PRIEST'S PARADOX

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In two recent papers(1) on the logic of paradox Priest has argued, among other things, that the naive theory, i.e. the collection of all theorems and proofs which result from the operation of naive proof procedures, is inconsistent.

His argument, remarkably enough, is reminiscent of the one put forward by Lucas(2) to the effect that there is something which a machine cannot do but the mind can do. Both Lucas and Priest put to use Gödel's first theorem to show that mechanism is not an option open to us if we are willing to preserve consistency. But if Priest's procedure is not novel, the drawn conclusion is. Instead of pressing, as Lucas does, that we should renounce mechanism qua false, Priest presses that we should embrace mechanism qua inconsistently true.

What I intend to show is that whatever the ground for a logic of paradox may be, a reasoning based on Gödel's first theorem is not it. I am not the first to say so, Chihara(3) has preceded me. If I feel justified to enter the arena, it is only because Priest has not been moved by Chihara's paper, which he considers as marred by an unmotivated and circular presumption of consistency.

I will therefore concentrate on making it evident that no presumption of consistency is required to show the unacceptability of Priest's argument, apart from that explicitely entertained by Priest himself for the derivation of the paradox: if the naive theory NT is consistent then it is inconsistent. Besides, I will argue that Priest's paradox requires, to go through, nothing less than a sheer and circular presumption of inconsistency.

(1) Graham PRIEST, 'The Logic of Paradox', *Journal of Philosophical Logic* 8 (1979), 219-241.

Graham PRIEST, 'Logic of Paradox Revisited', *Journal of Philosophical Logic*, 13 (1984), 153-179.

(2) J.R. LUCAS, 'Minds, Machines and Gödel', *Philosophy*, vol. XXXVI (1961).

(3) Charles S. CHIHARA, 'Priest, The Liar, and Gödel', *Journal of Philosophical Logic* 13 (1984) 117-124.

Before examining the paradox in some detail I want to make the following considerations: a) it goes without saying that if you characterize a concept by way of mutually inconsistent characterizations, what you get is a contradictory concept. Yet, to decree that the concept is contradictory it has to be shown that these characterizations are severally unavoidable, i.e. independently acceptable; b) the contradictory character of a concept is usually considered as a compelling reason for holding that the concept is empty, i.e. does not denote anything. Hence c) Priest's paradox, were it inevitable, is not likely to convince anybody (not already so convinced) of the existence of true contradictions and sound inconsistent theories; d) it is in no way obvious, contrary to Priest's belief, that if there is paradox, renouncing the consistency of NT is the solution. In other words, it is not at all clear that to assume the inconsistency of NT is sufficient to have a NT with the properties (axiomatizability and naive correctness) Priest apparently wants to ascribe to it. (I shall return to this later).

The paradox.

The derivation of the paradox consists of two claims and a definition (in fact a third disguised claim).

"Claim 1. Let T be any consistent theory which can represent all recursive functions, whose proof relation is recursive, and whose axioms and rules are naively correct. Then T is incomplete in the sense that there is a naively provable sentence that is not provable in the theory.

Claim 2. The naive theory can represent all recursive functions, and its proof relation is recursive.

The problem in nuce is that the naive theory is, by definition, such that anything which is naively provable is provable in it. Assuming its consistency, it would, therefore, seem to be both complete and incomplete in the relevant sense." (4)

I have allowed myself a full quotation. Indeed, Priest's exact wording is mandatory in view of my thesis that we are facing not a paradox but the appearance of one, which being induced by the sloppiness of the formulation, dissipates under tightening.

To start with I shall proceed to render Priest's presentation of the

(4) PRIEST 1984, p. 165.

paradox more transparent by a reformulation, which I am confident Priest would find unobjectionable.

Definition 1. NT=df the theory in which everything naively provable is provable.

Assumption 1. NT is consistent.

Claim 1. For all naively correct theories T and all sentences A, if A is the Godel sentence of T, then A is naively provable. Hence by def. 1 A is provable in NT.

Claim 2. NT can represent all recursive functions and its proof relation is recursive. Therefore NT is axiomatic (for if it is not, its proof relation cannot be recursive) and, by ass. 1, Godël incomplete.

Now, it should be stressed that even accepting claims 1 and 2 no paradox is going to emerge unless claim 1 is the statement that NT is Gödel complete. For this to be the case we need, what Priest seems to ignore, an extra explicit assumption.

Assumption 2. NT is naively correct.

Indeed, if NT is not naively correct then the provability in it of the Gödel sentence of all naively correct theories tells nothing against the undecidability of its own Gödel sentence, i.e. against its Gödel incompleteness.

Unlike Priest I have no definite idea as to the referent of "naively correct theory". In spite of this I think that it could be fairly assumed that a theory is naively correct if and only if all its theorems are naively correct. But then how could assumption 2 be justified? I see no way, unless we trivially conceive a theory, contrary to claim 2, simply as the unstructured collection of all naive proofs.

In order to justify assumption 2 definition 1 is of no help, considering that the first is not a consequence of the second. On the other hand we cannot so redefine NT: NT=df the axiomatic theory in which everything and only everything naively provable is provable. Quite patently no consistent theory can fall under such a definition: either it proves less or it proves more than everything naively provable. For, if it proves just what is naively provable then it cannot prove its own Gödel sentence, which is naively provable (since it is the Gödel sentence of a naively correct theory). So you cannot start off with such a new definition in a reductio ad absurdum of the hypothesis that NT is consistent. No reduction is either necessary or possible since NT has already been defined as inconsistent.

The fact is, what I am going to show in a moment, that Priest's derivation of the paradox is no more than a cumbersome way of positing such a definition from the very beginning. Let's see why.
In order that claim 1 should be the statement that NT is Gödel complete, it should read:
Claim 1′. For all naively correct theories T and all sentences A, if A is the Gödel sentence of T, then A is provable in the *naively correct* theory NT. Therefore NT does not contain any Gödel undecidable sentence, i.e. NT is Gödel complete.

I must immediately stress that claim 1′ is a version of claim 1 laid down according to the principle of charity. It is how claim 1 should read in order to have the paradox. But in fact claim 1′, and hence claim 1 if it is to be effective, does not make any sense. Supposing NT consistent, as we should do since this is the supposition which has to be reduced to absurdity, there is no way of making sense, and a fortiori of proving correct, the claim that the naively correct NT is godel complete in the sense of being capable of proving the Gödel sentence of all naively correct theories. There are two cases. Case 1: NT is axiomatic (or axiomatizable). Then claim 1 states that NT can prove its own Gödel sentence and hence that NT is inconsistent. Consequently claim 1 is the assumption of what has to be proved, i.e. the inconsistency of NT. Case 2. NT is non-axiomatizable. Then NT is not endowed with a recursive proof relation. Then claim 1 is either false or not inconsistent with claim 2. Claim 1 is false if it is the statement that NT is Gödel complete in the sense of being capable of proving *effectively* the Gödel sentence of all naively correct theories. Claim 1 is not inconsistent with claim 2 if it is the statement that NT is Gödel complete in the sense of being capable of proving *non-effectively* the Gödel sentence of all naively correct theories. For claim 2 is the statement that NT is Gödel incomplete in the sense of containing a sentence which is not *effectively provable* within it.

I have therefore shown that: i) claim 1 as formulated by Priest does not amount to Gödel completeness of NT and consequently is utterly ineffective as step in the derivation of the paradox; ii) claim 1 reformulated as the prima facie effective claim 1′ either is ineffective or amounts to a vicious presumption of the inconsistency of NT.

This, I think, is more than sufficient to dispose of Priest's paradox. Before closing I would like to expand on my statement d (p. 1) that it is in no way obvious that if there were a paradox, renouncing the consistency of NT would be the solution. What Priest has in mind is a theory which should be axiomatic, inconsistent and naively correct (hence complete in the sense of capable of proving the Gödel sentence of all naively correct theories). In other words, NT should be an inconsistent mechanism capable of generating all and only all naive theorems.

The point is that the existence of such a mechanism cannot be simply taken for granted, it has to be proved. I would be tempted to say: if the mechanism exists show it to me. Every recursive set of axioms (rules) is learnable, but nobody could learn something which is never presented to him. Priest seems to think that since a consistent NT with the desired properties (axiomatizability and naive correctness) cannot exist – the very concept is contradictory – then an inconsistent NT with the desired properties must exist.

But such a deduction is unacceptable, unless we want to say that a concept has a denotation provided it is contradictory. I suspect that even Priest would be unwilling to subscribe to such a bold statement.

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