# A CONSTRUCTIVE APPROACH TO DYADIC STANDARD DEONTIC LOGIC

by
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## § 1. *Introduction*

Anderson [1] and Kanger [7] have given a constructive approach to standard deontic logic (SDL), the so-called *Anderson Simplification* [4]. It may be asked whether there is a comparable construction in the case of dyadic standard deontic logic. Such a construction could offer us a method of choosing between the bewildering variety of possibilities when we go beyond SDL. In particular, Hansson [5] outlines three possible alternatives to SDL but offers no way to choose between them.

In the Anderson construction a special proposition letter (say v) called the *sanction* is introduced and the obligation operator O is defined through the use of this letter. In the following we shall proceed constructively by recognizing that in dyadic logic the sanction v must vary according to the conditions under which it is applied. It is shown that such an approach yields under appropriate definitions an interpretation of Hansson's dyadic standard deontic logic 3(DSDL3).

## § 2. *Construction of a dyadic standard deontic logic*

We construct a logic *DL* based on an underlying logic *L* which we assume is propositional calculus. In addition to the usual apparatus of *L* we shall assume that there are two other operators ∨ and ⊢ on *L*. The operator ∨ takes formulas of *L* into formulas of *L*, but the operator ⊢ takes formulas of *L* into formulas of Ł *L* where ⊢ A has the usual interpretation: A is a theorem of *L*. The formulas of *DL* are the following:

(a) the formulas of *L*, and
(b) the formulas that result from applying the connectives and the operators ∨ and ⊢ to the formulas of *DL*.

We assume that the usual rules of deduction of propositional logic apply including the deduction theorem and the rules of substitution and replacement. In addition we assume that if A is a theorem of *L* then $\vdash$ A is a theorem of *DL*.

In the following *we shall assume that all formulas* A, B, and C *are formulas in the basic logic L.* The results that we obtain can be then extended to all the formulas of *DL* by the rules of substitution and replacement.

We introduce now the dyadic operator O which has the following intended interpretation: given formulas A and B, O(A/B) is the obligation to see it that A is true under the condition B. Since v(B) is to be interpreted as the sanction to be applied under condition B, we define O(A/B) as follows:

$$(1)\ O(A/B)_{\text{def.}} = \vdash((\sim A \,\&\, B) \rightarrow v(B)).$$

In order to forestall a possible misunderstanding it may be observed that the foregoing does not say that a given conditional obligation, O(A/B), if true, is logically true, since the sanction $\vee$ is not fixed by logic but is, to some extent, arbitrary. But before we make further remarks about $\vee$, let us note some elementary consequences of (1).

First of all, like absolute obligations, conditional obligations are multiplicative, i.e.,

$$O(A \,\&\, B/C) \leftrightarrow O(A/C) \,\&\, O(B/C).$$

Moreover, if $\tau$ is a tautology and K is a contradiction, the following are true for any formula A:

$$O(A/K),\ O(\tau/A),\ \text{and}\ O(A/A).$$

Returning now to the sanction operator $\vee$, we may more completely characterize it as follows: First, because the standard rule of replacement applies, we may infer that

$$(2)\ \vdash(A \leftrightarrow B) \rightarrow \vdash(v(A) \leftrightarrow v(B)).$$

Now let $\tau$ be a tautology. It follows from (2) that v is (virtually) constant on tautologies and therefore as usual we may identify O(A/$\tau$) with O(A).

Because otherwise that identification would not make sense, we assume that v($\tau$) is *neither a contradiction nor a tautology.* Now, having assumed that v is defined on tautologies, we shall extend the operator v recursively to the other formulas of *L*.

If we answer the question what happens to the formula A → v(C) when the condition C is multiplied by B, we shall be able to make the required extension. In the following we shall assume that the answer depends on the logical relation between A and B, that is to say, on the logical validity of the formula (B → A). (Note that B → A is a decidable formula since *L* is complete). Taking our cue from examples such as Chisholm's paradox, we adopt the following as rules of inference in *DL*:

(3) (a) $\vdash(A \rightarrow v(B \,\&\, C)) \leftrightarrow \vdash(A \rightarrow v(C))$
unless $\vdash(B \rightarrow A)$.
(b) $\vdash(A \rightarrow v(B \,\&\, C)) \rightarrow \vdash \sim(B \,\&\, C)$
whenever $\vdash(B \rightarrow A)$.

We shall show in article 3 that these rules together with def. (2) are sufficient to determine that *DL* is a model of Hansson's DSDL3. But first we obtain some elementary consequences of (3).

An immediate consequence of (3) (b) is the following:

(4) If O(A/B) and O(~A/B) then $\vdash \sim$B. To prove (4) assume O(A/B) and O(~A/B) are both true. By multiplicativity of obligations we have O(K/B) which, by definition, is $\vdash(B \rightarrow v(B))$. The result (4) then follows from 3(b).

An immediate consequence of (4) is that conflicting obligations cannot occur. Such a consequence, pace von Wright [9], is necessary if deontic logic is to have any practical value.

It follows also that O(~A/A) is false unless $\vdash \sim$A.

To obtain further results it will be helpful to develop a pair of lemmas:

*Lemma 1*. If $\nvdash \sim(B \,\&\, C)$ then $\vdash(v(B \,\&\, C) \rightarrow v(C))$.

*Proof*. According to 3(b), if ~(B & C) is not provable then (B → v(B & C)) is not provable. The result then follows from substituting v(B & C) for A in 3(a).

*Lemma 2.* If $\sim O(\sim B/C)$ then $\vdash (v(C) \rightarrow v(B \,\&\, C))$.

*Proof.* The formula $\sim O(\sim B/C)$ implies $\nvdash (B \,\&\, C) \rightarrow v(C)$. The result then follows from (3) upon substituting (B & C) for B and v(C) for A.

These two lemmas enable us to obtain a theory of conditional obligation which is non ad-hoc and wholly constructive.

## § 3. *DL as a model of DSDL3*

We are now in a position to prove that a model of *DL* is a model of Hansson's DSDL3. In order for this to be the case the dyadic operator O(–/–) has to satisfy the following criteria: there is a reflexive, transitive and total relation R defined on the class of all possible worlds (or equivalently, all valuations on *L*) such that O(A/B) is valid in *DL* iff all worlds satisfying A contains all R-maximal worlds satisfying B. (Here x is an R-maximal world satisfying B provided that x satisfies B and moreover for all y satisfying B we have xRy.)

In order to prove that *DL* satisfies these criteria we first define R. We do this as follows: First divide the class *P* of all possible worlds into three equivalence classes $E_1$, $E_2$, and $E_3$ defined as follows: Let $E_1$ be the class of all possible worlds for which the formula v(*T*) is false whenever *T* is a tautology. Such worlds may be said to be *deontically perfect*. (We are assuming that v(*T*) is neither a tautology nor a contradiction.) Let $E_2$ be the class of deontically imperfect worlds in which, for every formula A, v(A) is false unless A is a tautology. Such worlds although not perfect may be considered not so imperfect as those worlds that belong to neither $E_1$ and $E_2$. The latter we denote by $E_3$. Finally we define R as follows:

(4) xRy iff $x \in E_i$ and $y \in E_j$ for some pair (i,j), $i \leq j$; $i,j = 1,2,3$.

It is easy to see that (4) defines a total, transitive and reflexive relation.

*Theorem 1.* All R-maximal elements are neither in $E_1$ or $E_2$.

*Proof:* Theorem 1 is an immediate consequence of the following:

*Lemma 3:* There is a world x such that for all formulas A, if A is not a tautology then v(A) is false for x.

*Proof:* Assume that no such world exists. Then there is a finite sequence of formulas $A_1$, $A_2$, ..., $A_n$ such that the formula

$$\sim v(A_1) \mathbin{\&} \sim v(A_2) \mathbin{\&} \dots \mathbin{\&} \sim v(A_n)$$

is a contradiction. It follows that

$$v(A_1) \vee v(A_2) \vee \dots \vee v(A_n)$$

is a tautology, whence v(τ) is a tautology contrary to the definition of *DL*.//

*Corollary:* A world x satisfying B is R-maximal (with respect to B) provided:

(a) x is in $E_1$,
or (b) x is in $E_2$ and there is no deontically perfect world satisfying B.

*Theorem 2.* A formula O(A/B) is valid in *DL* iff the class of all possible worlds satisfying A contains all R-maximal worlds satisfying B.

*Proof:* Suppose O(A/B) is valid in *DL*. Since if A is a tautology or B is a contradiction there is nothing to prove, suppose in addition that neither A not ~B is provable. Then it can be shown that A is not provably false. Let x be an R-maximal element satisfying B. Then — by definition — B is true for x and it follows from Theorem 1 that x is in $E_1$ or in $E_2$. If x is in $E_1$ then — since v(τ) is false for x — it follows that v(A) is false for x. Similarly, if x is in $E_2$ then by definition v(A) is false for x. Consequently — in either case — v(A & B) is false for x. Since $\nvdash A \rightarrow (\sim A) \mathbin{\&} B$ it follows that $\vdash (\sim A) \mathbin{\&} B \rightarrow v(A \mathbin{\&} B)$, whence A is true for x. This completes the proof.//

*Corollary. DL* is a logic of type DSDL3 of Hansson.

## § 4. *Conclusion*

We have given a rather natural construction that extends SDL to a model of Hansson's dyadic logic DSDL3. From this construction it appears that of all Hansson's logics, DSDL3 is the most natural extension. There may however be other dyadic logics superior to Hansson's, but it is beyond the scope of this paper to examine these here.

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