FREE INTUITIONISTIC LOGIC AND ITS S4 COUNTERPART*

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For the intuitionist, a proof that $\exists x A$ consists of proof that $A(a/x)$ for some term a and a demonstration that the term a denotes some element of the intended domain. Similarly, the intuitionist's account of a proof that $\forall x A$ also brings to mind the rendering of the quantifiers in free logic. A number of logicians – including Fourman and Scott, Posy, and Leblanc and Gumb – have been struck by this and have developed various free intuitionistic logics. In the context of computer programs, free logics provide a natural framework for expressing assertions about nonterminating function calls and other execution-time errors [9, 10]. Using a free intuitionistic type theory as the underlying logic, Plotkin has developed the denotational semantics of programming languages in terms of cop's without bottoms. Along a somewhat different line, Schapiro, Myhill, Goodman, and others, inspired by the translations of intuitionistic logic with the standard quantifiers (**IQC**) into quantified **S4** with increasing domains (**QS4**, have investigated both constructive and classical mathematics within the framework of **QS4**. **QS4** and related logics have also been proposed as "modal logics of programs", suitable for characterizing properties of operating systems.

In this paper, we present free first-order intuitionistic logic with equality (**IQC**$*=$) and its free **S4** counterpart under the Tarski-McKinsey translation (**QS4**$*=$**C**), which has increasing domains and invalidates the Fitch formula $a = a \supset \Box\, a = a$. Building upon the work of Fitting and others, we provide semantics and axiomatizations for both of our two logics, show that the Tarski-McKinsey translation preserves validity and invalidity, provide a tableaux system for **IQC**$*=$, and prove that Craig Interpolation Lemma holds for **IQC**$*=$. Finally, we sketch a correction to our defective proof of a

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version of the Compactness Theorem for evolving theories based on **QS4∗ =C** and certain other intensional logics.

A tableaux system and proof of the Craig Lemma for **QS4∗ =C** can be found in [6, 5]. Some of the reasons why the Craig Lemma is of importance in computer science are mentioned in [8]. The tableaux systems for **IQC∗ =** and **QS4∗ =C** and the algorithms that can be extracted from the proofs of the Craig Lemma can be readily automated, but we have not as yet written computer programs to implement them.

1. *Grammar*

As primitive logical constants, **QS4∗ =C** has '$\sim$', '$=$', '$\supset$', '$\forall$', and '$\Box$'. The signs '&', '$\vee$', '$\exists$', and '$\Diamond$' are defined in the usual manner. In **IQC∗ =**, all of the above nonmodal signs are taken as primitive. We understand *sentences* and *theories* (sets of sentences) to be defined as in [5]. $L_{S4}(I, P)$ ($L_I(I, P)$) is the language of **QS4∗ =C** (**IQC∗ =**) restricted to the set of individual parameters I and the set of predicate parameters P. A *quasi-sentence* is like a sentence except that it may contain free individual variables.

2. *Semantics*

We modify the mataphor theoretic semantics of [5], which stands between Leblanc's truth-value semantics [13] and model theoretic semantics, to suit the modal and intuitionistic cases. As for **QS4∗ =C**, let **Po** be the property of being a reflexive and transitive relation, Σ be a nonvoid set of possible worlds, R be a binary **Po**-relation on Σ, J and J′ be sets of individual parameters with J nonvoid, g be a function from J into J′, G be the naturally induced replacement function from $L_{S4}(J, P)$ into $L_{S4}(J', P)$, and d be a function which selects for each possible world $w \varepsilon \Sigma$ a subset of J′ as the set of designating terms in world w subject to the condition that $d(w) \subset d(w')$ for every $w, w' \varepsilon \Sigma$ such that wRw′. (Note that possibly $d(w) = d(w')$.) Finally, let α_{S4} be a function from the Cartesian product of Σ ant the atomic sentences of

L_{S4} (J', P) into {**T**, **F**} subject to the condition [5, 13] that for any world $w \varepsilon \Sigma$ and any $a, a' \varepsilon J'$,

(1) $\alpha_{S4}(w, a = a) = \mathbf{T}$,
(2) $\alpha_{S4}(w, a = a') = \mathbf{F}$ if $a \varepsilon d(w)$ but $\sim a' \varepsilon d(w)$, and
(3) $\alpha_{S4}(w, A) = \alpha_{S4}(w, A(a'/a))$ for any atomic sentence A if $\alpha_{S4}(w, a = a') = \mathbf{T}$.

A **QS4*=C**-*metaphor* is an 9-tuple of the form

$$M_{S4} = \langle J, P, \Sigma, R, J', f, d, \alpha_{S4}, Asgn_{S4} \rangle$$

where $Asgn_{S4}$ is like Asgn [4] with the obvious modifications for the modal case. That is, we understand a sentence $A \varepsilon L_{S4}(J', P)$ to be *true* on $Asgn_{S4}(M_{S4}, w)$ $(w \varepsilon \Sigma)$ if

(1_{S4}) $\alpha_{S4}(w, A) = \mathbf{T}$ when A is atomic,
(2_{S4}) B is not true on Asgn (M_{S4}, w) when $A = \sim B$,
(3_{S4}) B is not true on $Asgn_{S4}(M_{S4}, w)$ or C is when $A = B \supset C$,
(4_{S4}) $B(a/x)$ is true on $Asgn_{S4}(M_{S4}, w)$ for every $a \varepsilon d(w)$ when $A = \forall x B$,
(5_{S4}) B is true on $Asgn_{S4}(M_{S4}, w')$ for every $w' \varepsilon \Sigma$ such that wRw' when $A = \Box B$.

A sentence $A \varepsilon L_{S4}$ (J, P) is *true* on $M_{S4}(w)$ if G(A) is true on $Asgn_{S4}(M_{S4}, w)$. The theory S is **QS4*=C**-*satisfiable* if there is a **QS4*=C**-metaphor M_{S4} such that every sentence $A \varepsilon S$ is true on $M_{S4}(w)$ for some $w \varepsilon \Sigma$. A sentence A is a **QS4*=C**-*logical consequence* of a theory S if $S \cup \{\sim A\}$ is not satisfiable (i.e. not **QS4*=C**-satisfiable). A sentence A is **QS4*=C**-*valid* if A is a logical consequence of $\emptyset$. Other semantic notions are defined as in [5].

As for **IQC*=**, M_I is to be like M_{S4} except for α_I and $Asgn_I$. Let α_I be like α_{S4} subject to the additional requirement that $\alpha_I(w', A) = \mathbf{T}$ whenever $\alpha_I(w, A) = \mathbf{T}$ for some $w \varepsilon \Sigma$ such that wRw'.
Understand

$$M_I = \langle J, P, \exists, R, J', f, d, \alpha_I, Asgn_I \rangle$$

to be an **IQC*=**-*metaphor*, where truth on $Asgn_I(M_I, w)$ is defined recursively as follows:

$A \varepsilon L_I(J', P)$ is *true* on $Asgn_I(M_I, w)$ if

(1_I) $\alpha_I(w, A) = \mathbf{T}$ when A is atomic,
(2_I) either B is true on $Asgn_I(M_I, w)$ or C is when $A = B \vee C$,
(3_I) B and C are true on $Asgn_I(M_I, w)$ when $A = B \& C$,
(4_I) B is not true on $Asgn_I(M_I, w')$ for every $w' \varepsilon \Sigma$ such that wRw' when $A = \sim B$,
(5_I) either B is not true on $Asgn_I(M_I, w')$ or C is for every $w' \varepsilon \Sigma$ such that wRw' when $A = B \supset C$,
(6_I) $B(a/x)$ is true on $Asgn_I(M_I, w)$ for some $a \varepsilon d(w)$ when $A = \exists x B$,
(7_I) $B(a/x)$ is true on $Asgn_I(M_I, w')$ for every $w' \varepsilon \Sigma$ such that wRw' and every $a \varepsilon d(w')$ when $A = \forall x B$.

Let A be a sentence and let S be a theory of **IQC*=**. Understand $A \varepsilon L_I(J, P)$ to be *true* on $M_I(w)$ $(w \varepsilon \Sigma)$ if G(A) is true on $Asgn_I(M_I, w)$. S is **IQC*=**-*satisfiable* is there is an **IQC*=**-metaphor M_I such that for some $w \varepsilon \Sigma$ every $B \varepsilon S$ is true on $M_I(w)$. A is **IQC*=**-*valid* if A is true on M_I for every **IQC*=**-metaphor M_I and every $w \varepsilon \Sigma$ such that $A \varepsilon L_I(J, P)$. A is an **IQC*=**-*logical consequence* of S if A is true on $M_I(w)$ for every **IQC*=**-metaphor M_I and every $w \varepsilon \Sigma$ such that $S \cup \{A\} \subset L_I(J, P)$ and every $B \varepsilon S$ is true on $M_I(w)$. Other semantic notions are defined in the obvious manner.

Note that in **IQC*=** a truth-value assignment α_I is a total function. However, in a semantics for **IQC**, a truth-value assignment might best be left partial to avoid soundness problems. In this respect, the semantics of **IQC*=** is simpler than that of **IQC**.

3. *The Translation*

Let A, B, and C be quasi-sentences of **IQC*=**. The translation **N** of a quasi-sentence A of **IQC*=** into a quasi-sentence of **QS4*=C** is defined inductively as follows [17]:

$$
\begin{aligned}
\mathbf{N}(A) = \; & \Box A \text{ if } A \text{ is atomic,} \\
& \mathbf{N}(B) \vee \mathbf{N}(C) \text{ if } A = B \vee C, \\
& \mathbf{N}(B) \& \mathbf{N}(C) \text{ if } A = B \& C, \\
& \Box \sim \mathbf{N}(B) \text{ if } A = \sim B, \\
& \Box (\mathbf{N}(B) \supset \mathbf{N}(C)) \text{ if } A = B \supset C,
\end{aligned}
$$

$\exists x\, \mathbf{N}(B)$ if $A = \exists x\, B$, and
$\Box\, \forall x\, \mathbf{N}(B)$ if $A = \forall x\, B$.

We have:

THEOREM (McKinsey and Tarski, Rasiowa and Sikorski):
Let A be any sentence of **IQC**$*=$. A is **IQC**$*=$-valid iff $\mathbf{N}(A)$ is **QS4**$*=$**C**-valid.

Proof that the translation **N** preserves validity and invalidity readily follows from:

FITTING'S LEMMA: Let M_I and M_{S4} be as before and also be such that A is true on $\mathrm{Asgn}_I(M_I, w)$ iff $\Box A$ is true on $\mathrm{Asgn}_{S4}(M_{S4}, w)$ for any $w \in \Sigma$ and any atomic sentence $A \in L_I(J', P)$. Then for any $w \in \Sigma$:

(*a*) For any $A \in L_I(J', P)$, A is true on $\mathrm{Asgn}_I(M_I, w)$ iff $\mathbf{N}(A)$ is true on $\mathrm{Asgn}_{S4}(M_{S4}, w)$.

(*b*) For any $A \in L_I(J, P)$ A is true on $M_I(w)$ iff $\mathbf{N}(A)$ is true on $M_{S4}(w)$.

Proof: (*a*) Let $A \in L_I(J', P)$. The proof is by induction on the complexity of A. When A is atomic, the proof is immediate from the assumption. As the cases for the propositional connectives are similar to those in the literature, we give only the cases for the free quantifiers.

Case: $A = \exists x\, B$. By the induction hypothesis, $B(a/x)$ is true on $\mathrm{Asgn}_I(M_I, w)$ iff $\mathbf{N}(B(a/x))$ is true on $\mathrm{Asgn}_{S4}(M_{S4}, w)$ for any $a \in J'$. So $\exists x\, B$ is true on $\mathrm{Asgn}_I(M_I, w)$ iff $B(a/x)$ is true on $\mathrm{Asgn}_I(M_I, w)$ for some $a \in d(w)$ iff $\mathbf{N}(B(a/x)) = \mathbf{N}(B)(a/x)$ is true on $\mathrm{Asgn}_{S4}(M_{S4}, w)$ for some $a \in d(w)$ iff $\exists x\, \mathbf{N}(B) = \mathbf{N}(\exists x\, B)$ is true on $\mathrm{Asgn}_{S4}(M_{S4}, w)$.

Case: $A = \forall x\, B$. By the induction hypothesis, $B(a/x)$ is true on $\mathrm{Asgn}_I(M_I, w')$ iff $\mathbf{N}(B(a/x))$ is true on $\mathrm{Asgn}_{S4}(M_{S4}, w')$ for every $a \in J'$ and every $w' \in \Sigma$. So $\forall x\, B$ is true on $\mathrm{Asgn}_I(M_I, w)$ iff $B(a/x)$ is true on $\mathrm{Asgn}_I(M_I, w')$ for every $w' \in \Sigma$ such that wRw' and every $a \in d(w')$ iff $\mathbf{N}(B(a/x)) = \mathbf{N}(B)(a/x)$ is true on $\mathrm{Asgn}_{S4}(M_{S4}, w')$ for every $w' \in \Sigma$ such that wRw' and every $a \in d(w')$ iff $\Box\, \forall x\, \mathbf{N}(B) = \mathbf{N}(\forall x\, B)$ is true on $\mathrm{Asgn}_{S4}(M_{S4}, w)$.

(*b*) Let $A \in L_I(J, P)$. By part (a), $G(A) \in L_I(J', P)$ is true on As-

$gn_I(M_I, w)$ iff $\mathbf{N}(G(A)) \varepsilon L_{S4}(J', P)$ is true on $Asgn_{S4}(M_{S4}, w)$. So A is true on $M_I(w)$ iff G(A) is true on $Asgn_I(M_I, w)$ iff $\mathbf{N}(G(A)) = G(\mathbf{N}(A))$ is true on $Asgn_{S4}(M_{S4}, w)$ iff $\mathbf{N}(A)$ is true on $M_{S4}(w)$.□

4. *Axiomatization*

Our axiomatizations of **QS4∗ = C** and **IQC∗ =**, having modus ponens as the only rule of inference, are shown in Table 1. The soundness of **QS4∗ = C** is easily had. The proof of completeness uses Leblanc's construction for **QS4∗** (free **S4**) [13] supplemented with a treatment of equality as in Barnes and Gumb [1, 5]. The interested reader can find a brief account of Leblanc and Gumb's somewhat inaccessible work on the completeness and soundness of **IQC∗ =** in [14, 12]. Proof of the soundness of the identity-free version of **IQC∗ =** is immediate and completeness follows by the same considerations discussed in Barnes and Gumb [1, pp. 207-8]. (In our proof, Barnes and I appealed to a result in the literature which had a defective proof. The defect has since been discovered and corrected [12].)

5. *Tableaux*

A Kripke style tableaux system for **QS4∗ = C** has been developed and demonstrated correct with respect to our semantics [5, 6]. As for **IQC∗ =**, we employ Kleene tableaux much as in Fitting [2], and unless indicated otherwise, we follow Fitting's notational conventions. We understand **T**A (read 'true A') and **F**A (read 'false A') to be signed sentences if A is an (unsigned) sentence. Note that we do not follow Fitting's convention of taking A to be atomic.

Let S be as et of signed sentences containing at most one **F**-signed sentence, let $S_T = \{\mathbf{T}A: \mathbf{T}A \varepsilon S\}$, let eq$(a, b)$ be either $a = b$ or $b = a$, and let e.g. in reduction rule **KT** ∨, **T**A ∨ B ε S. As shown in Table 2, the reduction rules are as in Fitting except that we add closure and reduction rules for = and modify the reduction rules for ∀ and ∃ to suit the free case. In reduction rules **KT**∃ and **KF**∀ the individual parameter a is foreign to S (and to A).

A set of signed sentences is *closed* if either (1) F$a = a \varepsilon$ S for some individual parameter a or (2) **T**A, **F**A ε S for some sentence A. Other definitions and conventions are as in Fitting. The striking feature of Kleene tableaus is that the set of signed sentences generated contain at most one **F**-signed sentence. Proof that the Kleene **IQC**∗= tableaux system is sound and complete is similar to that sketched by Fitting for the Kleene **IQC** tableaux system.

6. *The Craig Interpolation Lemma*

A version of the Craig Lemma has been established for **QC**∗= (free logic with equality) [7] and for **QS4**∗=**C** and certain other modal logics [6]. Here, we establish the Craig Lemma for **IQC**∗= using Kleene tableaux much as in Fitting [2]. The Craig-Lyndon Interpolation Lemma for **IQC**∗= can be proven using the same construction.

Following Fitting, we write [A] ([S]) for the set of all individual, sentence, and predicate parameters occurring in the sentence A (the set of sentences S). Not that = is a logical constant (not a predicate parameter). As an immediate consequence of the lemma stated below, we have:

THEOREM (Craig Interpolation Lemma for **IQC**∗=):
If $\vdash A \supset C$, then there is a sentence B such that $\vdash A \supset B$, $\vdash B \supset C$, and $[B] \subset [A] \cap [C]$.

Before stating the lemma, we need some definitions. A *block* is a finite set of signed sentences having no more than one **F**-signed sentence. An *initial part* of a block is a subset of the **T**-signed sentences in the block. If S is a set of sentences, we take S_1 and S_2 to partion S. Let B and U be unsigned sentences and let S be a set of unsigned sentences. Understand U to be an {**T**S, **F**B}/{**T**S_1} (read 'U is an *interpolant* for the block {**T**S, **F**B} with respect to the initial part {**T**S_1}') if $[U] \subset [S] \cap [S, B]$ and both {**T**S_1, **F**U} and {**T**U, **T**S_2, **F**B} are inconsistent.

LEMMA: An inconsistent block has an interpolant with respect to every initial part.

Proof: As in Fitting, the proof is by induction on the length of the closed tableaux for the block. The basis and the cases in the induction step for the propositional connectives and for **KT** ∃ and **KF** ∀ are much as in Fitting. We modify Fitting's proof by using $\forall x(x = x)$ and its negation instead of **t** and **f**, thus obviating the need for rules of combination for **t** and **f**. We sketch the proof for the case in which the first reduction rule used is **KT** ∀. (The cases for **KT=**, **KF=**, and **KF** ∃, are similar [7].) The block is {**TS**, **T** $\forall x A$, **T** $\exists x(x=a)$, **FB**}, and {**TS**, **T**A(a/x), **FB**} is inconsistent. In each case, we construct the interpolant U.

Case 1: The initial part is {**T**S_1}. By the induction hypothesis, there is an I such that

I is an {**TS**, **T**A(a/x), **FB**}/{**T**S_1}.
Take U to be I.

Case 2: The initial part is {**T**S_1, **T** $\forall x A$, **T** $\exists x(x = a)$}. By the induction hypothesis, there is an I such that

I is an {**TS**, **T**A(a/x), **FB**}/{**T**S_1, **T**A(a/x)}.
Take U to be I.

Case 3: The initial part is {**T**S_1, **T** $\exists x(x = a)$}. By the induction hypothesis, there is an I such that

I is an {**TS**, **T**A(a/x), **FB**}/{**T**S_1}.

Case 3.1: $a \,\varepsilon\, [S_2, B]$. Take U to be $\exists x(x = a) \,\&\, I$

Case 3.2: $\sim a \,\varepsilon\, [S_2, B]$. Take U to be $\exists y I(y/a)$, where y is foreign to I.

Case 4: The initial part is {**T**S_1, **T** $\forall x A$}. By the induction hypothesis, there is an I such that

I is an {**TS**, **T**A(a/x), **FB**}/{**T**S_1, **T**A(a/x)}.

Case 4.1: $a \,\varepsilon\, [S_1]$. Take U to be $\exists x(x = a) \supset I$.

Case 4.2.: $\sim a \,\varepsilon\, [S_2]$. Take U to be $\forall y I(y/a)$, where y is foreign to I.□

Finally, note that the following result for **IQC**∗= (**IQC** with equality) does not hold for **IQC**∗= : If $\vdash A \supset C$, neither $\vdash \sim A$ nor $\vdash C$, and both A and C are =-free, then there is an =-free interpolant B. Consider $\vdash \forall x(A \,\&\sim A) \supset AxC$. A similar negative result holds for **QC**∗= and **QS4**∗=**C** [7, 6].

7. *A Note on The Compactness Theorem for Evolving Theories*

One version of the Compactness Theorem states that a theory is satisfiable iff all of its finite subsets are. Stated in this form, compactness holds for both **QS4∗ =C** and **IQC∗ =** in virtue of the (strong) completeness results mentioned above. However, my proof of a version of the compactness theorem for evolving theories [5, Theorem 8.2] based on **QS4∗ =C** and certain other intensional logics is defective, as Hanson [11] has pointed out. These evolving theories have a restriction **Pr** on their accessibility relation possessing, like the restriction **Po** for **QS4∗ =C** and **IQC∗ =**, the computable Kripke closure property [19]. Theorem 8.2 states that a **Pr**-evolving theory having an amenable counterpart is **Pr**-satisfiable iff all of its finite **Pr**-subsystems are. Hanson notes that there is a gap in the proof because it is never demonstrated that **Pr**-satisfiability is preserved under (strong) homomorphisms.

Since any property of relations **Pr** having the monotonic closure property (and hence the computable Kripke closure property) is closed under isomorphisms and subsystems [19], it follows that **Pr** is closed under homomorphisms. Proof that **Pr**-satisfiability is preserved under homomorphisms follows by an argument similar to that used in the proof of Theorem 3.1 [5]. Hence, Theorem 8.2 does hold. The proof can be made cleaner by adding, in the statements of Lemma 3.1, Theorem 3.1, and Corollary 3.1, the supposition that **Pr** is closed under homomorphisms to the supposition that **Pr** is closed under the preimages of homomorphisms. The proof of Corollary 3.1 should be amended as suggested by Hanson.

In any case, closure under homomorphisms should be added to Weaver and Gumb's list of laws [19, 5] of properties of relations having the monotonic and computable Kripke closure properties.

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Table 1: $\mathbf{QS4}^{*}=\mathbf{C}$ and $\mathbf{IQC}^{*}=$ Axiom Schemas

$\mathbf{QS4}{=}\mathbf{C}$	**Common Axioms**	$\mathbf{IQC}^{*}=$
	$A \supset (B \supset A)$	
	$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$	
		$A \supset (B \supset (A \,\&\, B))$
		$A \,\&\, B \supset A$
		$A \,\&\, V \supset B$
		$A \supset A \vee B$
		$B \supset A \vee B$
		$(A \supset C) \supset ((B \supset C) \supset (A \vee B \supset C))$
$(\sim A \supset \sim B) \supset (B \supset A)$		$(A \supset B) \supset ((A \supset \sim B) \supset \sim A)$
		$A \supset (\sim A \supset B)$
	$\forall x(A \supset B) \supset (\forall xA \supset \forall xB)$	
	$A \supset \forall xA$	
	$\forall xA(x/a)$ if A is an axiom	
		$\forall x(A \supset B) \supset (\exists xA \supset \exists xB)$
		$\exists xA \supset A$
	$a = a$	
	$a = a' \supset (A \supset A(a'/a))$ if A is atomic	
	$a = a' \supset (A(a'/a) \supset A)$ if A is atomic	
	$\forall x \exists y(y = x)$	
		$\forall xA \supset (\exists x(x = a) \supset A(a/x))$
		$A(a/x) \supset (\exists x(x = a) \supset \exists xA)$
$\Box(A \supset B) \supset (\Box A \supset \Box B)$		
$\Box A$ if A is an axiom		
$\Box A \supset A$		
$\Box A \supset \Box\Box A$		
$\exists x(x = a) \supset \Box\, \exists x(x = a$		

Table 2: **IQC** $\cdot=$ Kleene Tableaux Reduction Rules

KT$\vee$
$$\frac{S,\ \mathbf{TA} \vee \mathbf{B}}{S,\ \mathbf{TA} \mid S,\ \mathbf{TB}}$$

KF$\vee$
$$\frac{S_T,\ \mathbf{FA} \vee \mathbf{B}}{S_T,\ \mathbf{FA}}$$
$$\frac{S_T,\ \mathbf{FA} \vee \mathbf{B}}{S_T,\ \mathbf{FB}}$$

KT&
$$\frac{S,\ \mathbf{TA} \,\&\, \mathbf{B}}{S,\ \mathbf{TA},\ \mathbf{TB}}$$

KF&
$$\frac{S_T,\ \mathbf{FA} \,\&\, \mathbf{B}}{S_T,\ \mathbf{FA} \mid S_T,\ \mathbf{FB}}$$

KT~
$$\frac{S,\ \mathbf{T}{\sim}\mathbf{A}}{S_T,\ \mathbf{FA}}$$

KF~
$$\frac{S_T,\ \mathbf{F}{\sim}\mathbf{A}}{S_T,\ \mathbf{TA}}$$

KT$\supset$
$$\frac{S,\ \mathbf{TA} \supset \mathbf{B}}{S_T,\ \mathbf{FA} \mid S,\ \mathbf{TB}}$$

KF$\supset$
$$\frac{S_T,\ \mathbf{FA} \supset \mathbf{B}}{S_T,\ \mathbf{TA},\ \mathbf{FB}}$$

KT=
$$\frac{S,\ \mathbf{T}\mathrm{eq}(a,b),\ \mathbf{TA}}{S,\ \mathbf{TA}(b/a)}$$

KF=
$$\frac{S_T,\ \mathbf{T}\mathrm{eq}(a,b),\ \mathbf{FA}}{S_T,\ \mathbf{FA}(b/a)}$$

KT$\forall$
$$\frac{S,\ \mathbf{T}\forall x\mathrm{A},\ \mathbf{T}\exists x(x = \mathrm{A})}{S,\ \mathbf{T}(a/x)}$$

KF$\forall$
$$\frac{S_T,\ \mathbf{F}\forall x\mathrm{A}}{S_T,\ \mathbf{T}\exists x(x = a),\ \mathbf{FA}(a/x)}$$

KT$\exists$
$$\frac{S,\ \mathbf{T}\exists x\mathrm{A}}{S,\ \mathbf{TA}(a/x),\ \mathbf{T}\exists x(x = a)}$$

KF$\exists$
$$\frac{S_T,\ \mathbf{T}\exists x(x = a),\ \mathbf{F}\exists x\mathrm{A}}{S_T,\ \mathbf{FA}(a/x)}$$

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