MECHANICAL REASONING IN FUZZY LOGICS

Ewa ORŁOWSKA, Sławomir WIERZCHOŃ

1. *Introduction*

The basic motivation for dealing with the concept of fuzziness comes out from the observation that in real-life situations we use words, statements or instructions which have imprecise meaning. This type of impreciseness involves adjectives like "tall", "fast"; statements like "country X shows strong inflatinary trends", etc.

The core of the fuzzy set theory is Zadeh's principle of incompatibility (Zadeh, 1973). Roughly speaking the essence of this principle is that as the complexity and human content of a system increases, observer's ability to make precise and yet significant statements about its behaviour decreases; after reaching a certain threshold, precision and relevancy of our descriptions become incompatible attributes. Thus, the objective of the fuzzy set theory is to give a formal machinery to deal with such situations.

Developing this machinery one should be able to distinguish fuzziness from other types of uncertainty. At first fuzzy description is applicable to the situations when the events in question do not recur as well as there is a lack of reliable a priori probabilities. This shows that fuzzy approach is clearly distinguished from probabilistic approaches. However there are other kinds of inexactness different from fuzziness. For example in the case of medical diagnosis we are faced with at least two sources of ignorance: the vagueness of the definition of the diagnostic groups and the limitation of the information that a physician has at his disposal. The first type of ignorance is just the fuzziness while the second rather not (Smets, 1981).

Due to Zadeh, fuzziness is strongly related to the properties of a natural language as well as to human's ability of describing different concepts.

To be more concrete, suppose FOR is a set of formulas of a language and U is a universal set of objects to be described (universe of discourse). The meaning of a formula $A \in FOR$ is a subset $S(A)$

embedded in U, in which any element $u \in S(A)$ admits the description compatible with the formula A. The subset S(A) can be characterized by a function:

$$h_{S(A)} : U \to V$$

determining the grade of membership of any element $u \in U$ to S(A). The set V is referred to as the truth set and its structure depends on different applications. In artificial languages (e.g. computer language) the meaning of all formulas is well-defined and it suffices to take $V = \{0, 1\}$, i.e., any member $u \in U$ belongs or does not to a set S(A). However in natural languages we should take as V a partially ordered set instead of $\{0, 1\}$. In this way we get the commonly used definition of so-called membership function of a fuzzy set (i.e., a set without sharp boundaries) that is a generalization of the characteristic function of a crisp (non-fuzzy) set. More precisely, the following axiom has been established. (Negoita and Ralescu, 1975).

Axiom. The family FUZ(U) of all fuzzy subsets of a universe U is isomorphic to the class MEM(U) of all membership functions over U.

This axiom gave a great impulse for developing a mathematical tools for fuzzy sets theory, and the most convincing argument for dealing with it was Goguen's representation theorem which says that any system satisfying certain axioms is equivalent to the system of fuzzy sets (Goguen, 1974).

However, the definition of a fuzzy set (introduced above) seems to be inadequate. Due to Nahmias, (1979) to say that a fuzzy set A is characterized by a membership function is similar to saying that "a bicycle is characterized by two wheels, handlebars, a chain, etc. which never tells you what a bicycle is". In fact many researches develop a theory of membership functions and not the theory of fuzzy sets. (For stronger criticism of such an approach, see (Zeleny, 1980)).

To be more illustrative consider the following example. Let FOR be again a set of formulas and U be a universe of discourse. Suppose that each formula $A \in FOR$ is semantically well defined, i.e., it suffices to take as the truth set the set $\{0, 1\}$. Note that in this case the meaning S is a multi-valued mapping from T onto U. It was observed by Pawlak (1982) that even in this case it is not possible to describe any subset X of U. The only thing we can do, is to give so-called upper X^* and lower X_* description of the set X, i.e.,

$$X^* = \{A \in FOR: S(A) \cap X \neq \emptyset\},$$
$$X_* = \{A \in FOR: S(A) \subseteq A,\ S(A) \neq \emptyset\}.$$

Of course it is possible to introduce a generalized description d(X) of the set X defined by means of a membership function $f_{d(X)}$: $FOR \to [0, 1]$ such that:

$$X^* = \{A \in FOR\, f_{d(X)}(A) > 0\},$$
$$X_* = \{A \in FOR\, f_{d(X)}(A) = 1\}.$$

However the properties of $f_{d(X)}$ are rather different from the properties of the "classical" membership function.

This example illustrates a simple fact that not every function f: $U \to [0, 1]$ may be treated as a membership function of a fuzzy set. Another such example is the notion of the generalized set introduced by Aumann and Shepley (1974).

The above remarks lead us to the conclusion that there is no one and unique theory of fuzzy sets. We can rather talk about a collection of methods and attempts at dealing with inexactness. All of them take as the point of departure functions

$$h: U \to V$$

where, as previously U is a universe of discourse and V is at least a partially ordered set. The structure of this set hardly depends on context in which our "fuzzy" set works and on the particular kind of uncertainty we currently deal with.

In this connection various logics have been introduced in which elements of set V play a role of truth values and propositional operations admitted in the languages of these logics correspond to functions from the universe of discourse into set V. In section 2 we describe two of these logics and we present mechanical proof procedures for them. In section 3 we give a brief presentation of some of the possible generalizations of fuzzy propositional operations. Several different generalizations have been recently considered in the current literature, e.g. Yager (1980), Dombi (1982), Weber (1983).

2. *Deduction methods in fuzzy logics*

We consider the propositional language whose formulas are built up from the following symbols: propositional language whose formulas are built up from the following symbols: propositional variables taken from an infinite, denumerable set VAR; propositional constant 0 interpreted as a false proposition; propositional operations of negation ($\neg$), disjunction ($\vee$), conjunction ($\wedge$) and implication ($\rightarrow$). Let FOR be the set of all the formulas of the language. Semantics of the language is defined by means of notions of model and value of the formulas in a model. By a model we mean system $M = ([0,1], v)$ consisting of the closed real interval $[0,1]$ treated as the set of truth values and a valuation function $v: VAR \rightarrow [0,1]$ assigning truth values to propositional variables. Given a model M, we define the value of a formula A in model M ($val_M A$) as follows:

$$
\begin{aligned}
val_M p &= v(p) \text{ for } p \in VAR \\
val_M 0 &= 0 \\
val_M A \vee B &= \max(val_M A,\ val_M B) \\
val_M A \wedge B &= \min(val_M A,\ val_M B) \\
val_M A \rightarrow B &= \sup_{z \in [0,1]} \{z : \min(val_M A, z) \leqslant val_M B\} \\
val_M \neg A &= val_M (A \rightarrow 0)
\end{aligned}
$$

It follows from the above definitions that

$$
val_M A \rightarrow B = \begin{cases} 1 \text{ if } val_M A \leqslant Val_M B \\ val_M B \text{ otherwise} \end{cases}
$$

$$
val_M \neg A = \begin{cases} 1 \text{ if } val_M A = 0 \\ 0 \text{ otherwise} \end{cases}
$$

A formula A is true in a model M iff $val_M A = 1$. A formula is a tautology if it is true in every model. A set of formulas is true in a model M whenever every formula in this set is true in M.

We introduce a deduction system for the logic consisting of a set of decomposition rules. The rules enable us to associate with every formula a finite tree in which sequences of subformulas of the formula in question are assigned to every node. The leaves of the tree have associated with them sequences of what is called indecomposable formulas. For these sequences validity can be recognized syntactically and, moreover, there is a natural correspondence between validity of a formula and validity of these sequences of formulas. The system is inspired on the one hand by the Rasiowa-Sikorski system for the classical logic (Rasiowa and Sikorski, 1963) and on the other hand by the normal form theorem established in Nakamura (1963) for the fuzzy logic.

By an indecomposable formula we mean any formula of the form $p \to q$ or $(p \to q) \to q$, where p, q are propositional variables. In what follows we will use the abbreviation $A \Rightarrow B = (A \to B) \to B$. It can be easily seen that in any model M we have:

$$\mathrm{val}_M A \Rightarrow B = \begin{cases} 1 \text{ if } \mathrm{val}_M A > \mathrm{val}_M B \\ \mathrm{val}_M B \text{ otherwise} \end{cases}$$

Hence for any model M formulas $A \to B$ and $A \Rightarrow B$ satisfy the following condition:

$$\mathrm{val}_M(A \to B \vee A \Rightarrow B) = 1$$

We define the decomposition rules which enable us to decompose each formula into a family of sequences of indecomposable formulas. Rules denoted by $[\to(\circ)]$ and $[(\circ)\to]$, where $\circ = \vee, \wedge, \to, \Rightarrow$ provide a decomposition of formulas of the form $A \to (B_1 \circ B_2)$ and $(B_1 \circ B_2) \to A$, respectively, and in the resulting formulas $B_1 \circ B_2$ does not occur as a subformula. Let $S, S_1, \ldots, S_n$, $n \geq 1$, denote finite sequences of formulas. We admit the rules of the form $\dfrac{S}{S_1; \ldots; S_n}$ where formulas in sequences S_i consist of subformulas of formulas in S.

$$[\to(\to)] \quad \frac{S_1, A \to (B \to C), S_2}{S_1, A \to C, B \to C, S_2}$$

$$[\to(\Rightarrow)] \quad \frac{S_1, A \to (B \Rightarrow C), S_2}{S_1, A \to C, B \Rightarrow C, S_2}$$

$$[(\to)\to] \quad \frac{S_1, (A \to B) \to C, S_2}{S_1, B \to C, S_2; S_1; C, A \Rightarrow B, S_2}$$

$$[(\Rightarrow)\rightarrow] \quad \frac{S_1, (A \Rightarrow B) \rightarrow C, S_2}{S_1, B \rightarrow C, S_2; S_1, C, A \rightarrow B, S_2}$$

$$[\Rightarrow(\rightarrow)] \quad \frac{S_1, A \Rightarrow (B \rightarrow C), S_2}{S_1, B \rightarrow C, A \Rightarrow C, S_2}$$

$$[(\rightarrow)\Rightarrow] \quad \frac{S_1, (A \rightarrow B) \Rightarrow C, S_2}{S_1, A \rightarrow B, B \Rightarrow C, S_2}$$

$$[\Rightarrow(\Rightarrow)] \quad \frac{S_1, A \Rightarrow (B \Rightarrow C), S_2}{S_1, B \Rightarrow C, A \Rightarrow C, S_2}$$

$$[(\Rightarrow)\Rightarrow] \quad \frac{S_1, (A \Rightarrow B) \Rightarrow C, S_2}{S_1, A \Rightarrow B, B \Rightarrow C, S_2}$$

$$[\rightarrow(\vee)] \quad \frac{S_1, A \rightarrow (B \vee C), S_2}{S_1; A \rightarrow B, A \rightarrow C, S_2}$$

$$[\rightarrow(\wedge)] \quad \frac{S_1, A \rightarrow (B \wedge C), S_2}{S_1, A \rightarrow B, S_2; S_1, A \rightarrow C, S_2}$$

$$[(\vee)\rightarrow] \quad \frac{S_1, (A \vee B) \rightarrow C, S_2}{S_1, A \rightarrow C, S_2; S_1, B \rightarrow C, S_2}$$

$$[(\wedge)\rightarrow] \quad \frac{S_1, A \wedge B \rightarrow C, S_2}{S_1, A \rightarrow C, B \rightarrow C, S_2}$$

$$[\Rightarrow(\vee)] \quad \frac{S_1, A \Rightarrow (B \vee C), S_2}{S_1, A \Rightarrow B, C, S_2; S_1, A \Rightarrow C, B, S_2}$$

$$[\Rightarrow(\wedge)] \quad \frac{S_1, A \Rightarrow (B \wedge C), S_2}{S_1, A \Rightarrow B, A \Rightarrow C, S_2; S_1, A \Rightarrow B, C \rightarrow B, S_2; S_1, A \Rightarrow C, B \rightarrow C, S_2}$$

$$[(\vee)\Rightarrow] \quad \frac{S_1, (A \vee B) \Rightarrow C, S_2}{S_1, A \Rightarrow C, B \Rightarrow C, S_2}$$

$$[(\wedge)\Rightarrow]\ \frac{S_1, (A \wedge B) \Rightarrow C, S_2}{S_1, A \Rightarrow C, S_2 ; S_1, B \Rightarrow C, S_2}$$

$$[\vee]\ \frac{S_1, A \vee B, S_2}{S_1, A, B, S_2}$$

$$[\wedge]\ \frac{S_1, A \wedge B, S_2}{S_1, A, S_2 ; S_1, B, S_2}$$

$$[\rightarrow(\neg)]\ \frac{S_1, A \rightarrow \neg B, S_2}{S_1, A \rightarrow 0, B \rightarrow 0, S_2}$$

$$[(\neg)\rightarrow]\ \frac{S_1, \neg A \rightarrow B, S_2}{S_1, A \Rightarrow 0, B, S_2}$$

$$[\Rightarrow(\neg)]\ \frac{S_1, A \Rightarrow \neg B, S_2}{S_1, A \Rightarrow 0, B \rightarrow 0, S_2}$$

$$[(\neg)\Rightarrow]\ \frac{S_1, \neg A \Rightarrow B, S_2}{S_1, A \rightarrow 0, 0 \Rightarrow B, S_2}$$

$$[\neg]\ \frac{S_1, \neg A, S_2}{S_1, A \rightarrow 0, S_2}$$

The given rules are sound inference rules, namely the following semantic condition holds. Let $\vee S$ be the disjunction of all the formulas of sequence S.

Lemma 2.1

For every rule of the form $\frac{S}{S_1 ; \ldots ; S_n}$, $n = 1, 2, 3$, and for any model M

$$\mathrm{val}_M \vee S = \mathrm{val}_M((\vee S_1) \wedge \ldots \wedge(\vee S_n))$$

By a cyclic sequence we mean a sequence of the form: $A_1 \circ A_2, \ldots, A_{n-1} \circ A_n,\ A_n \circ A_1$, where $A_i \circ A_j$ denotes $A_i \rightarrow A_j$ or $A_j \Rightarrow A_i$ and at least one of the formulas in the sequence is $A_i \rightarrow A_j$.

Lemma 2.2

The disjunction of all the formulas of a cyclic sequence is a tautology.

A decomposition tree of a formula is obtained by successive applications of the given rules. To each node of the tree a sequence of formulas is assigned: the formula in question corresponds to the root; if a sequence S corresponds to a node s and if a rule of the form $\dfrac{S}{S_1; \ldots; S_n}$ can be applied to S, then sequences $S_1, \ldots, S_n$ correspond to the immediate successors $s_1, \ldots, s_n$ of s, respectively. The highest order of branching in the tree may be 3, it is obtained by application of rule $[\Rightarrow(\wedge)]$. We stop the proccess of decomposition in a node if the sequence assigned to it contains a cyclic subsequence or if all the formulas in the sequence are indecomposable. Such sequences are referred to as end sequences.

The following is a kind of completeness theorem for the given system of rules.

Theorem 2.3

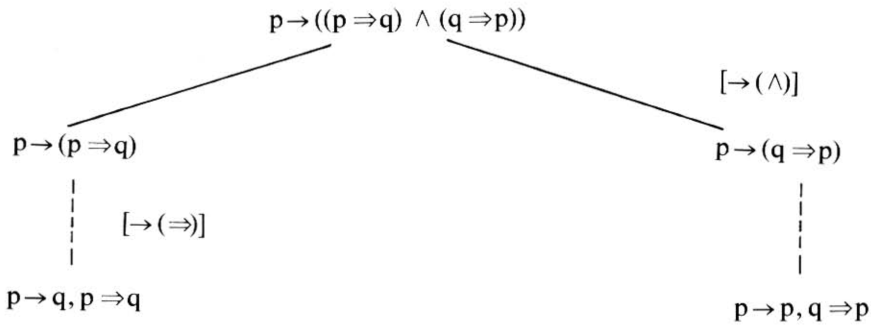
The following conditions are equivalent:

(a) Formula A is a tautology
(b) All the end sequences in a decomposition tree of A contain a cyclic subsequence.

The proof of these theorem can be easily obtained by using lemma 2.1 and lemma 2.2. Below are some simple examples of decomposition trees.

$$\neg(p \wedge \neg p)$$
$$\vdots \; [\neg]$$
$$p \wedge \neg p \rightarrow 0$$
$$\vdots \; [(\wedge)\rightarrow]$$
$$p \rightarrow 0, \quad \neg p \rightarrow 0$$
$$\vdots \; [(\neg)\rightarrow]$$
$$p \rightarrow 0, \; p \Rightarrow 0, 0$$

Sequence $p \to 0, p \Rightarrow 0$ is a cyclic sequence, so the given formula is a tautology.



The end sequences of the tree contain cyclic subsequences $p \to q$, $p \Rightarrow q$, and $p \to p$, respectively.

The resolution-style proof system for the first order fuzzy logic with classical quantifiers, the classical negation such that $val_M \neg A = 1 - val_M A$ and the classical implication $A \to B = \neg A \vee B$ is given in Lee (1972). In this logic a formula is considered to be true (false) in a model M whenever $val_M A \geqslant 0{,}5$ ($val_M A \leqslant 0{,}5$). As in the classical logic the resolution rule enables us to eliminate an inconsistent pair $A, \neg A$ for atomic A, from a pair of clauses and the factoring rule eliminates redundant disjuncts from a clause. Let for a set C of clauses RES(C) denote the set of clauses including C and closed with respect to the rules of resolution and factoring. The following theorems are proved by Lee.

Theorem 2.4

Let $C = \{A_1, \ldots, A_n\}$ be a set of clauses such that $val_M(A_1 \vee \ldots \vee A_n) = b$ and $val_M(A_1 \wedge \ldots \wedge A_n) = a$. Then for any clause $B \in RES(C)$ we have $a \leqslant val_M B \leqslant b$.

Theorem 2.5

The following conditions are equivalent:

(a) Set C of clauses is not true in any model
(b) Set C is not true in any model with the two-valued universe $\{0, 1\}$.

It follows that all the results concerning the resolution system for the classical two-valued logic (Robinson, 1965) can be extended to the fuzzy logic in question.

3. *Extensions of fuzzy logic*

The very broad definition of a fuzzy set does not give us any hints towards defining operations on fuzzy sets. The only indication may be that if the power set P(U) of a universe U is included in the class FUZ(U) of all fuzzy subsets of U then the operations on membership functions should reduce to the usual operations on characteristic functions for $V = \{0, 1\}$. For an axiomatic approach to this problem see (Bellman and Giertz, 1973). The broad class of such operations is provided by what is called triangular norms and triangular conorms introduced in Schweitzer and Sklar (1983).

A function $\sqcap : [0, 1]^2 \to [0, 1]$ is said to be a triangular norm (t-norm) if it satisfies the following conditions :

($\sqcap$1) $0 \sqcap 0 = 0$, $a \sqcap 1 = a$
($\sqcap$2) $a \leqslant a'$ and $b \leqslant b'$ imply $a \sqcap b \leqslant a' \sqcap b'$
($\sqcap$3) $\sqcap$ is commutative
($\sqcap$4) $\sqcap$ is associative.

The less t-norm $\sqcap_1$ is the function defined as follows:

$$a \sqcap_1 b = \begin{cases} a \text{ if } b = 1 \\ b \text{ if } a = 1 \\ 0 \text{ otherwise} \end{cases}$$

The greatest t-norm is the minimum operation.

A function $\sqcup : [0, 1]^2 \to [0, 1]$ is said to be a triangular conorm (t-conorm) if it satisfies the conditions of monotonicity, commutativity and associativity and moreover

($\sqcup$1) $1 \sqcup 1 = 1$, $a \sqcup 0 = a$.

The less t-conorm is maximum operation, and the greatest t-norm is defined as follows:

$$a \sqcup_g b = \begin{cases} a \text{ if } b = 0 \\ b \text{ if } a = 0 \\ 1 \text{ otherwise} \end{cases}$$

In general the laws of distributivity, absorbtion and idempotency do not hold for t-norms and t-conorms. The following theorem is given in (Weber, 1983).

Lemma 3.1

For any t-norm and t-conorm the following conditions hold:

(a) Distributivity laws imply absorbtion laws
(b) Absorption laws imply idempotence laws
(c) Idempotence laws imply t-norm equals minimum operation and t-conorm equals maximum operation.

This result tells us that it is impossible to fulfill the mentioned laws except for the ordinary fuzzy operations. t-norm and t-conorm are related according to the following laws.

Lemma 3.2

(a) $a \sqcup b = 1 - (1-a) \sqcap (1-b)$
(b) $a \sqcap b = 1 - (1-a) \sqcup (1-b)$

Several kinds of implication operators have been considered in the current literature. The following operation is a natural counterpart of the relative pseudo-complement in pseudo-Boolean algebras (Rašiowa and Sikorski, 1963).

$$a \to b = \sup_{z \in [0,1]} \{z : a \sqcap z \leqslant b\}$$

The greatest implication of this type is the operation:

$$a \underset{g}{\rightarrow} b = \begin{cases} 1 \text{ if } a < 1 \\ b \text{ otherwise} \end{cases}$$

The less implication is:

$$a \underset{l}{\rightarrow} b = \begin{cases} 1 \text{ if } a \leqslant b \\ b \text{ otherwise} \end{cases}$$

We can introduce counterparts of the other operations considered in the theory of lattices (Epstein and Horn, 1974).

$$a \leftarrow b = \inf_{z \in [0,1]} \{z: a \sqcup z \geqslant b\}$$

$$a \overset{Z}{\rightarrow} b = \sup_{z \in Z} \{z: a \sqcap z \leqslant b \text{ for } Z \subseteq [0,1]$$

$$a \overset{Z}{\rightarrow} b = \inf_{z \in Z} \{z: a \sqcup z \geqslant b\}$$

Complement operations can be obtained in different ways. First, any function $c: [0, 1] \rightarrow [0, 1]$ can be considered to be a complement if it satisfies the conditions:

(c1) $c(0) = 1,\ c(1) = 0$
(c2) $a \leqslant\ b$ implies $c(b) \leqslant c(a)$.

Second, complement operations can be defined by means of implications in a similar way to that developed in the lattice theory.

$$\urcorner a = a \rightarrow 0 \quad \sqcap\text{-complement}$$
$$\lrcorner a = a \leftarrow 1 \quad \sqcup\text{-complement}$$

It follows that these operations satisfy the following conditions:

$$\urcorner a = \sup_{z \in [0,1]} \{z: a \sqcap z = 0\} = \begin{cases} 1 \text{ if } a = 0 \\ 0 \text{ if } a > 0 \end{cases}$$

$$\lrcorner a = \inf_{z \in [0,1]} \{z: a \sqcup z = 1\} = \begin{cases} 0 \text{ if } a = 1 \\ 1 \text{ if } a < 1 \end{cases}$$

The new operations provide semantical counterparts of propositional operations in languages of many-valued logics, being generalizations of fuzzy logic. The important feature of these new logics is that they are non-distributive and non-complemented structures. It can be proved (Dubois and Prade, 1983 and 1980b) that if the excluded middle $a \sqcup c(a) = 1$ and non-contradiction law $a \sqcap c(a) = 0$ hold for a complement c, then $\sqcup$ and $\sqcap$ cannot be mutually distributive. However it is possible that none of these two properties holds e.g. for $a \sqcap b = a \cdot b$, and $a \sqcup b = a + b - a \cdot b$. For a deeper rationale for studying such non-distributive and non-complemented structures see (De Luca and Termini, 1972a, 1972b). To take the full advantage of these classes of logics investigations are necessary which can throw more light on a structure of the respective algebras. Selecting the adequate logics and designing proof methods for them seems to be an interesting subject for the further research.

4. *Applications*

Fuzzy logic has been developed to cope with ill-defined problems, especially to deal with so-called approximate reasoning (Zadeh, 1979).

As vague problems may differ in their nature one cannot look for a unique theory that allows to solve all such problems. This explains variety of approaches to what is termed "fuzzy logic".

However this logic can be successfully applied in different fields of artificial intelligence. For overview of such applications we refer the reader to (Dubois and Prade, 1980a). Let us mention some of them. Zadeh (1977) proposed PRUF – a meaning representation language for natural languages. FRIL (Balwin and Zhou, 1982) presents another application of fuzzy logic. It is a high level language for designing automatic inferential knowledge base systems. CARDIAG-2 (Adlassing and Kolarz, 1982) is an expert system providing medical diagnosis. References to another applications can be found in (Dubois and Prade, 1984).

Institute of Computer Science
Polish Academy of Sciences
P.O. Box 22
00-901 Warszawa PKiN
Poland

Ewa ORŁOWSKA,
Sławomir WIERZCHOŃ

REFERENCES

[1] Adlassing K.P., and Kolarz G. (1982). CARDIAG-2: Computer-assisted medical diagnosis using fuzzy subsets. In: *Approximate Reasoning in Decision Analysis*, (M.M. Gupta and E. Sanchez, eds.) North-Holland Publ. Co., Amsterdam, pp. 219-246.

[2] Aumann R.J., and Shepley L.S. (1974). *Values of Non-atomic Games*, Princeton University Press, Princeton, New Jersey.

[3] Baldwin J.F. and Zhou S.Q. (1982). *A fuzzy relational inference language* (to appear).

[4] Bellman R.E. and Giertz M. (1973). *On the analytic formalism of the theory of fuzzy sets*. Inf. Sci. 5, 149-157.

[5] De Luca A. and Termini S. (1972a). *Algorithmic aspects in complex system analysis*. Scientia 106, 659-671.

[6] De Luca A. and Termini S. (1972b). *Algebraic properties of fuzzy sets*. J. Math. Anal. and Appl. 40, 373-386.

[7] Dombi J. (1982). *A general class of fuzzy operators, the De Morgan class of fuzzy operators and fuzziness measures induced by fuzzy operators*. Fuzzy Sets and Systems 8, 149-163.

[8] Dubois D. and Prade H. (1980a). Fuzzy Sets and Systems: *Theory and Applications*, Academic Press, New York.

[9] Dubois D., Prade H. (1980b). New results about properties and semantics of fuzzy set-theoretic operators. In: *Fuzzy Sets: Theory and Applications to Policy Analysis and Information Systems* (P.P. Wang, S.K. Chany, eds.) Pergamon Press, pp. 59-75.

[10] Dubois D. and Prade H. (1983). *Fuzzy logics and the generalized modus ponens revisited*, Rapport du L.S.I. no 191 Avril 1984, Toulouse.

[11] Dubois D. and Prade H. (1984). *The management of uncertainty in expert systems: the possibilistic approach*. In: Proc. 10th Triennal IFORS Conf., Washington, D.C.

[12] Epstein G., Horn A. (1974). *P-algebras, an abstraction from Post algebras*. Algebra Universalis 4, 195-206.

[13] Goguen J.A. (1974). *Concept representation in natural and artificial languages: Axioms, extension and applications for fuzzy sets*. Int. J. Man-Mach. Studies, 6, 513-561.

[14] Lee R.C. (1972). *Fuzzy logic and the resolution principle*. Journal of the Association for Computing Machinery 19, 109-119.

[15] Nahmias S. (1979). Fuzzy variables in a random environment. In: *Advances in Fuzzy Set Theory and Applications*, (M.M. Gupta, R.K. Ragade and R.R. Yager, eds.), North-Holland Publ. Co., Amsterdam, pp. 165-180.

[16] Nakamura A. (1963). *A note on truth-value functions in the infinitely many-valued logics*. Zeitschrift für Mathematische Logik und Grundlagen der Mathematik 9, 141-144.

[17] Negoita C.V., Ralescu D.A. (1975). *Applications of Fuzzy Sets to System Analysis*, Birkhauser Verlag, Basel und Stuttgart.

[18] Pawlak Z. (1982). *Rough sets*. Intn. J. Comp. and Info. Sci. 11, 341-366.
[19] Rasiowa H., Sikorski R. (1963). *Mathematics of metamathematics*. Polish Scientific Publishers PWN, Warsaw.
[20] Robinson C.A. (1965). *Machine oriented logic based on the resolution principle*. Journal of the Association of Computing Machinery 12, 23-41.
[21] Schweitzer B. and Sklar A. (1983). *Probabilistic Metric Spaces*, North-Holland Publ. Co., Amsterdam.
[22] Smets P. (1981). *Medical diagnosis: fuzzy sets and degrees of belief*; Fuzzy Sets and Systems, 5, 259-266.
[23] Weber S. (1983). *A general concept of fuzzy connectives; negations and implications based on t-norms and t-conorms*. Fuzzy sets ans systems 11, 115-134.
[24] Zadeh L.A. (1973). Outline of a new approach to the analysis of complex systems and decision processes. In: *Multiple Criteria Decision Making*, (J.L. Cohrane and M. Zeleny, eds.), University of South Caroline Press, Columbia, S.C., 686-725.
[25] Zadeh L.A. (1977a). *PRUF – A meaning representation language for natural languages*. Int. J. Man-Mach. Stud. 10, 395-460.
[26] Zadeh L.A. (1979). A theory of approximate reasoning. In: *Machine Intelligence* (J.E. Hayes, D. Michie and L.I. Mikulich, eds.) vol. 9. Elsevier, New York, pp. 149-194.
[27] Zeleny M. (1980). *Fuzzy sets: Precision and relevancy*. In: (G.E. Lasker, ed.), Applied Systems and Cybernetics, Pergamon Press, New York, vol. VI, 2718-2721.
[28] Yager R.R. (1980). *On a general class of fuzzy connectives*. Fuzzy sets and systems 4, 235-242.