PARACONSISTENCY, PARACOMPLETENESS, AND INDUCTION(1)

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Introduction

Loosely speaking, classical deductive logic is constituted by the classical first-order predicate calculus, with or without equality, plus some of its standard extensions, such as some systems of set theory (Zermelo-Fraenkel, von Neumann-Bernays-Gödel, Kelley-Morse-Tarski,...) or usual type theory (essentially the higher-order predicate calculus)(2).

In this century, numerous new systems of deductive logic were developed, whose roots, in various cases, may be found even in Ancient Greece. Among these new systems, some are complementary to classical deductive logic (for example, classical modal logic, classical tense logic, and ordinary deontic logic). On the other hand, there exist others which are, so to say, rivals of classical logic, and whose principal aim is to replace it in some or in all domains of knowledge (for instance, intuitionistic logic, paraconsistent logic, and many-valued logic)(3).

It seems reasonable to admit that usual, extant science is founded on classical deductive logic, at least in principle. In other words, the deductive part of the empirical sciences can be in principle codified by the means of classical logic. Nonetheless, it is possible to show that the same task can be performed by several non-classical logics, even by logics which are rival of classical logic; this is, v.g. what does happen with some systems of paraconsistent logic. The explanation of this fact, at first sight surprizing, offers no difficulty at all, since some powerful systems of paraconsistent logic do contain, in a certain precise sense, classical deductive logic.

(1) This paper is a sequel to [9].

(2) On the notion of classical logic, non-classical logic, etc. one may consult [4].

(3) See [4] and [6].

So, a natural question originates: since there are non-classical deductive logics, are there non-classical inductive logics? The aim of the present note is to illustrate the possibility of building non-classical inductive logics, corresponding to some non-classical deductive logics. In order that our discussion be well determined and sufficiently precise, we shall accept as inductive logic, in its standard meaning, the contents of von Wright's book 'A Treatise on Induction and Probability' (see [12]). To make our problem still more definite, we introduce three systems of non-classical deductive logic, and try to discover what portion of [12] can be reconstructed on the basis of such systems.

1. *The propositional logics* β_0, β_1 *and* β_2

In this section we define three basic propositional logics: β_0, β_1 and β_2. Their underlying language has the following primitive symbols: 1) The connectives: $\rightarrow$ (implication), $\wedge$ (conjunction), $\vee$ (disjunction), and $\neg$ (negation); $\leftrightarrow$ (equivalence) is introduced by definition, as usual. 2) A denumerably infinite set of propositional variables: p, q, r, s,..., p', q', r', s',... 3) Parentheses. The concept of formula is defined as usual; the atomic formulas are the propositional variables, and the other formulas are called molecular.

Let us describe the postulates (axiom-schemes and primitive rule of inference) of β_0. They are the following (capital Latin letters stand for formulas):

1) $A \rightarrow (B \rightarrow A)$ 2) $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
3) $A, A \rightarrow B / B$ 4) $((A \rightarrow B) \rightarrow A) \rightarrow A$
5) $(A \wedge B) \rightarrow A$ 6) $(A \wedge B) \rightarrow B$ 7) $A \rightarrow (B \rightarrow (A \wedge B))$
8) $A \rightarrow (A \vee B)$ 9) $B \rightarrow (A \vee B)$
10) $(A \rightarrow C) \rightarrow ((B \rightarrow C) \rightarrow ((A \vee B) \rightarrow C))$
11) $(\neg A \rightarrow B) \rightarrow ((\neg A \rightarrow \neg B) \rightarrow A)$

Restriction: In postulate 11, A and B are supposed to be molecular.

Therefore, β_0 is classical propositional positive logic to which we have added scheme 11. If the restriction is dropped from the scheme, then β_0 reduces to the classical propositional calculus, as can be easily seen by the proofs of theorems 1-4 below.

The concepts of proof, deduction, the symbol $\vdash$, etc. are defined as in Kleene's book [7].

Theorem 1. - In β_0 *we have:* $\vdash \neg\neg A \rightarrow A$, *for* A *molecular.*

Proof. - Let us assume $\neg\neg A$. Then, it results that $\neg A \rightarrow \neg\neg A$ and $\neg A \rightarrow \neg A$ by positive logic. Therefore, taking into account scheme 11, $\neg\neg A \vdash A$, and, by the deduction theorem, $\vdash \neg\neg A \rightarrow A$.

Theorem 2. - In β_0: $\vdash A \rightarrow \neg\neg A$, *for* A *molecular.*

Proof. - Let us assume A. Hence, $\neg\neg\neg A \rightarrow A$, by positive logic, and $\neg\neg\neg A \rightarrow \neg A$, by the preceding theorem. So, $A \vdash \neg\neg A$, and $\vdash A \rightarrow \neg\neg A$.

Theorem 3. - In β_0: $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$, A *and* B *molecular.*

Proof. - In β_0 the scheme $(\neg\neg A \rightarrow B) \rightarrow ((\neg\neg A \rightarrow \neg B) \rightarrow \neg A)$, for B molecular, is valid. But, in β_0, $\vdash A \leftrightarrow \neg\neg A$, if A is not atomic. So, $\vdash (A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$, where A and B are not atomic.

Theorem 4. - If A *is a classical tautology and we replace its propositional variables by molecular formulas, obtaining the formula* A′, *then* A′ *is provable in* β_0.

Proof. - Consequence of the above theorems and of the fact that the classical propositional calculus can be axiomatized by the postulates 1-10, plus the scheme $(A \rightarrow B) \rightarrow ((A \rightarrow \neg B) \rightarrow \neg A)$ and $\neg\neg A \rightarrow A$ (cf. [7]).

Theorem 5. - In β_0, *if* A *is molecular, one has:*

$$\vdash \neg\neg A \leftrightarrow A, \quad \vdash \neg(A \wedge \neg A), \quad \vdash A \vee \neg A, \quad \vdash A \rightarrow (\neg A \rightarrow B).$$

Theorem 6. - In β_0, *the following formulas are theorems* (p *and* q *are propositional variables, i.e. atomic formulas)*:

$$\vdash \neg(p \vee q) \leftrightarrow \neg(q \vee p), \quad \vdash \neg(p \wedge q) \leftrightarrow \neg(q \wedge p),$$
$$\vdash \neg p \vee \neg\neg p, \quad \vdash \neg(\neg p \wedge \neg\neg p), \quad \vdash \neg\neg\neg p \leftrightarrow \neg p.$$

In what follows, F will denote the set of formulas of β_0.

Definition 1. - The function v, v: $F \to \{0,1\}$, *is a valuation of* β_0 *if, and only if* ($\Leftrightarrow$ *is a meta-linguistic abbreviation for equivalence), we have:*

1) $v(A \to B) = 1 \Leftrightarrow v(A) = 0$ or $v(B) = 1.$,
2) $v(A \wedge B) = 1 \Leftrightarrow v(A) = v(B) = 1$,
3) $v(A \vee B) = 1 \Leftrightarrow v(A) = 1$ or $v(B) = 1$,
4) $v(\neg A) = 1 \Leftrightarrow v(A) = 0$, when A is molecular.

Employing the ideas and methods of [9], it is not difficult to prove the following theorem:

Theorem 7. - β_0 *is strongly sound and strongly complete relatively to the semantics of valuation based on Definition 1.*

To prove the preceding theorem, it is convenient the lemma that follows:

Lemma. - If $\neg^* A$ *constitutes an abbreviation for* $\neg(A \vee A)$, *then* $\neg^*$ *has all properties of classical negation.* ($\neg(A \wedge A)$ *behaves also as the classical negation of* A.)

Proof. - Relative to $\to$, $\wedge$, $\vee$, and $\neg^*$, all postulates of classical positive logic are true, as well as $(\neg^* A \to B) \to ((\neg^* A \to \neg^* B) \to A)$. Consequently, we have classical propositional logic valid for such connectives.

There are valuations v of β_0 such that $v(A) = v(\neg A) = 1$, and valuations v′ such that $v'(B) = v'(\neg B) = 0$, where A and B are atomic. This means that, in the terminology of [9], β_0 belongs to the class of paraconsistent calculi and also to the category of paracomplete logics.

Now we introduce another propositional logic, which we call β_1. It is defined as β_0, with the difference that in scheme 11 A can not be atomic, but B may be.

Theorem 8. - The schemes $\neg(A \wedge \neg A)$ *and* $A \to (\neg A \to B)$ *are valid in* β_1.

Proof. - We have in β_1: $\vdash (A \wedge \neg A) \to A$ and $\vdash (A \wedge \neg A) \to \neg A$. Hence, $\vdash \neg(A \wedge \neg A)$. On the other hand, if A and $\neg A$ are assumed, we obtain that $A \wedge \neg A, \neg(A \wedge \neg A) \vdash B$, and, in consequence, that $\vdash A \to (\neg A \to B)$.

Definition 2. - v: $F \rightarrow \{0, 1\}$ is a valuation of β_1 if it is a valuation of β_0, and satisfies the following condition:

4') $v(A) = 1 \Rightarrow v(\neg A) = 0$, for every formula A.

Theorem 9. - β_1 *is strongly sound and strongly complete relatively to the semantics of* β_1*-valuations (the valuations of* β_1*).*

Proof. - See [9].

Theorem 10. - $p \vee \neg p$ (p *is a propositional variable, is not provable in* β_1.

Proof. - Corollary to Theorem 9.

In β_1 there exist valuations v for which $v(p) = v(\neg p) = 0$. But in β_1 no valuation v' satisfies the condition $v'(p) = v'(\neg p) = 1$. Thus, β_1 is paracomplete, though not paraconsistent (see [9]).

Finally, we introduce a third system β_2, which is the same as β_0, but with a different restriction on scheme 11: B can not be atomic, though A may be.

Theorem 11. - *In* β_2: $\vdash A \vee \neg A$ *and* $\vdash \neg(A \wedge \neg A) \rightarrow (A \rightarrow (\neg A \rightarrow B))$.

Proof. - We have:

$$A, \neg(A \vee \neg A) \vdash A \vee \neg A$$
$$A, \neg(A \vee \neg A) \vdash \neg(A \vee \neg A)$$
$$\neg(A \vee \neg A) \vdash \neg A$$

Similarly, $\neg(A \vee \neg A) \vdash \neg\neg A$.

Therefore, $\vdash \neg\neg(A \vee \neg A)$ and $\vdash A \vee \neg A$.

The proof of $\vdash \neg(A \wedge \neg A) \rightarrow (A \rightarrow (\neg A \rightarrow B))$ is immediate.

Definition 3. - v: $F \rightarrow \{0, 1\}$ is a β_2-valuation (or a valuation of β_2) if it is a β_0-valuation satisfying the following extra condition:

4'') $v(A) = 0 \Rightarrow v(\neg A) = 1$.

Theorem 12. - β_2 *is strongly sound and strongly complete with reference to the semantics based on the notion of* β_2*-valuation.*

Theorem 13. - In $\beta_2 : \nvdash \neg(p \wedge \neg p)$.

It is easy to verify that β_2 is paraconsistent, but not paracomplete (see [9]). β_2 is equivalent to system P_1 of [11].

Theorem 14. - β_2 *contains the calculus* C_1 *of* [3].

Employing the terminology of [9], we have in β_0:

$A \wedge \neg A$ *express* that A is true and not *well-behaved*;
$A \wedge \neg(A \wedge \neg A)$ *express* that A is true and *well-behaved*;
$\neg A \wedge \neg(A \wedge \neg A)$ *express* that A is false and *well-behaved*;
$\neg(A \vee \neg A)$ *express* that A is false and not *well-behaved*.

Theorem 15. - β_0, β_1 *and* β_2 *are finitely many-valued calculi.* β_0 *is four-valued and* β_1 *and* β_2 *are three-valued.*

Proof. - Consequence of the corresponding semantics of valuations of the calculi β_0, β_1 and β_2. (Observe that these semantics are two-valued, though not truth-functional.)

The extension of the propositional calculi studied to predicate calculi, with or without equality, offers no essential difficulties. Yet, this extension will not be considered in this paper.

Among other applications, β_0, β_1 and β_2 may be used as the starting point of paracomplete as paraconsistent set theories (cf. [3]), of a reconstruction of Meinong's theory of objects (see [10]), and of logics for the treatment of vague concepts in the senses of [1] and of [8].

2. *Nonstandard inductive logic*

According to von Wright [12], inductive logic has, broadly speaking, two fundamental parts, namely: elimination theory and confirmation theory.

Elimination theory is essentially a logic of conditions, especially as constructed by Keynes, Johnson, Broad, and von Wright. Such logic of conditions is conceived as encompassing the canons of eliminative induction, i.e. Mill's methods, in the new (extended and precise) forms given to them by von Wright.

The central characteristic which makes possible the classical theory of elimination, in von Wright's formulation, consists essentially in the circumstance that appropriate sets of conditions constitute Boolean algebras. The well-known definitions of necessary condition, sufficient condition, necessary and sufficient condition, etc. and their relevant properties, as well as von Wright's generalization of Mill's canons of induction, depend solely on that circumstance.

Nevertheless, our investigation of the systems β_0, β_1 and β_2 could be extended to show that to each of them is associated a corresponding logic of conditions. Moreover, one can check up that any of these logics of conditions is almost identical to a Boolean algebra. The sole significative difference reduces to the fact that the complement $\overline{P}$ of a condition P behaves approximately as a condition independent from P.

Really, with little effort one can extend to the logics of conditions associated with our systems practically all results of von Wright's theory of elimination. We omit the details here, since they are in general trivial.

Now, we pass to the question: what portion of classical confirmation theory can be reproduced in our logics? When one examines von Wright's postulates for the probability calculus, one immediately perceives that the existence of a classical (defined) negation in our logics is sufficient to assure us that the postulates can be reformulated within the field covered by our non-classical logics of conditions. In effect, von Wright's system of postulates does work because the classical logic of conditions constitutes a Boolean algebra. So, in our case, since the logics β_0, β_1 and β_2 contain in a certain sense classical logic, we are able to construct nonstandard confirmation theories similar to that of von Wright. Even sophisticated topics such as Bayes-Laplace's theory of inverse probability, Laplace's law of succession, Bernoulli's law of large members, and Keynes-Broad's theorem of confirmation are reproducible in our (possible) non-classical inductive logics.

Of course, all open philosophical problems of induction remain unsolved whether or not we change our deductive logic, and in consequence our inductive logic. In other words, a change of deductive logic, simply by itself, does not solve the basic questions connected with induction, which we are unable to settle within the domain of classical induction theory. However, what is important and significant is to note that induction can also be treated and investigated when we do not employ classical deductive logic as our *organon* of deductive inference, but instead logics very different from it, for instance paracomplete and paraconsistent systems.

We could demonstrate that other trends in the domain of induction could be treated within the scope of our non-classical logics. For example, the theory presented in [2] and all related later developments, de Finetti's subjectivistic stance (cf. [5]), and extant statistics.

The moral to be drawn from the present exposition is that non-classical deductive logics are really surprisingly strong: in the logics here studied, the derogation of principles so fundamental from the classical stance, such as the laws of contradiction and of excluded middle, does not hinder the functioning of the mechanism of induction.

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