MECHANICAL PROOF METHODS FOR POST LOGICS

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1. *Introduction*

The development of multiple-valued logic in its modern form began with the work of Post (1921) and Łukasiewicz (1920). The first algebra corresponding to the work of Post was formulated in Rosenbloom (1942). The theory of Post algebras and their various generalizations was extensively developed in a number of papers, for example Epstein (1960), Traczyk (1963, 1964, 1967), Dwinger (1968, 1975), Rasiowa (1969, 1973c), Rousseau (1970a), Sawicka (1971), Cat-Ho (1972), Epstein and Horn (1974a, 1974b), Maksimowa and Vakarelov (1974a), Malcew (1976), Muzio (1978), Perrine (1978), Romov (1978), Mirchewa and Vakarelov (1980), Pigozzi (1980). Post algebras play a role of semantical structures for a class of many-valued logics. Investigations on these logical calculi can be found in Rousseau (1967, 1969, 1970b), Rasiowa (1969, 1972), Perkowska (1971), Cat-Ho (1972), Saloni (1972), Epstein and Horn (1974c), Maksimowa and Vakarelov (1974b). Also in Rosser and Turquette (1952) the logic is introduced in which propositional operations can be expressed in terms of operations in a Post algebra.

Advances in multiple-valued logic have been inspired in large part by advances in computer technology and computer science. Among the various applications of these logics are: multiple-valued arithmetics and its applications to digital signal processing, logic design of multiple-valued switching circuits, design of microprogrammed processors and multiple-valued memory elements, parallel processing, natural language computer applications, representation of uncertain and incomplete knowledge. Detailed bibliography on these subjects can be found in Epstein et al. (1974), Wolf (1975), Rine (1977), Butler et al. (1979), Smith (1981).

In the present paper we describe theorem proving systems for the predicate calculus of the m-valued Post logic for an arbitrary finite $m \geqslant 2$. First, we present a Gentzen-style system consisting of rules of decomposition of formulas. The system is defined on the basis of the system for the classical predicate logic given in Rasiowa and Sikorski (1970). The rules enable us to assign to every formula a set of sequences of some simple formulas, whose validity can be recognized syntactically. Formalizations of this kind were also developed in Rousseau (1967, 1970) and Saloni (1972). For the logic of Rosser and Turquette the Gentzen-style axiomatization was provided by Kiryn (1966). Second, we describe a resolution-style system which is based on the system introduced in Robinson (1965) for the classical predicate logic. Roughly speaking, the resolution system provides a test for formulas in order to find out if they are contradictory. The similar system for the logic of Rosser and Turquette is given in Morgan (1974).

2. *Post algebra of order m*

A Post algebra of order $m \geq 2$ is an abstract algebra

$$(P, -, D_1, \ldots, D_{m-1}, \cup, \cap, \Rightarrow, e_0, \ldots, e_{m-1})$$

where P is a nonempty set; $e_0, \ldots, e_{m-1}$ are distinguished elements of P treated as zero-argument operations; $-$, $D_1, \ldots, D_{m-1}$ are unary operations; $\cup$, $\cap$, $\Rightarrow$, are binary operations and moreover the following conditions are satisfied:

(P1) $(P, \cup, \cap)$ is a distributive lattice;

(P2) e_0 is the zero element and e_{m-1} is the unit element of the lattice;

(P3) $(P, e_0, e_{m-1}, \cup, \cap, -, \rightarrow)$ is a pseudo-Boolean (Heyting) algebra;

(P4) $D_i(a \cup b) = D_i a \cup D_i b$;

(P5) $D_i(a \cap b) = D_i a \cap D_i b$;

(P6) $D_i(a \Rightarrow b) = (D_1 a \Rightarrow D_1 b) \cap \ldots \cap (D_i a \Rightarrow D_i b)$;

(P7) $D_i(-a) = -D_1 a$;

(P8) $D_i D_j a = D_j a$;

(P9) $D_i e_j = e_{m-1}$ if $i \leq j$ and $D_i e_j = e_0$ if $i > j$;

(P10) $a = (D_1 a \cap e_1) \cup \ldots \cup (D_{m-1} a \cap e_{m-1})$;

(P11) $D_1 a \cup - D_1 a = e_{m-1}$.

Let us observe that the class of all Post algebras of order $m \geq 2$ is equationally definable. The respective equations are: (P4), . . ., (P11); the lattice axioms of commutativity and associativity for operations $\cup$, $\cap$, and the abserption laws; distributivity laws for $\cup$ and $\cap$; and moreover the following axioms of pseudo-Boolean algebra:

$$a \cap (a \Rightarrow b) = a \cap b$$
$$(a \Rightarrow b) \cap b = b$$
$$(a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \cap c)$$
$$(a \Rightarrow a) \cap b = b$$
$$-(a \Rightarrow a) \cup b = b$$
$$a \Rightarrow (-(a \Rightarrow a)) = -a.$$

Let $\leq$ be the order in the lattice P. The following conditions hold in any Post algebra of order m.

(p12) $e_0 \leq e_1 \leq \ldots \leq e_{m-1}$;
(P13) $D_i a \leq D_j a$ if $j \leq i$;
(P14) If $a \leq b$ then $D_i a \leq D_i b$;
(P15) The set B_P of elements of the form $D_i a$ is closed with respect to operations $-$, $\cup$, $\cap$, $\Rightarrow$, and the algebra $(B_P, -, \cup, \cap, \Rightarrow)$ is a Boolean algebra.

In semantics of many-valued logic the m-element Post algebra P_m plays a special role. Algebra P_m is a Post algebra of order m in which set P of elements equals $\{e_0, \ldots, e_{m-1}\}$ and the operations are defined as follows:

$$e_i \cup e_j = e_{\max(i, j)}$$
$$e_i \cap e_j = e_{\min(i, j)}$$
$$e_i \Rightarrow e_j = e_{m-1} \text{ if } i \leq j \quad e_i \Rightarrow e_j = e_j \text{ if } i > j$$
$$-e_i = e_i \Rightarrow e_0$$
$$D_i e_j = e_{m-1} \text{ if } i \leq j \quad D_i e_j = e_0 \text{ if } i > j$$

The Boolean algebra corresponding to algebra P_m is the two-element Boolean algebra.

In the semantics of the m-valued Post logic we admit m truth values being elements of the algebra P_m. Propositional operations in the logic correspond to the algebraic operations in P_m. To provide semantics for first order language we consider generalized Post algebras with infinite operations of meet (P) $\cap$ and join (P) $\cup$. These operations satisfy the following conditions (Epstein (1960)) for every elements a, a_t, $t \in T$, in algebra P:

(P16) $a = (P) \bigcup_t a_t$ iff $D_i a = (B_P) \bigcup_t D_i a_t$;

(P17) $a = (P) \bigcap_t a_t$ iff $D_i a = (B_P) \bigcap_t D_i a_t$.

3. *The language of Post logic*

The language of Post logic is a first order language with classical operations of negation ($\neg$), disjunction ($\vee$), conjunction ($\wedge$), and implication ($\rightarrow$), with quantifiers $\forall$ and $\exists$, and with the special unary operations corresponding to the algebraic operations D_i. The formulas of the language are constructed from symbols taken from the following at least denumerable and pairwise disjoint sets:

VAR set of individual variables
$\{\neg, D_1, \ldots, D_{m-1}\}$ unary propositional operations
$\{\vee, \wedge, \rightarrow\}$ binary propositional operations
$\{\forall, \exists\}$ quantifiers
FUN_n $n = 0, 1, \ldots$ set of n-argument function symbols
REL_n $n = 1, 2, \ldots$ set of n-argument relation symbols

Elements of set FUN_0 play the role of individual constants. The set T of terms is the least set defined as follows:

$VAR \subseteq T$
$f \in FUN_n$, $t_1, \ldots, t_n \in T$ imply $f(t_1, \ldots, t_n) \in T$.

The set FOR of formulas of the language is the least set satisfying the following conditions:

$R \in REL_n$, $t_1, \ldots, t_n \in T$ imply $R(t_1, \ldots, t_n) \in FOR$
$A \in FOR$ implies $\neg A, D_i A \in FOR$ for all $i = 1, \ldots, m-1$
$A, B \in FOR$ implies $A \vee B$, $A \wedge B$, $A \rightarrow B \in FOR$
$A \in FOR$ implies $\forall xa$, $\exists xa \in FOR$.

As usually we assume that formulas do not contain redundant or overlapping quantifiers and we admit the standard definition of free and bound variable. A formula is said to be open if it does not contain quantifiers. A formula is closed if it does not contain free variables.

Formulas of the form $R(t_1, \ldots, t_n)$ are called atomic formulas.

The formulas of the m-valued Post logic take truth values from the set $\{e_0, \ldots, e_{m-1}\}$ of elements of the m-element Post algebra P_m. Propositional operations correspond to the respective algebraic operations and quantifiers to infinite operations in P_m. Intuitively, a formula D_iA says that the value of expression A is not less than e_i.

4. *Semantics of the language*

By a model we mean system $M = (U, m)$, where U is a nonempty set and m is a meaning function such that

$$m(f) \in U^{U^n} \text{ for } f \in FUN_n$$
$$m(R) \in \{e_0, \ldots, e_{m-1}\}^{U^n} \text{ for } R \in REL_n.$$

Thus meaning function assigns functions from the Cartesian product U^n into U to n-argument function symbols and functions from U^n into $\{e_0, \ldots, e_{m-1}\}$ to n-argument relations. Given a model M, by a valuation we mean a function $v: VAR \rightarrow U$ assigning elements of the universe of the model to individual variables. Let VAL(M) be the set of all the valuations corresponding to model M. Given a model M and a valuation $v \in VAL(M)$ we define function $val_{M,v}$ assigning elements of set U to terms and elements of algebra P_m to formulas:

$$val_{M,v}(x) = v(x) \text{ for } x \in VAR$$
$$val_{M,v}\, f(t_1, \ldots, t_n) = m(f)(val_{M,v}t_1, \ldots, val_{M,v}t_n) \text{ for } f \in FUN_n,\ t_1, \ldots, t_n \in T$$
$$val_{M,v}R(t_1, \ldots, t_n) = m(R)(val_{M,v}t_1, \ldots, val_{M,v}t_n) \text{ for } R \in REL_n,\ t_1, \ldots, t_n \in T$$
$$val_{M,v} \neg A = -val_{M,v}A$$
$$val_{M,v}\, A \vee B = val_{M,v}A \cup val_{M,v}B$$
$$val_{M,v}\, A \wedge B = val_{M,v}A \cap val_{M,v}B$$
$$val_{M,v}\, A \rightarrow B = val_{M,v}A \Rightarrow val_{M,v}B$$
$$val_{M,v}D_iA = D_i val_{M,v}A$$
$$val_{M,v}\, \forall xA = (P_m) \bigcap_{u \in U} val_{M,v_u}A$$
$$val_{M,v}\, \exists xa = (P_m) \bigcup_{u \in U} val_{M,v_u}A$$

where $v_u \in VAL(M)$ is the valuation such that $v_u(x) = u$ and $v_u(z) = v(z)$ for $z \neq x$.

A formula A is true in a model M if for any valuation $v \in VAL(M)$ we have $val_{M,v}A = e_{m-1}$. A set of formulas is true in a model M if all the formulas from this set are true in M. A formula A is e_k-valid if for every model M and for every valuation $v \in VAL(M)$ we have $val_{M,v}A \geqslant e_k$. A formula is a tautology if it is e_{m-1}-valid. The following is an easy consequence of the above definitions.

Lemma 4.1

The following conditions are equivalent:
(a) Formula A is e_k-valid
(b) Formula D_kA is a tautology.

5. *The Gentzen system for the Post logic*

The system presented in this section is a slight modification of the system given in Saloni (1972). A formula is said to be indecomposable if it is of one of the following forms: $D_iR(t_1, \ldots, t_n)$, $\urcorner D_iR(t_1, \ldots, t_n)$ for $i \in \{1, \ldots, m-1\}$. A sequence of formulas is indecomposable if it consists of indecomposable formulas. A sequence of formulas is fundamental if it contains a pair of formulas D_iA and $\urcorner D_jA$ for $i \leqslant j$ and for an atomic formula A. After Saloni we admit the following decomposition rules:

$$(\vee)\ \frac{T_1, D_i(A \vee B), T_2}{T_1, D_iA, D_iB, T_2} \qquad (\urcorner\vee)\ \frac{T_1, \urcorner D_i(A \vee B), T_2}{T_1, \urcorner D_iA, T_2; T_1, \urcorner D_iB, T_2}$$

$$(\wedge)\ \frac{T_1, D_i(A \wedge B), T_2}{T_1, D_iA, T_2; T_1, D_iB, T_2} \qquad (\urcorner\wedge)\ \frac{T_1, D_i(A \wedge B), T_2}{T_1, \urcorner D_iA, \urcorner D_iB, T_2}$$

$$(\rightarrow)\ \frac{T_1, D_i(A \rightarrow B), T_2}{T_1, \urcorner D_1A, D_1B, T_2; \ldots; T_1, \urcorner D_iA, D_iB, T_2}$$

$$(\urcorner\rightarrow)\ \frac{T_1, \urcorner D_i(A \rightarrow B), T_2}{T_1, D_1A, T_2; T_1, D_2A, \urcorner D_1B, T_2; \ldots; T_1, D_iA, \urcorner D_{i-1}B, T_2; T_1, \urcorner D_iB, T_2}$$

$$(\neg)\ \frac{T_1, D_i(\neg A), T_2}{\Leftarrow_1, \neg D_1 A, T_2} \qquad (\neg\neg)\ \frac{T_1, \neg D_i(\neg A), T_2}{T_1, D_1 A, T_2}$$

$$(D_j)\ \frac{T_1, D_i D_j A, T_2}{T_1, D_j A, T_2} \qquad (\neg D_j)\ \frac{T_1, \neg D_i D_j A, T_2}{T_1, \neg D_j A, T_2}$$

$$(\forall)\ \frac{T_1, D_i\ \forall x A(x), T_2}{T_1, D_i A(z), T_2}$$

$$(\neg\forall)\ \frac{T_1, \neg D_i\ \forall x A(x), T_2}{T_1, \neg D_i A(t), T_2, \neg D_i\ \forall x A(x)}$$

$$(\exists)\ \frac{T_1, D_i\ \exists x A(x), T_2}{T_1, D_i A(t), T_2, D_i\ \exists x A(x)}$$

$$(\neg\exists)\ \frac{T_1, \neg D_i\ \exists x A(x), T_2}{T_1, \neg D_i A(z), T_2}$$

where z is a free individual variable which does not occur in any formula above the line, and t is an arbitrary term. In the given rules T_1 denotes an indecomposable or empty sequence of formulas and T_2 is an arbitrary or empty sequence. The following lemma provides soundness of the given system of rules. Let disT be the disjunction of all the formulas from sequence T.

Lemma 5.1

For any rule of the form $\dfrac{T}{T_1; \ldots; T_n}$ $n \geqslant 1$

$$val_{M,v} disT = val_{M,v}(disT_1 \wedge \ldots \wedge disT_n)$$

A decomposition tree of a formula is obtained by successive applications of the rules. The highest order of branching in the tree may be m, it is obtained by the application of $(\neg\rightarrow)$ rule. To each node of the tree a sequence of formulas is assigned. We stop the process of decomposition in a node if the sequence assigned to this node is fundamental or indecomposable. Such sequences are called end sequences.

If T is an end sequence in a decomposition tree then dis T is a tautology iff sequence T is fundamental. If T is not an end sequence then it can be decomposed by exactly one of the given rules. If a rule of the form $\frac{T}{T_1;\ldots;T_n}$ was applied then dis T is a tautology iff dis T_j is a tautology for all $j = 1,\ldots,n$. We conclude that the following lemma holds.

Lemma 5.2

If a decomposition tree of a formula D_kA is finite then the following conditions are equivalent:

(a) Formula A is e_k-valid
(b) All the end sequences in the decomposition tree of formula D_kA are fundamental.

Now, we consider a case of infinite decomposition tree.

Lemma 5.3

If a decomposition tree of a formula D_kA is infinite then A is not e_k-valid.

Proof: If the decomposition tree of formula A is infinite then there exists an infinite branch. Let F be the set of all the indecomposable formulas occuring in the sequences of formulas assigned to the nodes of this branch. Since the branch is infinite, none of these sequences is fundamental. We define the model

$$M_0 = (U_0, m_0)$$

where $U_0 = T$

$$m_0(f)(t_1,\ldots,t_n) = f(t_1,\ldots,t_n)$$
$$m_0(R)(t_1,\ldots,t_n) = e_{i-1} \text{ if } D_iR(t_1,\ldots,t_n) \in F$$
$$m_0(R)(t_1,\ldots,t_n) = e_{m-1} \text{ if for all i } D_iR(t_1,\ldots,t_n) \notin F.$$

Let v_0 be the identity valuation. We will show that $val_{M_0,v_0} D_kA = e_0$. Suppose that $val_{M_0,v_0} D_kA = e_{m-1}$ and let G be the set of all the formulas B appearing in sequences of the infinite branch satisfying $val_{M_0,v_0} B = e_{m-1}$. To each formula B we assign a number ordB as follows:

$$\mathrm{ordR}(t_1, \ldots, t_n) = 1$$
$$\mathrm{ord}(B_1 \circ B_2) = \max(\mathrm{ord}B_1, \mathrm{ord}B_2) + 1 \text{ for } \circ = \vee, \wedge, \rightarrow$$
$$\mathrm{ord} * B = \mathrm{ord}B + 1 \text{ for } * = \neg, D_i, \forall, \exists.$$

Hence function ord reflects complexity of a formula expressed in terms of a number of successive propositional operations influencing each other. Let B_0 be the formula such that $\mathrm{ord}B_0 \leqslant \mathrm{ord}B$ for all $B \in G$. It is easy to see that B_0 is an indecomposable formula, and hence $B_0 \in F$. If $B_0 = D_i A_0$ for some atomic formula A_0 then by the definition of model M_0 we have $val_{M_0,v_0} B_0 = e_0$. Now, let $B_0 = \neg D_i A_0$. If for all j $D_j A_0 \notin F$ then $val_{M_0,v_0} B_0 = e_0$. If $D_j A_0 \in F$ for some j then j must be greater than i, for otherwise a fundamental sequence would appear on the infinite branch. Hence in any case $val_{M_0,v_0} B_0 = e_0$ what is in conflict with the fact that $B_0 \in G$.

By lemma 5.2 and lemma 5.3 we obtain the following theorem.

Theorem 5.4

The following conditions are equivalent:

(a) Formula A is e_k-valid
(b) All the end sequences in the decomposition tree of formula $D_k A$ are fundamental.

As a corollary we obtain a kind of completeness theorem for the given proof system.

Theorem 5.5

The following conditions are equivalent:

(a) Formula A is a tautology
(b) All the end sequences in the decomposition tree of formula $D_{m-1} A$ are fundamental.

6. *The Herbrand theorem*

A formula is said to be in prenex form if it consists of a sequence (possibly empty) of quantifiers whose scope is an open formula. For any formula A of the m-valued Post logic there exists a formula in prenex form equivalent to A. The prenex form of a formula can be obtained due to the following tautologies. Operation $\leftrightarrow$ of equivalence is defined as usually: $A \leftrightarrow B = (A \to B) \wedge (B \to A)$.

$$D_i\, \forall x A(x) \leftrightarrow \forall x D_i A(x)$$
$$D_i\, \exists x A(x) \leftrightarrow \exists x D_i A(x)$$
$$\neg\, \forall x A(x) \leftrightarrow \exists x \neg D_i A(x)$$
$$\neg\, \exists x A(x) \leftrightarrow \forall x \neg A(x)$$
$$B \vee \exists x A(x) \leftrightarrow \exists x(B \vee A(x))$$
$$B \wedge \forall x A(x) \leftrightarrow \forall x(B \wedge A(x))$$
$$B \wedge \exists x A(x) \leftrightarrow \exists x(B \wedge A(x))$$
$$B \vee \forall x A(x) \leftrightarrow \forall x(B \vee A(x))$$
$$\exists x A(x) \to B \leftrightarrow x(A(x) \to B)$$
$$\forall x A(x) \to B \leftrightarrow \exists x(A(x) \to B)$$
$$B \to \exists x A(x) \leftrightarrow \exists x(B \to A(x))$$
$$B \to \forall x A(x) \leftrightarrow \forall x(B \to A(x)).$$

Given a formula A in prenex form, by applying the Skolem method of elimination of quantifiers to all the formulas $D_i A$ for $i = 1, \ldots, m-1$, we obtain the set S(A) of formulas for which the following theorem holds.

Lemma 6.1

(a) If a formula A is true in a model $M = (U,m)$ then set S(A) is true in a model $M' = (U,m')$ such that m and m′ coincide on the function and relation symbols occuring in A.

(b) If set S(A) is true in a model M then A is true in M.

Let us observe that the following formulas are tautologies:

$$D_i(A \vee B) \leftrightarrow D_i A \vee D_i B$$
$$D_i(A \wedge B) \leftrightarrow D_i A \wedge D_i B$$
$$D_i(A \to B) \leftrightarrow (D_1 A \vee D_1 B) \wedge \ldots \wedge (D_i A \vee D_i B)$$

$D_i(\neg A) \leftrightarrow \neg D_1 A$
$D_i D_j A \leftrightarrow D_j A.$

Hence we have the following normal form theorem.

Lemma 6.2

Every open formula of the form D_iA can be expressed as a conjunction of disjunctions of indecomposable formulas.

The usual version of the Herbrand theorem for Post logic was proved in Perkowska (1971). In this paper we present the resolution-style Herbrand theorem. A formula is called D-clause provided that it is a disjunction of indecomposable formulas. Given a set C of D-clauses, the Herbrand universe HU(C) of set C is the set of all terms built up from the function symbols appearing in formulas of C, augmented by a single constant symbol if there is no constant symbol in these formulas. The Herbrand base HB(C) of set C is the set of all the substitution instances over HU(C) of all indecomposable formulas which are sub-formulas of formulas from C. Let $H\overline{B}(C)$ be the set of all the formulas $\neg A$ for $A \in HB(C)$, and let $H(C) = HB(C) \cup H\overline{B}(C)$. Let us observe that all the formulas in set H(C) are closed and hence their values in models do not depend on valuations. A set $X \subseteq H(C)$ is said to be C-inconsistent if $\neg D_iA \in X$ and $D_jA \in X$ for $i \leqslant j$; otherwise X is C-consistent.

A set $X \subseteq M(C)$ is said to be C-complete if for any $A \in H(C)$ we have $A \notin X$ implies $X \cup \{A\}$ is C-inconsistent. Each C-consistent and C-complete subset X of set H(C) determines the Herbrand model HM_X for set C:

$$HM_X = (HU(C), m_X)$$

where $m_X(f)(t_1, \ldots, t_n) = f(t_1, \ldots, t_n)$
$m_X(R)(t_1, \ldots, t_n) \geqslant e_i$ if $D_iR(t_1, \ldots, t_n) \in X$
$m_X(R)(t_1, \ldots, t_n) < e_i$ if $\neg D_iR(t_1, \ldots, t_n) \in X$

A formula $A \in H(C)$ is said to be inessential for a set $X \subseteq H(C)$ whenever the following conditions hold:

D_iA is inessential for X if $D_jA \in X$ for a certain $j > i$
$\neg D_iA$ is inessential for X if $\neg D_jA \in X$ for a certain $j < i$.

Lemma 6.3

(a) If $A \in X \subseteq H(C)$ then A is true in HM_X
(b) If $A \in H(C)$ is true in HM_X for a certain $X \subseteq H(C)$ then $A \in X$ or there is a formula $B \in X$ and that A is inessential for set $\{B\}$.

Lemma 6.4

If a set C of D-clauses is true in a certain model then there exists a Herbrand model HM_X for a certain $X \subseteq H(C)$ such that C is true in HM_X.

In the classical predicate logic the notion of semantic tree plays an important role in the proof of the Herbrand theorem. We define a semantic tree for a set C of D-clauses in a similar way. A semantic tree is a binary tree in which formulas from set H(C) are assigned to the nodes, with exception of the root, to which we assign the empty set. Given a node s, formulas D_iA and $\neg D_iA$ are assigned to the immediate successors of s whenever it can be done without causing inconsistency, that is the set of formulas corresponding to all the predecessors of s augmented by D_iA remains C-consistent, and also it remains C-consistent when augmented by $\neg D_iA$. We repeat this construction for all the formulas from the Herbrand base HB(C). It follows from the construction that in any semantic tree of a set C sets of formulas corresponding to its branches are C-consistent and C-complete.

We say that sets $X, Y \subseteq H(C)$ are D-equivalent whenever they are equal up to inessential elements, that is set X without formulas inessential for X equals set Y without formulas inessential for Y.

Lemma 6.5

(a) For any semantic tree sets of formulas corresponding to different branches of the tree are not D-equivalent
(b) For any C-consistent and C-complete set $X \subseteq H(C)$ and for any semantic tree of C there is a branch such that set of formulas corresponding to this branch and set X are D-equivalent.

We conclude that sets of formulas corresponding to the branches of a semantic tree for a set C provide a representation of all the Herbrand models for C.

Given a set C of D-clauses, let [C] be the set of all the closed formulas obtained from formulas from C by substituting terms from the Herbrand universe HU(C) for individual variables. By a failure point of a branch in a semantic tree of set C we mean an earliest node s on this branch such that set X of formulas corresponding to all the predecessors of s has the property: there exists a formula $A \in [C]$ such that A is not true in model HM_X.

Lemma 6.6

If for any model M a set C of D-clauses is not true in M then every branch in any semantic tree of C contains a failure point.

Proof: Let HM_X be a Herbrand model determined by a set X of formulas corresponding to a certain branch of a semantic tree for C. By the assumption set C is not true in model HM_X. Hence there is a formula $A \in C$ and a valuation v such that $val_{HM_X,v}A \neq e_{m-1}$. This valuation acts as a substitution of terms from the Herbrand universe for individual variables. Hence the formula A′ obtained from A by this substitution is an element of set [C]. Formula A′ is a disjunction of a finite number of indecomposable formulas, hence the branch in question contains the least initial segment such that the finite set Y of formulas corresponding to this segment is {A′} - complete, {A} - consistent and $val_{HM_Y}A' \neq e_{m-1}$. Clearly the last node of this initial segment is a failure point.

The following is the resolution-style Herbrand theorem for the Post logic.

Lemma 6.7

The following conditions are equivalent:

(a) A set C of D-clauses is not true in any model
(b) There exists a finite subset of set [C] which is not true in any model.

Proof: Let C be a set which is not true in any model and let us consider a semantic tree for C. By lemma 6.6 every branch of this tree contains a failure point. Let us consider a tree obtained from the semantic tree in question by rejecting all the nodes which are successors of the failure points. The resulting tree is finite, for otherwise by König's lemma there would exist a branch having no failure point. Each branch b of this tree is related to a certain F-clause from set C, namely to the formula A_b which is not true in the Herbrand model HM_{X_b} determined by set X_b of formulas corresponding to branch b. Moreover, each Herbrand model HM_{X_b} determines the substitution v_b of terms from the Herbrand universe for individual variables responsible for falsification of formula A_b. We define C′ to be the set of D-clauses obtained from all the formulas A_b by applying substitution $v_{b'}$, respectively. Clearly C′ is a finite subset of set [C], and for each branch b set C′ is not true in model HM_{X_b}. By lemma 6.4 set C′ is not true in any model. Now, let us assume that we are given set $C' \subseteq [C]$ which is not true in any model. Every formula A′ from set C′ is a substitution instance over HU(C) of a certain formula A from set C. Hence if A is true in a model M then so is A′. We conclude that if C is true in M then C′ is true in M, what completes the proof.

The Herbrand theorem provides a basis for a resolution theorem proving system for the Post logic.

7. *Resolution system*

The resolution system for the Post logic consists of two inference rules: resolution rule which, roughly speaking, enables us to eliminate an inconsistent pair of indecomposable formulas from a pair of D-clauses, and factoring rule which enables us to eliminate redundant disjuncts from a D-clause.

In what follows by a substitution we mean a function which assigns terms to individual variables. Any substitution can be extended in a natural way on the set of all terms and formulas.

$$\text{(r)}\ \frac{A_1 \vee D_i B_1,\ A_2 \vee \neg D_j B_2}{u(A_1 \vee p(A_2))} \text{ for } i \geqslant j \text{ resolution rule}$$

In this rule A_1 and A_2 are D-clauses; B_1 and B_2 are atomic formulas; u is a least (with respect to composition of functions) substitution which unifies B_1 and $p(B_2)$, that makes them equal; p is a permutation of variables such that $A_1 \vee D_iB_1$ and $p(A_2 \vee \neg D_jB_2)$ have no variables in common.

$$\text{(f)} \quad \frac{A \vee B_1 \vee B_2}{u(A \vee p(B_2))} \quad \text{factoring rule}$$

In this rule A is a D-clause; B_1 and B_2 are indecomposable formulas; u is the least substitution which unifies B_1 and $p(B_2)$; p is a permutation of variables such that B_1 and $p(B_2)$ have no variables in common.

Clearly the given rules preserve validity; given a set C of D-clauses, let RES(C) be the set of clauses including C and closed with respect to the rules (r) and (f).

By $\square$ we denote the empty D-clause containing no disjuncts. The empty clause represents falshood that is for any model M and for any valuation v we have $val_{M,v}\square = e_0$.

Lemma 7.1

For any D-clauses A,B which have no variables in common $[RES(\{A,B\})] = RES([A] \cup [B])$.

The following lemma is an analogue of the Skolem-Löwenheim theorem.

Lemma 7.2

For any nonempty set C of D-clauses if $RES(C) \subseteq C$ and $\square \notin C$ then there exists a model M such that C is true in M.

Proof: Let us suppose that for any model M set C is not true in M. Hence $\square \in C$ or by lemma 6.6 every branch in any semantic tree for C contains a failure point. The first condition contradicts the assumption. Let us consider the tree obtained from the semantic tree by dropping all the nodes which are successors of failure points. We will show that this tree consists of only one node. Suppose that the number of nodes is greater than 1, and assume that there is a branch b with $n \geqslant 2$ nodes. By the definition of semantic tree the $(n-1)$th node has two immediate successors with formulas of the form D_iA and $\neg D_iA$ assigned to them, respectively. Both of them are failure points.

Let X_b be the set of formulas corresponding to the first $n-1$ nodes of branch b and let $X_1 = X_b \cup \{D_iA\}$ and $X_2 = X_b \cup \{\neg D_iA\}$. By the definition of failure point there exist formulas $B_1, B_2 \in [C]$ which are not true in Herbrand models HM_{X_1} and HM_{X_2} respectively. It follows that $\neg D_iA$ occurs in clause B_1 as a disjunct and D_iA occurs in B_2. Let B be the formula obtained from B_1 and B_2 by applying the resolution rule and eliminating an inconsistent pair D_iA, $\neg D_iA$. By lemma 7.1 we have $B \in [C]$. Moreover, B is not true in the Herbrand model HM_{X_b}. Thus the $(n-1)$th node of branch b is a failure point, a contradiction. Hence the semantic tree in question consists of one node only (the root). But the only clause which is not true in the Herbrand model determined by the empty set of formulas is the empty clause. Hence $\Box \in C$, which contradicts the assumption.

The following theorem provides a kind of completeness of the given system.

Theorem 7.3

The following conditions are equivalent:

(a) For any model M set C of D-clauses is not true in M.
(b) There is a derivation of the empty clause $\Box$ from set C.

Proof: Assume condition (a). By lemma 7.2 either (i) $\Box \in C$ or (ii) RES(C) is not included in C. Clearly (i) implies (b). In case (ii) we construct a sequence of sets of D-clauses:

$$C_0 = C,$$
$$C_1 = RES(C_0) \cup C_0,$$

For $j \geq 1$ if C_j is defined then:

- If condition (c) $\Box \notin C_{j-1}$ and $\Box \in C_j$ or $C_{j-1} = C_j$ is satisfied then C_{j+1} is not defined,
- If condition (c) is not satisfied then $C_{j+1} = RES(C_j) \cup C_j$,
- If C_j is not defined then for every $k > j$ C_k is not defined.

It is easy to see that for each $j \geq 0$ the inclusion $C_j \subseteq C_{j+1}$ holds. If for every natural j set C_j is defined then we consider the union $\bigcup_j C_j$. We have $RES(\bigcup_j C_j) \subseteq \bigcup_j C_j$ and $\Box \notin \bigcup_j C_j$. By lemma 7.2 set $\bigcup_j C_j$ is not true in any model and hence C is not true in any model, a contradiction. If for a certain j we have $C_j = C_{j-1}$ then $\Box \notin C_{j-1}$ and $C_{j-1} = RES(C_{j-1}) \cup C_{j-1}$ and by lemma 7.2 set C_{j-1} is not true in any model, a contradiction. If $\Box \in C_j$ for a certain j then condition (b) holds. Hence (a) implies (b). Clearly, we also have (b) implies (a).

8. ω^+-*valued Post logic*

A natural generalization of the m-valued Post logic is the ω^+-valued Post logic, whose formulas may assume values from a linearly ordered set of type ω^+. A semantic basis for the logic is provided by generalized Post algebras of order ω^+. These algebras are linearly ordered Heyting algebras (Horn (1969)) with a chain $\{e_i\}_{0 \leqslant i \leqslant \omega}$ of type ω^+ of constants and with one-argument operations D_i for $1 \leqslant i < \omega$ which have properties similar to those of the operations $D_1, \ldots, D_{m-1}$ in Post algebras of any finite order $m > 2$. The elements $D_i a$ are all the complemented elements in these lattices. Every element a is uniquely reppresented as an infinite join $a = \bigcup_{i=1}^{\infty} D_i a \cap e_i$. The generalized Post algebras of order ω^+ and the ω^+-valued predicate logic were investigated by many authors, for example Sawicka (1971), Rasiowa (1973c), Maksimowa and Vakarelov (1974). The Hilbert style axiomatization of the logic contains, among others the ω-rule with infinitely many premises:

$$\frac{D_i A \text{ for every } 1 \leqslant i < \omega}{A}$$

The mechanical proof methods for the logic are given in Orłowska (1976, 1978). The applications of the logic are related to theory of programs, see for example. Rasiowa (1973a, 1973b, 1974). Survey papers on multiple-valued logics in which their applications in computer science are stressed are Epstein et al. (1974), Wolf (1975), Butler et al. (1979), Smith (1981).

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