II, while, say, the original notion of logical consequence found in Bolzano did not obey III (cf. van Benthem, 1984c). Clearly, then, cumulation of information has not always been taken for granted in the logical tradition.

Nevertheless, few logical studies have investigated nonmonotonicity as such. And yet, quite natural considerations induce a nonmonotone (in sense I) version of classical logic, employing the partial perspective of the above Section 3. The following results have a certain analogy with the basic nonmonotonic features of the earlier "data logic" (again, cf. Veltman, 1981), but the presentation given here shows that these features already arise wholly within a classical logical framework.

Consider the propositional language with &, ~, and → only, interpreted as before in models $\langle I, \sqsubseteq, V\rangle$ with a partial order $\langle I, \sqsubseteq\rangle$. As we have seen, classical logic required the additional constraints of Persistence and Stability, while intuitionistic logic still retained Persistence. What this perspective suggests, evidently, is a "minimal logic" without either presupposition.

Theorem: The minimal logic of &, ~, and → is given by the following principles:

(1) the usual "structural" rules for $\vdash$, such as reflexivity, transitivity, and (sense III) monotonicity with respect to antecedents
(2) $\varphi \,\&\, \psi \vdash \varphi$; $\varphi \,\&\, \psi \vdash \psi$; $\varphi, \psi \vdash \varphi \,\&\, \psi$
(3) $\varphi, \varphi \to \psi \vdash \psi$; if $P, \varphi \vdash \psi$, then $P \vdash \varphi \to \psi$, provided that P is a set of premises consisting of negations and implications only
(4) $\varphi, \sim\varphi \vdash \psi$; if $P, \varphi \vdash$ *false*, then $P \vdash \sim\varphi$, with the same proviso.

Comments: First, negation may be regarded as a special case of implication, with $\sim\varphi$ = (def) $\varphi \rightarrow$*false*. (Here, *false* is assumed to imply everything; as usual.) Thus, principle (4) becomes a consequence of (3), and we may omit it henceforth, concentrating on conjunctions and implications (with the *false*). Now, the only difference between the logic presented here and the usual calculus for "minimal logic" resides in the Conditionalization rule: in general, here, conditionals can only be introduced in persistent contexts.

In fact, this may be the proper setting to observe that, by the above result, unrestricted Conditionalization is actually equivalent to Persistence – a result implicit in Gabbay's proof that intuitionistic logic is the weakest logic satisfying the "Deduction Theorem".

Proof: A formula φ is persistent everywhere if and only if the following inference is valid: $\varphi \vdash true \rightarrow \varphi$ (with *true* any universal validity). Now, if Conditionalization holds generally, the latter is a trivial consequence of $\varphi \vdash \varphi$; φ, $true \vdash \varphi$ – and hence Persistence holds. Conversely, if every formula is persistent, then Conditionalization is valid. For, suppose that $P, \varphi \vdash \psi$; we need to show that $P \vdash \varphi \rightarrow \psi$. Let P hold at any stage i. Consider any $j \sqsupseteq i$ where φ holds. By Persistence, P still holds at j, and it follows that ψ holds there, by the assumption. Q.E.D.

Proof of the Theorem: Obviously, all principles mentioned are valid. (Notice the role of the proviso on Conditionalization: only negations and implications are automatically persistent.)

Conversely, suppose that $\varphi_1, \ldots, \varphi_n \nvdash \psi$. Consider the Henkin model of all deductively closed consistent sets of formulas, with a relation defined by

$$S_1 \sqsubseteq S_2 \quad \text{if} \quad \text{all } \rightarrow\text{-formulas in } S_1 \text{ are also in } S_2.$$

Thus, S_2 agrees with S_1's long-range predictions, while perhaps disagreeing on atomic facts. This relation is a reflexive transitive order. With a valuation V defined by $V(p)$ = (def) $\{S \mid p \text{ in } S\}$, one now arrives at the basic Truth Lemma:

$$\langle I, \sqsubseteq, V\rangle \models \varphi[S] \quad \text{iff} \quad \varphi \text{ is in } S.$$

Proof: For proposition letters, and for the &-case, the assertion is obvious. For $\varphi = \psi \rightarrow \chi$, the argument is as follows. If $\psi \rightarrow \chi$ is in S, and $S \sqsubseteq S'$, then $\psi \rightarrow \chi$ is in S' (definition of $\sqsubseteq$). Now, if ψ is true at S', then ψ is in S' (inductive hypothesis), and hence χ is in S' (deductive closure of S'): that is, χ is true at S' (again by the inductive hypothesis). So, $\psi \rightarrow \chi$ is true at S. If, however, $\psi \rightarrow \chi$ is not in S, then $\{\alpha \rightarrow \beta \mid \beta \text{ in } S\} \cup \{\psi\}$ does not derive χ (otherwise, $\psi \rightarrow \chi$ would be in S after all, by the special Conditionalization rule), and hence its deductive closure is a counter-example to the truth of $\psi \rightarrow \chi$ at S. Q.E.D.

Thus, in the deductive closure S of the above original premises $\varphi_1, \ldots, \varphi_n$, ψ will become refuted.

The counter-example obtained in this way is not quite correct yet, as $\sqsubseteq$ is not necessarily a partial order: it might lack anti-symmetry. But, standard modal "unraveling" techniques will transform the counter-example obtained into a partially ordered one. Q.E.D.

This proof does not answer all questions of semantic interest concerning the above logic. Notably, the relational structures produced by the unraveling method may be quite exotic infinite partial orders. One obvious issue, then, is whether some smoother class of models suffices. For instance, it is known that the earlier intuitionistic logic needs only *finite* partial orders for its models. But, that is not the case here.

Example: The following principle is valid on all finite partial orders:

$$\sim\sim p \rightarrow \sim\sim(\mathit{true} \rightarrow p).$$

It is not derivable in the minimal logic, however, being refutable in an ascending omega-sequence with infinite alternation of truth and falsity for p. (Compare the closely related McKinsey Axiom of S4.1.)

All the same, when "nonstandard" models are admitted with possible loops, it turns out that finite models suffice after all. This may be seen by embedding our language, with its intended semantics, into that of the modal logic *S4* – as was also possible in the intuitionistic case. It then follows that our minimal logic is even *decidable*.

Another question, not to be answered here, remains. Can the minimal logic be modelled using partial orders having chains of at most order type omega?

Summarizing then, the basic logic of our earlier partial perspective is nonmonotone in sense I. Further logics arise by concentrating upon special classes of statements, satisfying additional preservation properties – of which Persistence and Stability are only two examples.

One can certainly think of other attractive candidates. (Close to the above interests, it might be useful to determine the logic of just Stability, in the earlier form "$\Box\Diamond\varphi \to \varphi$," or its interesting variant "$\Box\Diamond\varphi \to \Diamond\Box\varphi$" proposed by Roeper; cf. van Benthem, 1984a.)

Next, one may want to extend the above investigation to richer languages. For instance, *disjunction* could be added by the earlier classical definition ($\Box\Diamond(\varphi \vee \psi)$). In the absence of Persistence, curious phenomena arise then. For instance, not even $\varphi \models \varphi \vee \psi$ will be valid: as was noted before, the conclusion is persistent, whereas the premise might be valid at just this one stage. In general, once Persistence is given up as a general principle, one has to be extremely careful with traditionally "evident" inferences.

Thus, there is a good case for considering two kinds of disjunction here: "choice disjunction" (one of the disjuncts right now) and "eventual choice" as above. (Again, in the absence of Persistence, these have become mutually independent notions.) In the resulting logic, it is choice disjunction which will still obey its two ordinary natural deduction rules.

But, then, there is also a case for adding *two* negations, one strong ($\sim$ as above) and one weak, expressing mere absence of verification at the present stage. A simple modification of the above completeness argument will produce the new minimal logic. (Maximally consistent sets of formulas are needed this time. Notice that the growth relation $\sqsubseteq$ will certainly not be set-theoretic inclusion now: weak negations may disappear when knowledge grows.) The additional rules obtained are the classical natural deduction rules for weak negation $\neg$, together with the "interaction principle" $\sim\varphi \vdash \neg\varphi$.

By way of check upon this outcome, one may notice that the double negation $\sim\neg$ defines the earlier necessity operator $\Box$ ("in all $\sqsubseteq$-extensions"), and then derive the usual *S4*-axioms for it from the above rules.

Example: A derivation of $\Box\varphi \to \Box\Box\varphi$, that is, $\sim\neg\varphi \to \sim\neg\sim\neg\varphi$:

$$\sim\neg\varphi,\ \neg\sim\neg\varphi \vdash \textit{false}$$
$$\sim\neg\varphi \vdash \neg\sim\neg\varphi \to \textit{false}$$
$$\sim\neg\varphi \vdash \sim\neg\sim\neg\varphi$$
$$\vdash \sim\neg\varphi \to \sim\neg\sim\neg\varphi.$$

This completes our exploration of the minimal classical logic without the presupposition of monotonicity. There are many obvious further questions to be asked in the above spirit, of course.

By way of conclusion, another distinction should be mentioned, however, one found in Moore, 1983. The above failures of monotonicity usually have a modal flavor: as our information grows, epistemic possibilities disappear, and hence statements expressing these possibilities are lost. This is the "auto-epistemic" source of nonmonotonicity. In addition, however, there are also nonmonotonic effects of "default reasoning"; that is, turning absence of information, temporarily, into additional (negative) premises. For instance, in logical semantics, the role of ceteris paribus clauses in current theories of counterfactuals would fall under the latter heading.

Right now, the perspective of this report has nothing to offer for the latter case, which seems to involve nonmonotonicity at levels II and III, rather than I.

Indeed, nonmonotonicity at level I is perfectly compatible with monotonicity at level III, as it is still entirely up to us how we want to define the notion of consequence. (Recall the earlier observation about Bolzano versus Tarski consequence at the beginning of this section.) Nevertheless, there may be good reasons for investigating non-Tarskian, nonmonotone alternatives at level III. This topic has been neglected in mainstream logic, though not in the more methodological areas of the philosophy of science, where Carnap and others have studied the great variety of notions of "confirmation," "derivation," "explanation," etcetera, occurring in scientific, and ordinary reasoning. (Cf. van Benthem, 1984c, on this point.)

For instance, the analogy with standard deduction may be mistaken in studies of default reasoning, whose explications in terms of "minimal (closed) models" are more reminiscent of semantic analyses of confirmation, as given by Hempel and other philosophers of science.

Even so, the logical study of nonmonotonicity at level III, or more generally, of the variety of "argumentative connections" certainly deserves closer attention.

Postscript

One obvious limitation of this study has been the omission of quantificational logic. For instance, once quantifiers over individuals are introduced, issues of Persistence soon become more problematic (witness Barwise and Perry, 1983). Nevertheless, our claim about the utility of classical paradigms is not invalidated here. For instance, it is very easy to extend the earlier Beth tableau analysis to cover strong consequence in predicate logic. Similarly, predicate-logical Henkin models can be used to obtain "partial" decomposition clauses for quantified statements as well. And finally, modal predicate logic is still an excellent testing ground for the difficulties that arise for any theory coming to terms with the notion of partial information supporting or refuting statements of generality.

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