A COMPANION TO MODAL LOGIC – SOME CORRECTIONS

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At three places in our recently published [3] we offer proofs of certain results which we have since found to be fallacious. The purpose of this paper is to diagnose the errors in these purported proofs and to show how they can be replaced by valid ones. Fortunately, in each case the result itself is correct. The results in question are the non-compactness of K4.3W and of S4.3.1, discussed on pp. 105-9, and the completeness of K4, discussed on pp. 145-8. There is also a consequential error in Exercise 6.6 on p. 110.

I

KW (also known as G – see [1]) is the system obtained by adding to the minimal normal modal system K the single extra axiom

W $L(Lp \supset p) \supset Lp$

K4.3W is obtained by the further addition of

$\mathbf{D1}_0$ $L((Lp \cdot p) \supset q) \vee L((Lq \cdot q) \supset p)$

On p. 105 we noted (correctly) that all frames for K4.3W are transitive, irreflexive and weakly connected. But we also claimed, incorrectly, that all generated frames for the system are finite. To see that this is not so, consider the frame <W,R> in which W consists of the natural numbers and a single 'infinite' world, ω, and for any j and $k \in$ W, jRk iff $j > k$. Let us call this frame the ω-*frame*. It may be pictured thus:

$0 \; 1 \; \ldots \; \underleftarrow{i \; \ldots \; \omega}$

This frame is transitive, irreflexive and weakly connected, it is generated (by ω), and it is obviously not finite; but it is easy to check that neither **W** nor $\mathbf{D1}_0$ can be falsified at any world in it, even at ω, and therefore that it is a frame for K4.3W. What is more to the point here, however, is that our argument for the non-compactness of K4.3W on pp. 106-7 proceeded by claiming that although every finite subset of

(Λ) $\{Mp, MMp, \ldots, M^i p, \ldots\}$

is simultaneously satisfiable in some frame for K4.3W, Λ itself is not simultaneously satisfiable in any frame for the system. But this is incorrect, since every wff in Λ is true at ω in a model based on the ω-frame in which p is true at every finite world. (1)

Nevertheless K4.3W *is* non-compact, and this can be proved by a similar argument using another set of wff in place of Λ. We can in fact use a set which was suggested by Fine for proving the non-compactness of KW, and which is mentioned for this purpose in Exercise 6.5 on p. 110. This is

(θ) $\{Mp_1, L(p_1 \supset Mp_2), \ldots, L(p_i \supset Mp_{i+1}), \ldots\}$

We shall show that what we incorrectly claimed about Λ really does hold about θ, viz. (1) that it is not simultaneously satisfiable in any frame for K4.3W, but (2) that every finite subset of it is simultaneously satisfiable in some frame for K4.3W (and that therefore the whole set is K4.3W-consistent); and the existence of such a set is sufficient to prove non-compactness.

To prove (1) we first define a *chain* in a frame <W,R> as a (finite or infinite) set of distinct members of W, $w_0, w_1, \ldots, w_i, \ldots$ such that w_0 R w_1 R ... R w_i R ... (Note that the ω-frame, though it contains infinitely many chains, contains no chains of infinite length; for even if we take ω as the w_0 of the definition, w_1 must then be some finite world, and the chain will terminate in a finite number of steps.)

We next show that any transitive irreflexive frame in which θ is simultaneously satisfiable contains at least one infinite chain. The proof is this. Suppose that in some transitive irreflexive model <W,R,V> there is some $w_0 \in$ W at which every wff in θ is true. Then since Mp_1 is true at w_0, there is some $w_1 \in$ W such that w_0 R w_1 and p_1

(1) That Λ could not be used to prove the non-compactness of K4.3W in fact follows easily from theorem 3 on p. 41 of Fine's [2]. For all the members of Λ are made up from a single variable, and Fine's theorem shows that K4.3W has the property he calls 'weak compactness', *viz* that any consistent set of wff made up from a finite number of variables *is* simultaneously satisfiable on one of its frames.

is true at it; and since R is irreflexive, $w_1 \neq w_0$. Now suppose there is a chain in the model, $w_0,\ldots,w_i$, such that p_i is true at w_i. Then since $L(p_i \supset Mp_{i+1})$ is true at w_0 and R is transitive, $p_i \supset Mp_{i+1}$ is true at w_i, and therefore so is Mp_{i+1}. Hence there must be some $w_{i+1} \in$ W such that w_i R w_{i+1} and p_{i+1} is true at it; and since R is irreflexive and transitive, w_{i+1} must be distinct from w_i and any earlier world in the chain. This is sufficient to show that the frame of the model contains at least one infinite chain.

No frame containing an infinite chain, however, can be a frame for K4.3W (or even for KW). For consider a model based on any such frame in which p is false at every world in the chain in question and true everywhere else. Since the chain is infinite, Lp will be false at every world in it, and therefore $Lp \supset p$ will be true at every such world. Since p is true at every world not in the chain, $Lp \supset p$ is true at all those worlds as well. Thus $Lp \supset p$ is true everywhere in the model, and therefore so is $L(Lp \supset p)$. As a result, **W** is false at every world in the chain. This shows that the frame is not a frame for any system that contains **W**, and thereby proves (1).

To prove (2) we note that any finite subset of θ is a subset of some (finite)

$$\theta_n = \{Mp_1, L(p_1 \supset Mp_2), \ldots, L(p_{n-1} \supset Mp_n)\}$$

It is therefore sufficient to exhibit a frame for K4.3W in which θ_n is simultaneously satisfiable; and such a frame is provided by the frame in which W = $\{w_0,\ldots,w_n\}$ and w_i R w_j iff $i<j$. For every finite strict linear ordering is a frame for K4.3W, and if each p_i $(1 \leqslant i \leqslant n)$ is true at w_i and false everywhere else, then every wff in θ_n is true at w_0.

As we noted earlier, (1) and (2) suffice to prove the non-compactness of K4.3W. But the proof we have given in fact establishes the non-compactness of KW and every extension of it which is contained in K4.3W. The reason is that every frame for KW, and therefore for any extension of KW, must be irreflexive and transitive; so the proof of (1) shows that θ is not simultaneously satisfiable in any frame for any such extension. And since the frame used in the proof of (2) is a frame for K4.3W, it is also a frame for any system contained in it.

II

There is a similar situation with S4.3.1, i.e. S4+

D1 $L(Lp \supset q) \vee L(Lq \supset p)$

and

N1 $L(L(p \supset Lp) \supset p) \supset (MLp \supset p)$

We claimed on p. 109 that the set

(Λ) $\{\alpha_0,\ldots,\alpha_i,\ldots\} \cup \{MLp\}$

where $\alpha_0 = p$, $\alpha_1 = M{\sim}p$, and each α_{i+1} $(i \geqslant 1) = M({\sim}p \cdot M(p \cdot \alpha_{i-1}))$, is not simultaneously satisfiable in any frame for S4.3.1. But this is not so. Consider the frame which is like the ω-frame except that R is reflexive (i.e. jRk iff $j \geqslant k$). Let us call this the *reflexive* ω*-frame*. Like the ω-frame it contains no infinite chains, though ω can see infinitely many worlds. We shall show firstly that it is a frame for S4.3.1, and then that all the wff in Λ can be true together at ω.

The reflexive ω-frame is clearly reflexive, transitive and connected, and as is well known, every such frame validates all theorems of S4.3 (i.e. S4 + **D1**). So to show that it is a frame for S4.3.1 all we need to show is that **N1** is valid on it. In fact we can show that the stronger formula

J1 $L(L(p \supset Lp) \supset p) \supset p$

(from which **N1** easily follows by PC) is valid on it, and thus that it is a frame even for K3.1 (i.e. S4.3.1 with **J1** replacing **N1**). It is convenient here to consider **J1** in its obviously equivalent form

J1′ $L({\sim}p \supset M(p \cdot M{\sim}p)) \supset p$

Suppose that at some $w \in$ W, **J1′** is false. then at w, $L({\sim}p \supset M(p \cdot M{\sim}p))$ is true and p is false (and therefore ${\sim}p$ is true). Since R is reflexive and transitive, we then have ${\sim}p \supset M(p \cdot M{\sim}p)$ true at every $w' \leqslant w$; and so, since ${\sim}p$ is true at w, $M(p \cdot M{\sim}p)$ is also true there. Hence w can see some world at which both p and $M{\sim}p$ are true; and since p is false at w, this must be some world, say n, which is less than w, and therefore finite. Moreover, since $M{\sim}p$ is true at n, n must be related to some m at which ${\sim}p$ is true; and since ${\sim}p$ is false at n,

$m<n$, and *a fortiori* $m<w$. Hence $\sim p \supset M(p \cdot M\sim p)$ is true at m; and so, since $\sim p$ is true at m, $M(p \cdot M\sim p)$ is also true at m, which must therefore be related to some k, with $k<m$, at which p and $M\sim p$ are true. Thus the position with w and n repeats itself with m and k, and as a result there must be an infinite chain of worlds beginning with w, n. But this is impossible, since there are only finitely many worlds between n and 0. Thus **J1** cannot be falsified at any world in the frame, and hence neither can **N1**.

The reflexive ω-frame is therefore a frame for S4.3.1; but if we let p be true at ω and at every even world, and false at every odd world, then every wff in Λ will be true at ω. Thus our purported proof of the non-compactness of S4.3.1 fails.

As with K4.3W, however, we can repair the proof by using another set in place of Λ. To define this set, we let $\alpha_i = p_i \supset M(\sim p_0 \cdot \ldots \cdot \sim p_i \cdot p_{i+1})$. Then the set is

$$(\Psi) \quad \{MLp_0, \sim p_0, M(\sim p_0 \cdot p_1)\} \cup \{L^i\alpha_i : i \geqslant 1\}$$

By using Ψ we can in fact prove the more general result that every modal system that contains **N1** and is contained in K3.1 (i.e. S4 + **D1** + **J1**) is non-compact. (Since **N1** is derivable frm **J1**, this range of systems clearly includes S4.3.1.)

Let S be any system in this range. Our proof will have the same structure as the one given above for K4.3W; i.e. we shall show (1) that Ψ is not simultaneously satisfiable in any frame for S, but (2) that every finite subset of it is simultaneously satisfiable in some frame for S.

To prove (1), suppose that in some model <W,R,V> there is some $w_0 \in W$ at which every wff in Ψ is true. Then in that model: (a) Since MLp_0 is true at w_0, w_0 must see some world w^* at which Lp_0 is true. (b) Since $\sim p_0$ is true at w_0, w^* cannot see w_0. (c) Since $M(\sim p_0 \cdot p_1)$ is true at w_0, w_0 must see some world w_1 at which p_0 is false and p_1 is true. But given a chain $w_1,\ldots,w_n (n \geqslant 1)$, such that w_0 R w_1 and that each $p_i (1 \leqslant i \leqslant n)$ is true at w_i, the truth of $L^n\alpha_n$ at w_0 requires that α_n is true at w_n, and therefore that w_n can see some w_{n+1} at which p_{n+1} is true and each of $p_0,\ldots,p_n$ is false; and since each world among $w_1,\ldots,w_n$ has one of $p_1,\ldots,p_n$ true at it, w_{n+1} must be distinct from any of its predecessors in the chain. There must therefore be an infinite chain of worlds, beginning with w_1, throughout which p_0 is false. From this it follows (d) that w^* cannot see any world in the chain in question, because Lp_0 is true at w^*. In summary, the frame of any such model must contain (i) an infinite chain of worlds $w_1,\ldots,w_i,\ldots$, and (ii) a pair of worlds w_0 and w^* such that w_0 can see both w_1 and w^* but w^* can see neither w_0 nor any world in the chain. (This is not meant to exclude the possibilities that some or all worlds in the chain might also see earlier members of the chain, or w^*, that w_0 might see other worlds in the chain as well as w_1, or that w_0 might be identical with one of the worlds in the chain; but these possibilities do not affect the proof.) It is, however, impossible for any such frame to be a frame for S. For consider a model in which p is false at w_0, true and false alternately throughout the chain (false at all the odd worlds if w_0 is identical with one of these, false at all the even worlds otherwise), and true everywhere else in the frame; then **N1**, which is a theorem of S, will be false at w_0. This may be seen as follows: Let w be any world in the model. If p is true at w, then obviously so is $L(p \supset Lp) \supset p$. If p is false at w, then w is either w_0 or some world in the chain, and in either case w can see some world at which p is true which can in turn see some world at which p is false; thus at w, $L(p \supset Lp)$ is false, and therefore again $L(p \supset Lp) \supset p$ is true. Thus $L(p \supset Lp) \supset p$ is true everywhere in the model, and so $L(L(p \supset Lp) \supset p)$ is true at w_0. Next, since w^* cannot see either w_0 or any world in the chain, Lp is true at w^*, and so MLp is true at w_0. However, p is false at w_0, and therefore so is **N1**.

This suffices to prove (1).

To prove (2) we first note that every finite subset of Ψ is a subset of some (finite)

$$\Psi_n = \{MLp_0, \sim p_0, M(\sim p_0 \cdot p_1), L\alpha_1,\ldots, L^n \alpha_n\}$$

Now consider the frame $\langle W,R \rangle$ where $W = \{w_1,\ldots,w_{n+2}\}$ and R is linear over W in the sense that for any w_i and $w_j \in W$. $w_i R w_j$ iff $i \leqslant j$. It is a straightforward matter to show that every finite linear frame, and therefore $\langle W,R \rangle$, is a frame for K3.1; so, since K3.1 contains S, $\langle W,R \rangle$ is a frame for S. And if we let p_0 be true at w_{n+2} only, and each p_i $(1 \leqslant i \leqslant n+1)$ be true at w_i only, then every wff in Ψ_n will be true at w_1.

(1) and (2) together establish the non-compactness of S.

(It should be noted that a consequential error occurs in Exercise 6.6 on p. 110 of [3]. The set mentioned there – the Λ of p. 109 with the omission of *MLp* – does not yield a non-compactness proof for K3.1 in the way claimed. We have, of course, just shown that Ψ can be used for this purpose, but in fact, if we were only concerned to prove the non-compactness of systems in the range from K1.1 (i.e. S4 + **J1**) to K3.1, the simpler set $\{p_0\} \cup \{L\alpha_i : i \geqslant 0\}$ would do instead.)

III

The third error concerns our claim to have given on p. 147 a proof of the completeness of KW with respect to the class of all finite transitive irreflexive frames, using the method of filtrations. The proof we give here to replace our defective one is due to Warren Goldfarb, and we are grateful to him for permitting us to use it. (2)

Briefly, the argument on p. 147 proceeds by taking an arbitrary non-theorem of KW, α, and then defining a filtration $<W^*,R^*,V^*>$ of the canonical model $<W,R,V>$ for KW, through Φ_α (the set of all sub-formulae of α). We defined W* by letting it consist of precisely one final world in each equivalence class in W with respect to Φ_α: i.e. a world which is not related (by R) to any world in its own equivalence class. We defined R* by saying that for any w and $w' \in W^*$, $w\, R^*\, w'$ iff (a) $w \neq w'$ and (b) for every wff $L\beta \in \Phi_\alpha$, if $V(L\beta, w) = 1$ then both $V(L\beta, w') = 1$ and $V(\beta, w') = 1$. V* was defined in the usual way, as the restriction of V to the members of W*. Now as far as we can see, there was nothing wrong in our proof that $<W^*,R^*,V^*>$ *is* a filtration of $<W,R,V>$ through Φ_α. Nor was there any fault in the proof in the preceding passage (pp. 146-7) that every relevant equivalence class contains some final world. But for the overall proof to succeed, R* must be irreflexive and transitive, and we were incorrect in claiming that it is in all cases. It is, indeed, obvious by condition (a) that R* is irreflexive, but there is nothing in our definition to prevent there being two distinct worlds, w and w', in W* such that both $w\, R^*\, w'$ and $w'\, R^*\, w$, and in such a case, since we cannot have $w\, R^*\, w$, transitivity fails.

(2) Goldfarb's proof, of which we have learned from a private communication, dates from 1979. Goldfarb also has a completeness proof, using a similar method, for K1.1 (S4Grz in his terminology – see [3], pp. 111-162).

A simple counter-example may make this clearer. Let α be p. Then Φ_α is $\{p\}$, and the canonical model for KW will split into two equivalence classes of worlds, (A) those in which p is true (i.e. which contain p), and (B) those in which p is false (i.e. which contain $\sim p$). Now it is easy to show that $\{p, L\sim p, M\sim p\}$ is KW-consistent, and also that any maximal KW-consistent set of wff which includes it is final in A. Similarly, $\{\sim p, Lp, Mp\}$ is KW-consistent, and any maximal KW-consistent set which includes it is final in B. So let W* = $\{w, w'\}$, where w and w' include these two sets respectively. Then since $w \neq w'$, condition (a) for $w\,\mathrm{R}^*w'$ is satisfied; and since Φ_α contains no wff of the form $L\beta$, condition (b) is also (trivially) satisfied. Thus we have $w\,\mathrm{R}^*w'$. And for the same reason we also have $w'\,\mathrm{R}^*w$. Moreover, not only does this particular R* fail to be transitive in such a case, but no R* whatsoever which is suitable in the sense explained on p. 138 of [3] could be both irreflexive and transitive when defined over the w and w' we have just been considering. For condition (1) for suitability, as applied to the present case, requires that if in the canonical model for KW we have $w\,\mathrm{R}u$ for any $u \approx w'$, then we must have $w\,\mathrm{R}^*w'$. Now since w contains $M\sim p$, it must be related in the canonical model to some world which contains $\sim p$, i.e. to some world in B. Thus if R* is suitable, we have $w\,\mathrm{R}^*w'$. Similarly, since w' contains Mp, we have $w'\,\mathrm{R}v$ for some $v \in \mathrm{A}$, and therefore $w'\,\mathrm{R}^*w$. As a result, if R* is transitive we have both $w\,\mathrm{R}^*w$ and $w'\,\mathrm{R}^*w'$, and so it cannot be irreflexive.

Nevertheless, it is possible, following Goldfarb, to define an R* which, though not suitable in every case, yields for any wff which fails on the canonical model for KW a model <W*, R*, V*> which is finite, transitive and irreflexive, and for which we can prove an analogue of the fundamental theorem for filtrations (Theorem 9.1 on p. 139 of [3]), sufficient to show that that wff also fails on <W*,R*,V*>.

As before, we start from an arbitrary non-theorem of KW, α. We then define <W*,R*,V*> as follows. W* is to consist of one world from each equivalence class with respect to Φ_α in the canonical model (though this time it need not be a final world in its class). V* is again to be simply the restriction of V to W*. But R* is now defined by saying that for any w and $w' \in \mathrm{W}^*$, $w\,\mathrm{R}^*w'$ iff

(a′) For some $L\gamma \in \Phi_\alpha$, $L\gamma \in w'$ and $\gamma \notin w'$

and

(b′) For every $L\beta \in \Phi_\alpha$, if $L\beta \in w$, then both $L\beta \in w'$ and $\beta \in w'$.

(Note that (b′) is similar to our original (b), but stated in terms of membership rather than value-assignments; (a′), however, differs considerably from (a).)

Now R*, as thus defined, is both irreflexive and transitive. It is irreflexive because if we ever had $w\,R^*\,w$, then by (a′) we should have to have some $L\gamma(\in \Phi_\alpha)$ in w where γ was not in w; but (b′) requires that on the contrary γ *is* in w. And it is also transitive. For suppose we have both $w_1 R^* w_2$ and $w_2 R^* w_3$. Then since $w_2 R^* w_3$, (a′) ensures that we have $L\gamma \in w_3$ and $\gamma \notin w_3$ for some $L\gamma \in \Phi_\alpha$, and thus (a′) is satisfied for $w_1 R^* w_3$. And furthermore (since $\vdash_{KW} Lp \supset LLp$) whenever we have $L\beta$ in w_1 we also have $LL\beta$ in w_1. Thus by (b′), since $w_1 R^* w_2$, if any $L\beta \in \Phi_\alpha$ is in w_1, $L\beta$ is also in w_2; and by (b′) again, since $w_2 R^* w_3$, we then have both $L\beta$ and β in w_3. Hence (b′) also holds for $w_1 R^* w_3$.

Our original proof in [3] proceeded by showing that the model we defined there was a filtration of the canonical model for KW through Φ_α. Now the model we have just defined will not be such a filtration in every case, since R* will not always be suitable. To see that this is so, consider again the case decribed earlier, in which α is p and w and w' include $\{p, L{\sim}p, M{\sim}p\}$ and $\{{\sim}p, Lp, Mp\}$ respectively. If R* is defined in Goldfarb's way, then both w and w' will be dead ends in $\langle W^*, R^*, V^*\rangle$, since there is no $L\gamma \in \Phi_\alpha$ and so condition (a′) always fails. Hence in this model R* is irreflexive and (trivially) transitive; but, as we showed above, no R* which is suitable can be both irreflexive and transitive for these w and w'. However, the reason for appealing to the suitability requirement is simply to be able to apply the fundamental theorem for filtrations; and when R* is defined in Goldfarb's way, although we cannot prove that $\langle W^*, R^*, V^*\rangle$ is a filtration, we can, as we remarked earlier, prove that an analogue of the fundamental theorem, which will do all the necessary work, holds for it. For we can prove that if $\langle W, R, V\rangle$ is the canonical model for KW and $\langle W^*, R^*, V^*\rangle$ is defined as above, then for every wff $\beta \in \Phi_\alpha$ and every $w \in W^*$, $V^*(\beta, w) = 1$ iff $\beta \in w$. (This result in fact still holds where in place of Φ_α we have any set of wff closed under sub-formulae.)

The only non-trivial step in the proof is the inductive step for L. For this we assume that $L\gamma \in \Phi_\alpha$, take as our inductive hypothesis that $V^*(\gamma, w) = 1$ iff $\gamma \in w$ (for every $w \in W^*$), and show that in that case $V^*(L\gamma, w) = 1$ iff $L\gamma \in w$.

(i) Suppose first that $L\gamma \in w$. Consider any $w' \in W^*$ such that $w R^* w'$. By condition (b′), we have $\gamma \in w'$, and hence by the inductive hypothesis, $V^*(\gamma, w') = 1$, for any such w'. So by the value-assignment rule for L, we have $V^*(L\gamma, w) = 1$.

(ii) Suppose now that $L\gamma \notin w$. Then by the axiom **W**, $L(L\gamma \supset \gamma) \notin w$. So in the canonical model for KW there must be some $u \in W$ such that $w R u$ and $L\gamma \supset \gamma \notin u$, and therefore $L\gamma \in u$ and $\gamma \notin u$. Now let w' be the world in W^* which is equivalent to u with respect to Φ_α. Since $\gamma \in \Phi_\alpha$ we then have $\gamma \notin w'$, and so, by the inductive hypothesis, $V^*(\gamma, w') = 0$. So in order to show that $V^*(L\gamma, w) = 0$, all that remains to be proved is that $w R^* w'$. The proof is this:

To show that condition (a′) holds, we note that, as we have shown above, $L\gamma \in u$ and $\gamma \notin u$. But $L\gamma$ and γ are both in Φ_α. Therefore $L\gamma \in w'$ and $\gamma \notin w'$.

To show that condition (b′) holds, consider any $L\beta \in \Phi_\alpha$ which is in w. Since $\vdash_{KW} Lp \supset LLp$, we then have both $LL\beta$ and $L\beta$ in w, and so, since $w R u$, we have $L\beta \in u$ and $\beta \in u$. Hence, since both $L\beta$ and β are in Φ_α we have $L\beta \in w'$ and $\beta \in w'$ as required.

This completes the proof.

In summary: Given that α is a non-theorem of KW, we know by the general theory of canonical models that for some $w \in W$ in the canonical model $\langle W,R,V \rangle$ for KW, $\alpha \notin w$. We have shown how to define a model $\langle W^*,R^*,V^* \rangle$ in which W^* is finite because Φ_α is, in which R^* is irreflexive and transitive, in which $\alpha \notin w$ for some $w \in W^*$, and for which every wff in Φ_α (and therefore α itself) is false at every world that does not contain it. This is sufficient to prove that KW is complete with respect to the class of all finite transitive and irreflexive models.

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