INTENSIONAL LOGIC AND SEMANTIC VALUE GAPS

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According to Frege and Carnap, a well-formed expression of a language may have two semantic values called *Sinn* and *Bedeutung* by Frege [1] and mentioned as *intension* and *extension* by Carnap [2]. In our days, Carnap's terminology is videly accepted (sometimes 'reference' is used instead of 'extension'). In most cases, 'meaning' and 'intension' are used synonymously, although in some writings the term 'intension' refers only to the set-theoretic representation of meaning (a function from possible worlds into a set of extension). I shall use the term 'factual value' instead of 'extension' (since it seems to me somewhat perverse to speak of *the extension* of a sentence or an individual name). Of course, by the *factual value* of an individual term I mean the object (if any) denoted by the term and by the factual value of a (declarative) sentence I mean its truth value (if it has one). The factual value of any extensional functor (including predicates) is assumed to be a function (in the set-theoretical sense) which may be called its *extension*.

It may happen that a well-formed (meaningful) expression of a language has no factual value. I shall use the term 'semantic value gap' to refer to this phenomenon. The simplest case of a semantic value gap is perhaps a definite description without an actual denotation. If such a term occurs in a sentence in a *de re* position – as in the traditional example 'The present king of France is bald' –, then the sentence has no truth value (at least, it is claimed to be so by Frege [1]). This is an example of the *truth value gap*.

However, semantic value gaps are not restricted to names and sentences. Some philosophers and some linguists argue that there are predicates which are *undefined* for some objects. For example, colour predicates are undefined for numbers, the predicate 'ruminant' is inapplicable to inanimate physical objects, and mathematical predicates (such as 'has a quadratic divisor') are undefined for physical objects. Even in the language of a science, some predicates and operations are partial ones: Think of division in arithmetic, square root operation in the field of real numbers, limit operations, differentiation and integration of functions in analysis. Accepting this view, we have new sources of semantic value gaps. Moreover, we can distinguish *the emptiness of the extension* and *the lack of the extension* of a monadic predicate. For example, if our domain of individuals is the set of the natural numbers, then the extension of the predicate 'even prime number greater than two' is empty, whereas the predicate 'green' – being totally undefined for numbers – has no extension at all. Similarly, the predicate 'child of Chronos' has no extension in worlds other than that of Greek mythology.

However, since Frege, there has been a constant effort to drive out semantic value gaps from the realm of logic. In my view, this is the wrong policy. Firstly, it seems that the appearance of semantic value gaps is a real phenomenon. Secondly – and this is the most important argument – a logical system permitting semantic value gaps provides a natural means for a fine differentiation of meanings. To begin with simple examples, we all have learned that pairs of tautologies such as "$A \supset A$" and "$B \supset B$" are not distinguishable in logic, both being true in all logically possible worlds (and at all moments of time), and hence, their intensions must coincide. Similarly, sentences of the form A and "$A \& (B \vee \sim B)$" are logically synonymous: their truth values coincide in all logically possible worlds. Most linguists are not content with these results of logic. They argue that there *is* a significant difference between the meanings of these sentences. Now assume that there is a logically possible world w in which sentence A has a truth value but B has none. Then – assuming that the truth value gap is hereditary via truth functions – "$B \supset B$" and "$A \& (B \vee \sim B)$" have no truth value in w, but A is true or false and "$A \supset A$" is true in w. By this, our linguists are satisfied: it is possible that "$A \supset A$" is, but "$B \supset B$" is not, true in a world w, and hence, they are not synonymous. Similarly for the pair A and "$A \& (B \vee \sim B)$". (Of course, the case that "$B \supset B$" is false, or that A is true and "$A \& (B \vee \sim B)$" is false remains impossible.) As a consequence, it is not automatically guaranteed that A and "$A \& (B \vee \sim B)$" are interchangeable *salva veritate* in all contexts. (They are surely not so as arguments of intensional functors such as 'thinks that', 'sees' etc.)

A more striking example: Are the following tautological sentences synonymous?

Bill likes or does not like steak.
Steak likes or does not like Bill.

In general: are sentences of the form

"$Fab \vee \sim Fab$" and "$Fba \vee \sim Fba$"

synonymous? If we admit the possibility that the two-place predicate F is defined (true or false) for the couple (a, b) but is undefined for (b, a), our answer is NO. Here lies the advantage of permitting partial functions as factual values (extensions) of predicates.

As an objection against accepting partially defined predicates, one might say that it is highly uncertain to limit the domain in which a predicate is defined (is true or false). Most people perhaps agree that 'ruminant' is not defined for inert objects, but even these might be confused in answering the question whether this predicate is applicable to amoebas or protozoa. It seems to be a metaphysical dogma that every predicate P has itw own 'applicability domain', a domain of objects such that P is true or false of them. Is this a stronger dogma than the assumption that every predicate has a clear-cut truth domain? If you think that the latter is not a metaphysical dogma, but an unavoidable idealization for practising logic, then anyone who (like myself) wishes to maintain the use of partially defined predicates will answer that this is another (hardly avoidable, but at any rate, useful) idealization for the purpose of *better* practising logic. Moreover, we can assume that in our actual world, every predicate is totally defined (applicable to all assumed existing objects). To enjoy the benefits of a logic with semantic value gaps it is sufficient to assume that there are possible worlds in which the factual values of some functors are partial functions. That is, you are requested to accept only *the possibility* of partial functions, leaving open the problem of the actual existence of such functions. However, I should like to stress that the notion of partially defined predicates originates from a linguistic intuition, and hence, it does not lack a real, empirical base.

In what follows, I shall outline a semantic system of tensed intensional logic with factual value gaps. My first approach to this subject was presented at a workshop 1979 (this was published in Hungarian 1980, a Russian version is available in [3]). A somewhat modified version in English [4] appeared in Studia Logica 1981. In the present paper, the basic ideas are the same as in my first approach 1979, but the details are refined and simplified in several respects. Before the systematic exposition, it may be appropriate to make a few informal, preliminary remarks.

Concerning type theory, my point of departure is the usual system of extensional types. Thus, o and ι are extensional types – the types of sentences and individual terms, respectively – and if α and β are extensional types, so is $(\alpha\beta)$. However, I shall not introduce a new symbol 's' for senses – as R. Montague did in [5] – for creating new types of *names of intensions*. Instead, I shall distinguish extensional and intensional functor types (the latter will be called *operator types*). The latter are defined as follows: if α, $\beta_1,\ldots, \beta_k$) are extensional types ($k \geqslant 1$), then $(\alpha;\ \beta_1;\ldots;\ \beta_k)$ is an operator type. In this way, I shall avoid the unlimited iteration of intensions which endows Montague's system with a highly platonic character. The intuitive difference between a functor belonging to type $(\alpha\beta)$ and an operator belonging to type $(\alpha;\beta)$ is, as one might guess, that the first one operates on the *factual value*, and the second one on the *intension*, of its argument.

As primitive logical symbols, I shall use (besides parentheses) the lambda operator (λ), the identity sign (=), the descriptor (**I**) for forming names from monadic predicates, the intensor (^) for forming names of propositions from sentences, and two temporal operators '**since**' and '**till**'. Other connectives ($\sim$, &, $\vee$, $\supset$), quantifiers, modal operators, and the usual past and future tense operators (**P**, **F**) can be introduced by definitions.

An intensional language may contain *nonlogical constants* in all extensional and operator types. On the other hand, *variables* will be permitted only in the extensional types. But I shall use two sorts of variables in all extensional types called *extensional* and *intensional* variables, respectively. In the metalanguage, I shall refer with Roman letters to the extensional variables, and with Greek letters to the intensional ones. The main difference of the two sorts of variables is as follows: an expression of form "$(\lambda x\ A)$" is an extensional functor, whereas "$(\lambda \xi\ A)$" is an operator (an intensional functor). Thus, if A belongs to type α, and x and ξ belong to type β, "$(\lambda x\ A)$" belongs to type $(\alpha\beta)$, whereas "$(\lambda\xi\ A)$" belongs to the operator type $(\alpha;\beta)$.

An identity "$(A = B)$" will be accepted as well-formed only if the expressions A and B belong to the same *extensional* type. Quantification will be defined by means of λ-abstraction and identity. As a consequence, *only extensional variables* (of the extensional types) *are quantifiable*. Intensional variables will be proved to be eliminable from closed terms and formulas. This means that intensional variables are only auxiliary tools which are useful, for instance, for expressing incomplete natural language expressions, but they disappear as soon as the full sentence is built up.

The semantics begins in the usual way. We shall provide for all extensional types α a domain $D(\alpha)$ of factual values and a domain $Int(\alpha) = {}^{I}D(\alpha)$ of intensions where I is an index set of form $W \times T$, W is the set of worlds, and T is the set of time moments. (I use '${}^{B}A$' to denote the set of functions from B into A.) As regards operator types $Int\ (\tau;\beta)$ will be defined as ${}^{Int(\beta)}Int(\tau)$, where β is an extensional type. Furthermore, a function d defined on W will provide the set of actual individuals for all worlds $w \in W$ (with the proviso $d(w) \subseteq D(\iota)$). Quantification in type ι will be restricted to $d(w)$ at index $i = (w, t)$ (a remarkable difference from Montague's intensional logic).

The factual value of an extensional functor of type $(\alpha\beta)$ will be a partial function from $D(\beta)$ to $D(\alpha)$. I assume that the factual value gap is *hereditary* in extensional contexts; i.e., if A and B are well-formed extensional terms, some occurrence of B in A does not lie in the scope of an intensional operator in A, and B is without factual value at an index i, then so is A. The semantic rules are in accord with this assumption. Of course, the value gap need not be hereditary via intensional operators. In the formal semantics, all value gaps will be filled in by distinguished elements called *zero etities*. For each extensional type α, its domain $D(\alpha)$ will contain a zero entity denoted by $\Theta(\alpha)$.

If A is a sentence, "$^\wedge A$" is a term denoting its intension and belonging to type ι. In accordance with this syntactic rule, the domain $D(\iota)$ will include sentence intensions as well. The set of actual individuals of the world w – denoted by $d(w)$ – always includes all sentence intensions, but it is possible that it contains no other (real) objects.

In the last section of this paper, I shall mention an extension of the system by accepting "$(A = B)$" as well-formed in the case when A and B belong to the same *operator* type. In this way, quantification of *intensional* variables is definable. Then, our ontological commitment will be somewhat higher than before, but it still remains far from that of Montague's system. Iteration of intensions remains impossible. (Other interesting extensions of the system will not be treated here.)

§ *1. Type theory*

1.1. Type symbols. I shall use 'o' (omicron) and 'ι' (iota) for denoting the logical type (or category) of the (declarative) *sentences* (formulas) and (individual) *names* (terms), respectively. If α and β are type symbols, I shall use "$(\alpha\beta)$" for denoting the type of *extensional* functors which, combined with an argument of type β, yield an expression of type α. Finally, "$(\alpha;\beta)$" will denote the type of *operators* (intensional functors) which might be combined with arguments of type β to yield an expression of type α. More formally and exactly:

The set EXTY is the smallest set of symbols such that:

(i) $o, \iota \in$ EXTY and
(ii) $\alpha, \beta \in$ EXTY $\Rightarrow$ "$(\alpha\beta)$" $\in$ EXTY.

And the set OPTY (of operator types) is the smallest set of symbols such that

(iii) $\alpha, \beta \in$ EXTY $\Rightarrow$ "$(\alpha;\beta)$" $\in$ OPTY and
(iv) ($\tau \in$ OPTY and $\beta \in$ EXTY) $\Rightarrow$ "$(\tau;\beta)$" $\in$ OPTY.

Finally, the set of all type symbols is to be

$$\text{TYPE} =_{df} \text{EXTY} \cup \text{OPTY}.$$

In what follows, I shall omit the outermost parentheses surrounding type symbols.

1.2. By a *type-theoretical structure* let us mean an eight-tuple

$$S = (U, W, T, <, D, \Theta, Int, d)$$

where U, W, T are nonempty sets, $<$ is an ordering on T, D and Θ are functions with domain EXTY, Int is a function defined on TYPE, d is a function from W, and the conditons (i) to (vi) below are fulfilled. (Intuitively: U is a set of individuals, W is a set of (labels of) possible worlds), T is a set of time moments, $<$ is the relation *earlier than* (between time moments), the function D provides a domain of objects for all extensional types, Θ selects an object called zero entity from each of these domains, *Int* provides a domain of intensions for all types, and the function d provides a domain of individuals for each world $w \in W$.)

(i) T is a set linearly ordered by $<$.

(ii) $D(o) = \{0, 1, 2\} = 3$, and $\Theta(o) = 2$. (Here '0' and '1' represent the truth values falsity and truth, respectively, and '2' represents the truth value gap.)

(iii) $D(\iota) = U \cup {}^I 3 \cup \{U\}$ where

$$I = W \times T,$$

and $\Theta(\iota) = U$. (The members of U may be regarded as primitive "real" individuals, and the members of ${}^I 3$ may be called sentence intensions.)

(iv) If $\alpha, \beta \in$ EXTY,

$$D(\alpha\beta) =_{\mathrm{df}} \{f \in {}^{D(\beta)}D(\alpha) : f(\Theta(\beta)) = \Theta(\alpha)\},$$

and let $\Theta(\alpha\beta)$ be the function such that for all $b \in D(\beta)$,

$$\Theta(\alpha\beta)(b) = \Theta(\alpha).$$

(v) If $\alpha \in$ EXTY, $Int(\alpha) = {}^I D(\alpha)$. If $\tau \in$ TYPE and $\beta \in$ EXTY, then

$$Int(\tau;\beta) = {}^{Int(\beta)}Int(\tau).$$

(vi) For all $w \in W$, ${}^I 3 \subseteq d(w) \subseteq U \cup {}^I 3$.

§ 2. *Grammar*

By an *intensional language* let us mean a quintuple

$$L^{\mathrm{int}} = (LC, Var, Con, Op, Cat)$$

satisfying the following conditions (G1) to (G5):

(G1) LC is the set of the *logical constants* of L^{int}:
$LC = \{(,), \lambda, =, \mathbf{I}, {}^\wedge, \mathbf{since}, \mathbf{till}\}$.

(G2) Var is the set of *variables* of L^{int}:
$Var = \bigcup_{\alpha \in \mathrm{EXTY}} (Var^{\mathrm{ext}}(\alpha) \cup Var^{\mathrm{int}}(\alpha))$

where $Var^{ext}(\alpha) = \{x_{\alpha n} : n \in \omega\}$ and $Var^{int}(\alpha) = \{\xi_{\alpha n} : n \in \omega\}$.

(G3) *Con* is the set of (nonlogical) *extensional constants* of L^{int}:
$Con = \bigcup_{\alpha \in \mathrm{EXTY}} Con(\alpha)$

where $Con(\alpha)$ is a (possibly empty) denumerable set of symbols called *constants* of type α.

(G4) *Op* is the set of (nonlogical) *operators* (intensional constants) of L^{int}:
$Op = \bigcup_{\tau \in \mathrm{OPTY}} Op(\tau)$

where $Op(\tau)$ is a (possibly empty) denumerable set of symbols called *operators* of type τ.

(G5) *Cat* is the set of the well-formed expressions of L^{int}:
$Cat = Cat.ext \cup Cat.int$,

$Cat.ext = \bigcup_{\alpha \in \mathrm{EXTY}} Cat(\alpha)$, $Cat.int = \bigcup_{\tau \in \mathrm{OPTY}} Cat(\tau)$, where the sets $Cat(\alpha)$ and $Cat(\tau)$ are inductively defined by the items (S1) to (S7) below. For the sake of brevity, the category of an expression A will be indicated by writing "A_α" where α is a type symbol.

(S1) $\alpha \in \mathrm{EXTY} \Rightarrow Var^{ext}(\alpha) \cup Var^{int}(\alpha) \cup Con(\alpha) \subseteq Cat(\alpha)$,
and $\tau \in \mathrm{OPTY} \Rightarrow Op(\tau) \subseteq Cat(\tau)$.
(S2) $\alpha, \beta \in \mathrm{EXTY} \Rightarrow$ "$C_{\alpha\beta}(B_\beta)$" $\in Cat(\alpha)$, and
$(\tau \in \mathrm{TYPE}, \beta \in \mathrm{EXTY}) \Rightarrow$ "$C_{\tau;\beta}(B_\beta)$" $\in Cat(\tau)$.
(S3) $\alpha, \beta \in \mathrm{EXTY} \Rightarrow$ "$(\lambda x_\beta A_\alpha)$" $\in Cat(\alpha\beta)$, and
$(\tau \in \mathrm{TYPE}, \beta \in \mathrm{EXTY}) \Rightarrow$ "$(\lambda \xi_\beta A_\tau)$" $\in Cat(\tau;\beta)$.

(Here $x \in Var^{ext}(\beta)$, and $\xi \in Var^{int}(\beta)$.)

(S4) $\alpha \in \mathrm{EXTY} \Rightarrow$ "$(A_\alpha = B_\alpha)$" $\in Cat(o)$.
(S5) "$IA_{o\iota}$" $\in Cat(\iota)$.
(S6) "${}^\wedge A_o$" $\in Cat(\iota)$.
(S7) "$(A_o$ **since** $B_o)$" $\in Cat(o)$, "$(A_o$ **till** $B_o)$" $\in Cat(o)$.

Free and *bound* occurrences of a variable in a term are distinguished as usual. Also, *open* and *closed* terms are defined in the canonical way.

§ 3. *Semantics*

3.1. By a *projection* (or *interpreting function*) of L^{int} into a type-theoretical structure S let us mean a function σ on $Con \cup Op$ such that:

(i) if $C \in Con(\iota)$, $\sigma(C) \in U \subseteq D(\iota)$,
(ii) if $C \in Con(\alpha)$, $\alpha \neq \iota$, then $\sigma(C) \in Int(\alpha)$, and
(iii) if $C \in Op(\tau)$, $\sigma(C) \in Int(\tau)$.

Remark. The constants of type ι are to be considered as rigid terms (like proper names); this is the reason of the difference between clauses (i) and (ii). Non-rigid individual terms may be expressed as descriptions, see later on.

3.2. By an *assignment* (or *valuation*) of (the variables of) L^{int} in the structure S let us mean a function v on Var such that for all $\alpha \in$ EXTY, if $x \in Var^{\text{ext}}(\alpha)$, then $v(x) \in D(\alpha)$, and if $\xi \in Var^{\text{int}}(\alpha)$, then $v(\xi) \in Int(\alpha)$. Givel L^{int} and S, let us denote by $S(V)$ the set of all assigments of Var in S. – If $x \in Var^{\text{ext}}(\alpha)$, $a \in D(\alpha)$, $\xi \in Var^{\text{int}}(\alpha)$, $f \in Int(\alpha)$, v, v_1, $v_2 \in S(V)$, and for $\beta \in Var$,

$$v_1(\beta) = \begin{cases} a, \text{ if } \beta = x, \\ v(\beta) \text{ otherwise}; \end{cases} \qquad v_2(\beta) = \begin{cases} \text{f, if } \beta = \xi, \\ v(\beta) \text{ otherwise}; \end{cases}$$

then we write "$v[x:a]$" for v_1, and "$v[\xi:f]$" for v_2.

Remark. As one sees, the possible values of a variable v_α are factual values of type α, whereas those of ξ_α are intensions of type α. This shows the *semantic* difference between the two sorts of variables.

3.3. Intensions. Let σ and v be a projection and an assignment of L^{int} into S, respectively. For all $A \in Cat$, we define the *intension* of A in S, according to σ and v, denoted by "$\text{int}^S_{\sigma,v}(A)$", by the recursive clauses (I1) to (I7) below. I shall write "$\text{int}_v(A)$" instead of "$\text{int}^S_{\sigma,v}(A)$" assuming S and σ to be fixed. – If $A \in Cat.ext$, then $\text{int}_v(A)$ is a function defined on $I = W \times T$; hence, it can be defined by determining $\text{int}_v(A)(i)$ for all $i \in I$. I shall use this possibility in some clauses below.

The category of an expression A will be indicated by a type symbol superscript – as introduced in § 2, (G5) – at its first occurrence in a rule. Further, if $a \in D(\alpha)$ and $w \in W$, then "$[a]_w$" will be used in the following sense:

$$[a]_w = \begin{cases} \Theta(\iota), \text{ if } a \in D(\iota) - d(w), \\ a \text{ otherwise.} \end{cases}$$

Of course, the case $a \in D(\iota) - d(w)$ may occur only if $\alpha = \iota$.

(I1.1) If $x \in Var^{\text{ext}}(\alpha)$, $\xi \in Var^{\text{int}}(\alpha)$, and $i = (w, t) \in I$, then $\text{int}_\nu(x)(i) = [\nu(x)]_w$, and $\text{int}_\nu(\xi)(i) = [\nu(\xi)(i)]_w$.

(I1.2) If $C \in Con(\alpha)$, then

$$\text{int}_\nu(C)(i) = \begin{cases} [\sigma(C)]_w \text{ if } \sigma = \iota \text{ and } i = (w, t) \in I, \\ \sigma(C)(i) \text{ otherwise.} \end{cases}$$

(I1.3) If $C \in Op$, $\text{int}_\nu(C) = \sigma(C)$.

(I2.1) $\text{int}_\nu(\text{``}C_{\alpha\beta}(B_\beta)\text{''}(i) = [\text{int}_\nu(C)(i)(\text{int}_\nu(B)(i))]_w$, where $i = (w, t)$.

(I2.2) $\text{int}_\nu(\text{``}C_{\iota;\beta}(B_\beta)\text{''}(i) = [\text{int}_\nu(C)(\text{int}_\nu(B))(i)]_w$, where $i = (w,t)$.

(I2.3) If $\tau \neq \iota$, $\text{int}_\nu(\text{``}C_{\tau;\beta}(B_\beta)\text{''}) = \text{int}_\nu(C)(\text{int}_\nu(B))$.

(I3.1) For all $b \in D(\beta) - \{\Theta(\beta)\}$,

$\text{int}_\nu(\text{``}\lambda x_\beta A_\alpha)\text{''})(i)(b) = \text{int}_{\nu[x:b]}(A)(i)$,

and $\text{int}_\nu(\text{``}(\lambda x A)\text{''})(i)(\Theta(\beta)) = \Theta(\alpha)$.

(I3.2) For all $f \in Int(\beta)$,

$\text{int}_\nu(\text{``}(\lambda \xi_\beta A_\tau)\text{''})(f) = \text{int}_{\nu[\xi:f]}(A)$.

(I4)

$$\text{int}_\nu(\text{``}(A_\alpha = B_\alpha)\text{''})(i) = \begin{cases} 2 \text{ if } \text{int}_\nu(A)(i) = \Theta(\alpha) \text{ or } \text{int}_\nu(B)(i) = \Theta(\alpha), \\ 1 \text{ if } \text{int}_\nu(A)(i) = \text{int}_\nu(B)(i) \neq \Theta(\alpha), \\ 0 \text{ otherwise.} \end{cases}$$

(I5) $\text{int}_\nu(\text{``}\mathbf{I}A_{o\iota}\text{''})(i) = u_0$ provided $i = (w,t)$ and $\{u \in d(w)\colon \text{int}_\nu(A)(i)(u) = 1\} = \{u_0\}$;

in other cases $\text{int}_\nu(\text{``}\mathbf{I}A)(i) = \Theta(\iota)$.

(I6) $\text{int}_\nu(\text{``}{}^\wedge A_o\text{''})(i) = \text{int}_\nu(A)$.

(I7) $\text{int}_\nu(\text{``}A_o \textbf{ since } B_o)\text{''}) = \Phi$, $\text{int}_\nu(\text{``}A_o \textbf{till } B_o)\text{''}) = \Psi$,

where Φ and Ψ are functions on I defined as follows:

$$\Phi(w,t) = \begin{cases} 2 \text{ if for all } t' < t,\ \text{int}_\nu(A)(w,t') = 2, \text{ or for all } t' < t, \\ \quad \text{int}_\nu(B)(w,t') = 2, \\ 1 \text{ if for some } t' < t,\ [\text{int}_\nu(B)(w,t') = 1 \text{ and for all } t'' \\ \quad \text{such that } t' < t'' < t,\ \text{int}_\nu(A)(w,t'') = 1], \\ 0 \text{ otherwise.} \end{cases}$$

$$\Psi(w,t) = \begin{cases} 2 \text{ if } & \text{for all } t' \text{ such that } t < t', \text{ int}_v(A)(w,t') = 2, \text{ or} \\ & \text{for all } t' \text{ such that } t < t', \text{ int}_v(B)(w,t') = 2, \\ 1 \text{ if } & \text{for some } t' \text{ such that } t < t', [\text{int}_v(B)(w,t') = 1, \\ & \text{and for all } t'' \text{ such that } t < t'' < t', \text{ int}_v(A)(w,t'') = 1], \\ 0 & \text{otherwise.} \end{cases}$$

Remarks. (i) It is easy to show that if $A \in Cat(\iota)$, then for all $i \in I$, $\text{int}_v(A)(i) \in d(w)$ where $i = (w,t)$. – (ii) Extensional variables and individual constants as well as names of sentence intensions of form "$^\wedge A_o$" are rigid terms; cf. (I1.1), (I1.2), and (I6).

3.4. If $A \in Cat.ext$, then

$$|A|_{vi} =_{\text{df}} \text{int}_v(A)(i)$$

will be called the *factual value* of *A at i* ($i \in I$), according to S, σ, and v. If $A \in Cat(\alpha) \subseteq Cat.ext$, then $|A|_{vi} \in D(\alpha)$. (Remember that the current term for 'factual value' is 'extension'.)

3.5. For $A \in Cat$, $\text{int}_v(A)$ may be regarded as the *contextual intension* of A according to S, σ and the "context" v. Of course, if A is *closed*, $\text{int}_v(A)$ does not depend on v. The *absolute intension* of A (according to S and σ) might be defined as a function $\|A\|$ from $S(V)$ by

$$v \in S(V) \Rightarrow \|A\|(v) = \text{int}_v(A).$$

(Contextual and absolute intension is called by Montague 'sense' and 'meaning', respectively, cf. [5].)

3.6. The central notions of semantics. The couple (S, σ) is said to be an *interpretation* of the language L^{int} iff S is a type-theoretical structure and σ is a projection (an interpreting function) of L^{int} into S. The quadruple (S, σ, v, i) is said to be a *representation* of the set $\Gamma \subseteq Cat.ext$ (of L^{int}) iff (S, σ) is an interpretation of L^{int}, $v \in S(V)$, $i \in I = W \times T$, and for all $\alpha \in \text{EXTY}$,

$$A \in \Gamma \cap Cat(\alpha) \Rightarrow |A|^S_{\sigma v i} \neq \Theta(\alpha).$$

If, in addition,

$$A \in \Gamma \cap Cat(o) \Rightarrow |A|^S_{\sigma v i} = 1$$

holds, then we say that (S, σ, v, i) is a *model* of Γ.

Let K be a class of interpretations of L^{int}, and $\Gamma \subseteq Cat(o)$. Then Γ is said to be *K-satisfiable* iff Γ has a K-model (S, σ, v, i) where $(S, \sigma) \in K$.

A sentence $A \in Cat(o)$ is said to be a *strong K-consequence* of Γ iff every K-representation of Γ is a representation of $\{A\}$ and every K-model of Γ is a model of $\{A\}$; in symbols: $\Gamma \mid\models_K A$. – Further, A is said to be a *weak K-consequence* of Γ iff A is false in no models of Γ; in symbols: $\Gamma \models_K A$. The sentence A is said to be *K-valid* (*K-irrefutable*) iff A is a strong (a weak, resp.) K-consequence of the empty class of formulas. K-valid sentences are true in all K-interpretations, whereas K-irrefutable ones are false in no K-interpretations.

Terms B and C are said to be *K-synonymous* iff they belong to the same (extensional or intensional) category, and their absolute intensions coincide in all K-interpretations. We denote this relation by "$B :_K = :C$". – If K is the class of all interpretations of L^{int}, we omit the subscript 'K' in the notations above.

Remark. If $A \in Cat.ext$, then "$(A = A)$" is K-irrefutable (for every class K of interpretations), but it need not be K-valid (for $|(A = A)|_{vi} = 2$ might be possible). However, "$((\lambda x \cdot x) = (\lambda x \cdot x))$" *is* valid (i.e., K-valid for every class K).

§ 4. *Some semantical metatheorems*

4.1. The law of replacement. Assume that A, B belong to the same category, A is a part of $C \in Cat$, and denote "$C[B /\!/ A]$" the expression obtained from C by replacing an occurrence of A not preceded immediately by 'λ' by B. Then:

$$(A :_K = :B) \Rightarrow (C :_K = :C[B /\!/ A])$$

for all class K of interpretations of L^{int}.

Let us say that *B is substitutable for the (extensional or intensional) variable $\mathfrak{Y}$ in A* iff $\mathfrak{Y}$ and B belong to the same (extensional) type, $A \in \text{Cat}$, and whenever $\mathfrak{Z}$ is a variable occurring free in B, and "$(\lambda \mathfrak{Z} \cdot C)$" is a part of A, then no free occurrence of $\mathfrak{Y}$ in A stands in "$(\lambda \mathfrak{Z} \cdot C)$". Let us denote by "$A[B/\mathfrak{Y}]$" the expression obtained from A by substituting B for all free occurrences of $\mathfrak{Y}$.

4.2. The law of intensional lambda-conversion. If B is substitutable for the *intensional* variable ξ in A, then

$$(\lambda \xi A)(B) := :A[B/\xi].$$

Corollary: The eliminability of intensional variables. If $A \in Cat.ext$ is a *closed* term, then there is a term A' containing no intensional variables such that

$$A := :A'.$$

A term $B \in Cat.ext$ is said to be a *rigid* one iff $\text{int}_v(B)$ is always a constant partial function on I. Rigid terms are the extensional variables, the constants of type ι, the sentence-intension names of form "$^\wedge A_o$", and all extensional terms involving only bound variables and logical constants.

By an *intensional operator* let us mean any term of $Cat.int$ as well as any of the logical constants **since**, **till**, and the intensor '$^\wedge$'.

4.3. The law of extensional lambda-conversion. Assume that the *extensional* variable x has some free occurrence in $A \in Cat.ext$, B is substitutable for x in A, and one of the following two conditions is fulfilled:

(i) No free occurrence of x in A is a part of an argument of an intensional operator.

(ii) B is a rigid term.

Then:

$$(\lambda x \cdot A)(B) := :A[B/x].$$

The term $A \in Cat.ext$ is said to be *pure extensional* iff it involves no intensional operators and no intensional variables.

4.4. The hereditaryness of factual value gaps in extensional contexts. Assume that $A \in Cat(\alpha) \subseteq Cat.ext$, A is pure extensional, $B \in Cat(\beta)$, B is a part of A, and no free variable of B is bound in A. Then:

$$|B|_{vi} = \Theta(\beta) \Rightarrow |A|_{vi} = \Theta(\alpha),$$

for all interpretations (S, σ), for all $v \in S(V)$, and for all $i \in I$.

§ 5. *Definition of classical connectives and operators*

In the following definitions, the category of a term will be indicated by a type symbol subscript at its first occurrence. Some parentheses will be omitted if no misunderstanding arises by their omission.

5.1. The symbols '$\uparrow$', '$\downarrow$', and '$\sim$' (Verum, Falsum, and Negation, respectively) are to be introduced as follows:

$$\uparrow \;=_{\mathrm{df}}\; \text{``}(\lambda p_o p) = (\lambda p \cdot p)\text{''}; \quad \downarrow \;=_{\mathrm{df}}\; \text{``}(\lambda p_o p) = (\lambda p \cdot \uparrow)\text{''};$$
$$\sim \;=_{\mathrm{df}}\; \text{``}\lambda p_o(p = \downarrow)\text{''};$$

where p is the first member of $Var^{\mathrm{ext}}(\mathrm{o})$. – We write

"$A \neq B$" for "$\sim(A = B)$".

5.2. '**P**' and '**F**' (past and future tense operators) are defined as follows:

$$\mathbf{P} =_{\mathrm{df}} \text{``}\lambda\pi_o(\uparrow \textbf{ since } \pi)\text{''}; \quad \mathbf{F} =_{\mathrm{df}} \text{``}\lambda\pi_o(\uparrow \textbf{ till } \pi)\text{''};$$

where π is the first member of $Var^{\mathrm{int}}(\mathrm{o})$.

5.3. The modal operators '$\Box$' and '$\Diamond$' (necessity, possibility) are defined by

$$\Box =_{\mathrm{df}} \text{``}\lambda\pi_o({}^\wedge\pi = {}^\wedge\uparrow)\text{''}; \quad \Diamond =_{\mathrm{df}} \text{``}\lambda\pi_o({}^\wedge\pi \neq {}^\wedge(\pi \neq \pi))\text{''};$$

where π is as above. Note that "$\sim\Diamond(\sim A)$" and "$\Box(A)$" are not synonymous, and that "$\Box(A)$" and "$\Diamond(A)$" never take 2 as their factual value.

5.4. Quantifiers. If $C \in Cat(\mathrm{o}\alpha)$, the formula

$$(C = \lambda x_\alpha \uparrow)$$

expresses that the (perhaps higher-order) predicate C holds *true for all* members of $D(\alpha)$, whereas

$$(C = \lambda x_\alpha(Cx = Cx))$$

expresses that C is *false for no* members of $D(\alpha)$. (In a semantics without value-gaps, the two formulas are synonymous.) It is the latter which we shall call the universal quantification of C. However, in the case $\alpha = \iota$ we want to restrict quantification to $d(w)$ instead of $D(\iota)$. By this, the general definition of the quantifier $\forall_\alpha$ (of type α) is as follows:

$$\forall_\alpha =_{\mathrm{df}} \text{``}\lambda P_{\mathrm{o}\alpha}[\lambda x_\alpha((x = x) = Px) = \lambda x((x = x) = (Px = Px))]\text{''}.$$

If $\alpha \neq \iota$, this is synonymous with the shorter one:

$$\forall_\alpha =_{\mathrm{df}} \text{``}\lambda P_{\mathrm{o}\alpha}[P = \lambda x_\alpha(Px = Px)]\text{''}.$$

Here P is the first member of $Var^{ext}(o\alpha)$.

Now we can introduce the following abbreviations:

$$\text{"}\forall x_\alpha \cdot A_o\text{" for "}\forall_\alpha(\lambda x_\alpha \cdot A_o)\text{"},$$
$$\text{"}\exists x_\alpha \cdot A_o\text{" for "}\sim \forall x_\alpha \cdot \sim A_o\text{"},$$
$$\text{"}\mathbf{I}x_\iota \cdot A_o\text{" for "}\mathbf{I}(\lambda x_\iota \cdot A_o)\text{"}$$

And we have the following valuation rule:

$$|\forall x_\alpha A_o|_{vi} = \begin{cases} 2 \text{ if for all } a \in D(\alpha),\ |A|_{v[x:a],i} = 2, \\ 0 \text{ if for some } a \in D(\alpha),\ |A|_{v[x:a],i} = 0, \\ 1 \text{ otherwise.} \end{cases}$$

In the case $\alpha = \iota$, "$a \in D(\alpha)$" is to be replaced here by "$a \in d(w)$", where $i = (w,t)$.

Now we can introduce '&' (conjunction) as Montague did in [5]:

$$\& =_{df} \text{"}\lambda p_o \cdot \lambda q_o \cdot \forall h_{oo}[p = (hp = hq)]\text{"}$$

where h is the first member of $Var^{ext}(oo)$, and p, q are the first two members of $Var^{ext}(o)$. Of course, we write "$A \& B$" instead of "$\&(A)(B)$". If one of A, B takes the factual value 2, then so does "$A \& B$".

The functors '$\vee$' and '$\supset$' can be introduced by using '$\sim$' and '&' as usual. We do not need the biconditional, since "$(A_o = B_o)$" does the same job.

5.5. Subordination. Let us note that if B, $C \in Cat(o\alpha)$, then

$$\forall x_\alpha(Bx \supset Cx)$$

does not mean that all B's are C's. (It only means that no B's belong to the falsity-domain of C; i.e., it is not excluded that C is undefined for some B's.) To express the latter, we shall introduce the functors '$\mathbf{sub}_\alpha$' as follows:

$$\mathbf{sub}_\alpha =_{df} \text{"}\lambda P_{o\alpha}\lambda Q_{o\alpha}\forall R_{o\alpha}[(R = Q) \supset \forall x_\alpha(Px \supset \Diamond(Rx))]\text{"}$$

where P, Q, R are the first three members of $Var^{ext}(o\alpha)$.

We write "$B_{o\alpha}$ **sub** $C_{o\alpha}$" instead of "$\mathbf{sub}_\alpha(B_{o\alpha})(C_{o\alpha})$.
Then "B **sub** C" abbreviates the formula

$$\forall R[(R = C) \supset \forall x(Bx \supset \Diamond(Rx))]$$

which takes the value

2, if the factual value of B or of C is $\Theta(o\alpha)$,
1, if all B's are C's, and
0 in the remaining cases.

Note that "B **sub** C" and "$(\lambda x \cdot {\sim}Cx)$ **sub** $(\lambda x \cdot {\sim}Bx)$" are not synonymous. The same holds for

"B **sub** $\lambda x(Cx \vee {\sim}Cx)$" and "$C$ **sub** $\lambda x(Bx \vee {\sim}Bx)$".

Thus, if we translate a sentence of the form "Every B is a C" as "B **sub** C", we get that the following sentences are not synonymous:

Every boy is or is not a pupil.
Every pupil is or is not a boy.

Again, this is a remarkable result of value-gap semantics.

A final abbreviation:

"B **equ** C" stands for "$(B$ **sub** $C)$ & $(C$ **sub** $B)$.

§ 6. *Extended intensionality*

The semantics explained on the previous pages makes our ontological commitment to admit intensional objects as moderate as it is possible at all. A shortcoming of the system: the metalanguage statement "$\mathrm{int}_v(A) = \mathrm{int}_v(B)$" is not expressible in the object language if $A, B \in \mathit{Cat.int}$. (If $A, B \in \mathit{Cat.ext}$,

$$^\wedge(A = A) = {}^\wedge(B = B) \; \& \; {}^\wedge(A = B) = {}^\wedge(A = A)$$

is suitable for expressing their synonymity.) The way of the correction is obvious: let us extend the syntactic rule (S4) – the use of identity symbol – for intensional terms:

(S4^{i}) If $A, B \in \mathit{Cat}(\tau) \subseteq \mathit{Cat.int}$, then "$(A = B)$" $\in \mathit{Cat}(o)$.

The corresponding semantic rule:

(I4^{i}) If $\tau \in \mathrm{OPTY}$, then

$$|(A_\tau = B_\tau)|_{vi} = \begin{cases} 1 \text{ if } \mathrm{int}_v(A) = \mathrm{int}_v(B), \\ 0 \text{ otherwise.} \end{cases}$$

The increase in our ontological commitment is clearly indicated by the fact that our intensional variables became quantifiable: if $C \in Cat(o;\alpha)$ and $\xi \in Var^{int}(\alpha)$, then

$$(C = (\lambda\xi \cdot \xi = \xi))$$

expresses that C holds true for all $f \in Int(\alpha)$. Thus, we can write

$$\text{``}\forall\xi_\alpha \cdot A_o\text{''} \quad \text{for} \quad \text{``}(\lambda\xi_\alpha A_o) = (\lambda\xi \cdot \xi = \xi)\text{''}.$$

Of course, the intensional variables are no longer eliminable.

Another advantage of the extended use of identity is that one can quantify in type ι on $U \cap d(w)$. Let us call the members of $U \cap d(w)$ the *real objects* of the world $w \in W$. We define:

$$\mathbf{real} =_{df} \text{``}(\lambda x_\iota \cdot \forall\pi_o(x \neq {}^\wedge\pi))\text{''},$$

where π and x are the first members of $Var^{int}(o)$ and $Var^{ext}(\iota)$, respectively. Then

$$\forall x_\iota(\mathbf{real}(x) \supset A_o)$$

or, in a stronger form,

$$\mathbf{real\ sub}\ (\lambda x_\iota A_o)$$

expresses the universal quantification of A over the real objects of $d(w)$.

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