ANALYSING THE MEANING OF MODALITIES IN S4 AND S5

Professor E.K. VOYSHVILLO

The laws and rules of a logical system are justified and the very system has well grounded extra-logical applications only when the propositions of its language make sense and it is quite clear.

The possible-worlds semantics for modal systems, the relational and topological ones clarify to a certain extent the meaning and sense of modal propositions. In this way it becomes obvious that these statements refer not only to a given/actual/world, but also to a set of worlds which are accessible from it and constitute a certain vicinity. So far it is not clear, however, why the actual world and its vicinity are always related to a model structure and what the latter is both from an ontological and gnoseological /epistemological/ viewpoint.

Neither is it clear what the very possible worlds and the accessibility relation between them are, and what determines the differences of various systems as regards this relation. Especially complicated seems the problem about the meaning and sense of propositions with complex, iterated modalities, e.g. the system S4.

Our purpose here is to throw some light on these problems. In the language of S4 statements with complex modalities /of the type $\Diamond\Box A$, $\Box\Diamond\Box A$ stc./ relating to a certain world β imply the occurrence of some changes in this world. These changes result in the appearance of some new necessities, against the background of falling out possibilities; as we shall see, certain necessities may fall out, too. For the sake of explaining this sort of processes we take for granted that in these worlds definite diachronic laws exist which are not usually accounted for when investigating the laws of science and the very notion of such a law.

Let us first of all consider possible worlds in general. A world β is interpreted in a rather natural way as the set of facts relating to the individuals of a non empty class, in which some properties and relations have been defined, i.e. the facts refer to a class of individuals forming a relational structure.

This is based on a theory of predication which views S4 and S5 as the result of abstraction as regards the structure of simple /elementary/ propositions, just as classical propositional logic is considered from the standpoint of predicate calculus. If possible worlds are domains of the real world that contain finite sets of individuals, then simple sentences of theories /having in mind the well known method of quantifier-elimination/ are their atomic propositions; the abstraction from the structure of the latter yields the elementary propositions of S4 and S5.

In the language the set of facts is a usual classical Carnap-style state description /s.d./ Here it is much more convenient, however, to interpret it not as a conjunction, but rather as a set, its elements being atomic /or elementary – in the case of a propositional language/ statements and their negations. Infinite s.d. should also be admitted here. But unlike the semantics of classical logic, in the analysis of modal propositions we ought to bear in mind that facts, corresponding properties and relations are bound together and law-governed, just like in reality. As far as the expression of these bonds in the language is concerned, the usual laws are formulated as follows. $\forall x (A(x) \rightarrow B(x))$, where $A(x)$ and $B(x)$ are formulas of classical predicate logic and "$\rightarrow$" is a relevance implication, e.g. that of the well known system *E* /which seems to convey the meaning of necessities, characteristic of the laws of nature/.

We ought to have in mind the existence of another kind of laws as well. Without going into detail, we should at least note the following. The description of a world β can be presented as $\Gamma \cup \alpha$ where α is the classical s.d., while Γ is the class of above-mentioned laws, as well as of their non-factual consequents/under a more elaborate description/. Among such consequents of the law:
$\forall x(A(x) \rightarrow B(x))$ is the formula $\overline{\Diamond}(A(a_i) \;\&\; \overline{B}(a_i))$ for any individual a_i.

It is essential to point out that Γ determines a set of various possibilities concerning factual states of the world. Thus, under the law $\forall x(A(x) \rightarrow B(x))$ we must exclude s.d. in which both $A(a_i)$ and $\overline{B}(a_i)$ hold simultaneously for any individual a_i. If M is the set of all classical s.d., then Γ defines in it the subset M_Γ /which is not empty thanks to the consistency of Γ/. This subset is itself the model structure of S5, having in mind that the accessibility relation R holds for every $a_i, a_j \in M_\Gamma$.

In this way worlds in a formal semantics correspond to classical s.d. α. It is precisely in the framing of the model structure that the laws Γ express themselves. Naturally, the truth-conditions of propositions in S5 ought to be defined relative to these worlds α just as in relational semantics with the only difference that the valuation function of variables in a world α is determined by this very world. So we have $v(\varrho, \alpha) = T \Leftrightarrow \varrho \in \alpha$. Bearing this in mind, it is more convenient to use the notion of truth as a predicate /as in the well known definition of truth/. Thus, instead of the above formulation, we have $T‘\varrho’/\alpha \Leftrightarrow \varrho \in \alpha$. Generally speaking for an arbitrary formula A instead of $v(A, \alpha) = T$ and $v(A, \alpha) = F$ we are going to use the complex symbols $T‘A’/\alpha$ and $F‘A’/\alpha$ where A is the symbol for names of propositions

Moscow State University ‘Lomonosov’ Professor E.K. VOYSHVILLO

The usual definitions of truth conditions in S5 are quite natural. If Γ forbids s.d. α in which $\overline{A}$ holds, then it is only natural to state that $\overline{\Diamond}\,\overline{A}$. In this case the opposite /taking into consideration the reduction of double negation/: $\Box A$ means that T ‘A’/α for all $\alpha \in M_\Gamma$. However, from the viewpoint of meaning it is not correct to use the notion of truth of A in α if A in its turn is a modal proposition, because classical s.d. do not determine such propositions. Following the formal semantics and taking the set M_Γ to be the model structure of S5 in regard to the above mentioned R, we should correlate the truthfulness or falsity of modal propositions to M_Γ itself, and not to separate worlds in it.

It is natural to consider as such worlds the above stated $\beta = \Gamma \cup \alpha$ /or – abbreviated: Γ, α/ and – as a model structure $\Theta_\Gamma = \{\beta : \beta = \Gamma \cup \alpha, \alpha \in M_\Gamma\}$.

“$T‘\varrho’/\Gamma, \alpha$” is defined as before by “$\varrho \in \alpha$”. The definitions of $T‘(A \,\&\, B)’/\beta$; $T‘(A \vee B)’/\beta$; $T‘\overline{A}’/\beta$ are as usual. As for $T‘\Box A’/\beta \Leftrightarrow \forall \beta_1(R\beta\beta_1 \Rightarrow T‘A’/\beta_1$ the symbols $R\beta\beta_1$ imply that $\beta, \beta_1 \in \Theta_\Gamma$ /remember that in S5 the truth conditions are defined for the worlds of a certain Θ_Γ /. Thus $T‘\Box A’/\Gamma, \alpha \Leftrightarrow \forall \alpha_1(\alpha_1 \in M_\Gamma \Rightarrow T‘A’/\Gamma, \alpha)$. Obviously, if A contains no modal operators, then $T‘A’/\Gamma, \alpha$ is equivalent to $T‘A’/\alpha$.

It is easy to note that in the definition of $T‘\Box A’/\beta$ the fact that β belongs to a certain Θ_Γ is not essentially important. Neither is the notion of a model structure indispensable for the definition of the universal validity of a formula A, i.e. $\models A$. It can be shown that the usual definition of “$\models A$” as $\forall \Theta_\Gamma \forall \beta(\beta \in \Theta_\Gamma \Rightarrow T‘A’/\beta)$ is equivalent to "$\dot{\forall}\beta T$'A'$/\beta$". So, by analogy to classical logic, we have $\models A \Leftrightarrow \dot{\forall}\beta T$'A'$/\beta$ /a formula A is universally valid in S5, iff it is true in all possible worlds/.

A modal proposition, e.g. $\Box A$ relating to a certain world β is determined /if true/ or not determined by this very world. But it conveys the meaning of states of affairs in a set of worlds, in some way related to the former, accessible from it and constituing a vicinity of its. This set of worlds is the model structure itself. In formal semantics the vicinity of β differs from the model structure insofar as the former is only a part of the latter, because of the fact that α alone, and not $\Gamma \cup \alpha$ is taken in its capacity of a world. Then the vicinity of α is regarded from the viewpoint of Γ, i.e. within M_Γ.

When passing over to S4, the formal definitions of possible worlds, truth conditions and universal validity are preserved. What needs alteration is the definition of accessibility R. The latter should be interpreted as diachronic change in the world, in the process of which $\overline{\Box} A$ obtains in β, but it may well happen that $\Box A$ obtains in β_1, accessible from β. To put it in another way, some possibilities disappear, in this case: $\Diamond \overline{A}$ in β / an indication of this is the fact that the formula $\Diamond A \supset \Box \Diamond A$ is not universally valid in S4 /. Under this interpretation of the accessibility relation we can explain the complex, iterated modalities $\Diamond \Box$ and $\Box \Diamond \Box$. They account for the possibility of some changes in Γ, at least as a result of the appearance of non-factual consequences of the given laws.

The thought of new laws emerging against the background of already existing ones is in the main stream of the interpretation of Γ and the very possible worlds as aggregate amounts of our knowledge of the external world. Then, for instance, $\Box \Diamond \Box A$ under $\overline{\Box} A$ obtaining in β can be interpreted as follows. In reality there is a law, in view of which $\Box A$ obtains; but it is not known to us so far, i.e. it is not included in Γ which corresponds to the state of our knowledge β. Sooner or later, however, it is to be discovered, but then it is hard to reasonably interpret the case of $\Diamond \Box A$ which obtaines in β under $\overline{\Box} \Diamond \Box A$ in β. If we abandon the belief that finally the unknown is discovered, then the fact that $\Diamond \Box A$ obtains in β /under $\overline{\Box} \Diamond \Box$ A in β/ is all right in *case* there is an objective, but unknown to us law which determines $\Box A$. But then, on the contrary, it is hard to explain the case when $\Box \Diamond \Box A$ holds in β /under $\overline{\Box} A$ obtaining in β/. We consider the laws in Γ to be objective, taking for granted that under the changes in question the laws themselves do not come into being, nor disappear. This is the reason why relevance implication should be made use of for the sake of expressing the corresponding bonds in reality. If we used strict implication $<$ /resp. the formula $\forall x(A(x) < B(x))$ or its equivalent $\Box(\forall x(\overline{A}(x) \vee B(x))$ instead of the formula $\forall x(A(x) \rightarrow B(x))$/ then a consequence $\forall x(\Box B(x))$ would lead to the appearance of laws $\forall x \Box(\overline{A}(x) \vee B(x))$ and $\forall x \Box(A(x) \vee B(x))$, bearing in mind that the conjunction of these two is equivalent to $\forall x(\Box B(x))$.

More precisely, the set of worlds accessible from β given in a moment of time t_0, i.e. β_{t0}, can be characterised as follows. For one thing, its elements are successive states of the world β_t, $t_0 \leq t$; next – for every β_t there is a set of its possible alternatives at a given time t: $\Theta_{\Gamma_t} = \{\beta : \beta = \Gamma_t \cup a, a \in M_{\Gamma_t}\}$ where M_{Γ_t} is defined just as the above introduced M_Γ. Obviously, $\beta_t \in \Theta_{\Gamma_t}$, but it is important to identify it as the actual state of the world at time t, differentiating it from the rest of the elements of Θ_{Γ_t} in their capacity of alternative states. Any two elements of Θ_{Γ_t}, e.g. β_i and β_j are accessible from each other; from every element $\beta_i \in \Theta_{\Gamma_t}$ any element of $\Theta_{\Gamma_{t1}}$ is accessible, provided that $t < t_1$.

The above mentioned changes of worlds are of two main types: some alterations of β_t are determined by Γ_t, others are contingent.

Obviously the delineated set of worlds is the model structure of S4. But it is also possible to exclude this notion from some variants of semantic theories of S4, as in the case of S5.

As regards the changing of worlds and particularly of Γ in time, now it should be admitted that in Γ there are such laws and consequences of laws, according to which the appearance of new necessities may depend on contingent changes in the world. Thus, a consequence of laws may be $A/t \supset \forall t_1 \ (t < t_1 \supset \overline{\Diamond} \overline{A}/t_1)$ which means that it is impossible to rule out the property or state A that has come into being at a certain moment of time, /e.g. some disease, traumata, death, the results of some accidents etc./ According to the usual definition $\overline{\Diamond} \overline{A}$ is equivalent to $\Box A$, but obviously we have here some special kind of necessity: it can be called "negative" to mark the difference from the one when necessity means that a situation A is determined by one law or another.

Now, let it be the case that TA/β_t but $F\Box A/\beta_t$, then $T\Box A/\beta_{t1}$ for any $t_1>t$ and $T\Box\Diamond\Box A/\beta_t$. This means, in keeping with the truth conditions, that TA/β_{it1} for any $\beta_i\in\Theta_{\Gamma_{t1}}$ and for any $t_1>t$. Here $\Theta_{\Gamma_{t1}}$ is the vicinity of some actual world β_t, accessible from it. The statement TA/β_{it1} for any β_i from $\Theta_{\Gamma_{t1}}$ means that the situation A obtains in β_{t1} and at that moment of time (t_1) it could not but be so. But amongst the alternatives of β_t there is β_{jt} such that FA/β_{jt} /thus at the moment t obtains a situation A in the actual world β_t, but this is contingent, and not determined by anything/. In the case when A/β_t – under the above noted relation – we have only $T\Diamond\Box A/\beta_t$. World-changes may proceed in such a way that it may happen even FA/β_t for any time $t_1>t$, but this would not imply that $\Box\bar{A}$ in t_1, because in each $\Theta_{\Gamma_{t1}}$ there may be β_{it1} such that TA/β_{it1} /althought the situation A does not obtain in β_{t1}, it might be the case that it should obtain for every t_1/ The possibility of $\Box A$ coming to being at a moment t is not realised in the coures of development. /We have here an instance of necessity, "contingent by origin", so to say./

It is easy to realise that the formulated semantic theory is adequate to S4. However, it is somewhat limited as regards the explication of non-logical modalities. It is precisely this part of it that contains the theory of possibilities and necessities only for properties and relations. As for events, they cannot in principle be characterised as "necessary... at any time", or "obtaining... at any time". An event A is either necessary or not at some moment of time t.

For the sake of the corresponding extension of our theory the existence of certain diachronic laws should also be taken into account:

$$\forall t(A/t \to \exists t_1((t<t_1 \,\&\, \forall t_2(t<t_1<t_2 \supset A/t_2)) \to B/t_1))$$

where A stands for some properties or states, while B is an event. If A obtains necessarily at every moment t_2, then B, too, is necessary at t_1.

When a state A obtaines at a moment t there is a time t_1 such that if A coninues to be at each moment t_2 in the open interval tt_1 /i.e. the interval does not include t and t_1 themselves/, then the event B comes into being, as a result of some process taking place during this interval of time.

For example: A/t – someone is alive at t; t_1 – the moment of his death; tt_1 – the period of time when he is still alive. Or A/t means, e.g. "The Sun is hot at t", tt_1 is the interval of time, during which the Sun is still hot, but gradually cools down, t_1 being the moment when it has become cold.

The fact that B/t_1 is determined at $t<t_1$ /i.e. the determination of future, as was pointed above/ means that $\Box B/t_1$. If we have a consequence of the laws $\Box(A/t \supset \forall t_2(t<t_1<t_2 \supset A/t_2))$ and the fact A/t (i.e. T'A'/β_t), and therefore A/t_2 for any t_2 $t<t_1<t_2$, then $T\Box B/\beta_{t1}$.

As far as the event B is concerned, however, it is no use interpreting the above metastatement with the supposition that $\Box B$ must continue to be true even in the consecutive states of the world β_{t3} /$t_3>t_1$/; it only means that for any $\beta_{it1}\in\Theta_{r_{t1}}$ we have T'B'/β_{it1}, (i.e. B has taken place in some actual world β_{t1} and it could not but be so.)

In order to reflect this in a semantic theory, we have to once again use the notion of a modal structure. The latter is conceived of as a set of worlds, accessible from a certain one β_{t0} during a time interval (t_0, t), where $t_0\leq t$, i.e. the interval may be reduced to a moment of time t_0.

Now we can say that $T\Box B/\beta_t$ is justified for β_{t1} as for the structural element, corresponding to the interval tt_1; this holds for all the worlds of the structure $\Theta_{r_{t1}}$, which is itself the set of worlds, accessible from β_{t1} at a moment t_1 /this shows that the modal characteristics of events are modalities of the S5-type./

In the process of development of worlds the necessity of B in β_{t1} is determined for any t_2, $t<t_2<t_1$. This implies that $T\Box\Diamond\Box B/\beta_{t2}$. But in β_t /i.e. in a moment t of the process of development of this world/, we have $T\Diamond\Box B/\beta_t$ for β_t as for the structural element, corresponding to the interval (t, t_1).

The difference between β_t and β_{t2} consists in the following. In Θ_{r_t} there is β_{it} such that F A/β_{it} /a situation A has come to be in a moment t, i.e. in a world β_t by chance, it might have been the case that $\overline{A}$ obtains), while for any $\beta_{it2}\in\Theta_{r_{t2}}$ we have T'A'/β_{it2}. The latter means that A is necessary in β_{t2} as in an element of the model structure, corresponding to the interval (t_2' t_2''), where t_2' is a moment which follows immediately after t /we might as well presuppose the line of time to be somehow cut in a discrete succession of moments/ while t_2'' is the moment which immediately precedes t_1.

Thus we have a necessity /of a situation and resp. of a statement A/ which manifests itself in the process of development of the world in a certain moment t_2' and disappears in t_1.

In the kind of analysis that has been carried out here we have tried to show some additional /in respect to the possible worlds semantics/ aspects of modal notions which seem important from philosophical and methodological point of view. A way is shown in which necessity and possibility are determined by the laws of reality./ A statement □A is true in a certain world β not because A holds in all the possible worlds, accessible from β, but – vice versa – this is the case because the very necessity of the situation A is determined in β itself./ In conection with the analysis of complex /iterated/ modalities ◇□, □◇□ it becomes clear how the emergence of a situation or an event and the necessity of this may be determined at preceding moments of time; the bond is revealed between necessity and chance, as well as between situations, processes and events. The structure is made clear of some processes of development of the world, in which certain possibilities and necessities disappear so that others should come into being.

Moscow State University 'Lomonosov' Professor E.K. VOYSHVILLO
Chair of Logic,
Department of Philosophy
/Translated by Hristo Smolenov/