A RESULT FOR COMBINATORS, BCK LOGICS AND BCK ALGEBRAS

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Introduction

A *BCK* logic is an implicational logic based on modus ponens and the following axiom schemes:

Axiom B $\mathscr{A} \supset \mathscr{B} . \supset . (\mathscr{C} \supset \mathscr{A}) \supset (\mathscr{C} \supset \mathscr{B})$
Axiom C $\mathscr{A} \supset (\mathscr{B} \supset \mathscr{C}) . \supset . \mathscr{B} \supset (\mathscr{A} \supset \mathscr{C})$
Axiom K $\mathscr{A} \supset (\mathscr{B} \supset \mathscr{A})$.

In this paper, as a corollary of a theorem on combinators, we prove the following, perhaps surprising result:

Theorem 2 If $\mathscr{A} \supset \mathscr{B}$ and $\mathscr{C}$ are theorems of *BCK* logic, then there is a substitution instance $\mathscr{A}_1 \supset \mathscr{B}_1$ of $\mathscr{A} \supset \mathscr{B}$ and once $\mathscr{C}_1$ of $\mathscr{C}$, such that $\mathscr{A}_1 = \mathscr{C}_1$. $\mathscr{B}_1$ is therefore a theorem of *BCK* logic.

In other words substitution instances of any pair of theorems of *BCK* logic can be used as minor and major premises in the rule modus ponens.

Theorem 2 of course does not hold in classical or intuitionistic implicational logics. For example we cannot use instances of the following as minor and major premises for modus ponens:

$$\mathscr{A} \supset \mathscr{A}$$
$$\mathscr{A} \supset (\mathscr{A} \supset \mathscr{B}) . \supset . \mathscr{A} \supset \mathscr{B} .$$

Combinators and Functional Characters

The Axioms B, C and K are named after the combinators B, C and K because they are directly related to the "principal functional characters" of these combinators.

The combinators B, C and K are operators with the following basic properties:

$$BXYZ = X(YZ) \qquad \text{-(1)}$$
$$CXYZ = XZY \qquad \text{-(2)}$$
$$KXY = X \qquad \text{-(3)}$$

Other BCK-combinators [1] are given by:

If X and Y are BCK-combinators so is (XY).

Each combinator has associated classes called *functional characters* (fcs); the most general of these is called the *principal functional character*.

Functional characters are built up from certain basic ones and the rule:

If α and β are fcs so is $F\alpha\beta$.

The term F has the following rule:

Rule F $\qquad F\alpha\beta X\ ,\ \alpha Y \vdash \beta(XY)$

αY can be interpreted as "Y has (or is an element of) fc or type α". $F\alpha\beta$ is then interpreted as the fc, type (or class), of all functions from α into β.

The rule then says: If X has fc $F\alpha\beta$ and Y has fc α then XY has fc β.

The principal functional characters (pfcs) of B, C and K are given by:

$$\vdash F(F\alpha\beta)(F(F\gamma\alpha)(F\gamma\beta))B$$
$$\vdash F(F\alpha(F\beta\gamma))(F\beta(F\alpha\gamma))C$$
$$\vdash F\alpha(F\beta\alpha)K.$$

Note that these are the most general possible under (1), (2), (3) and Rule F. The one for K, together with Rule F, for example, gives:

$$\alpha X,\ \beta Y \vdash \alpha(KXY)$$

[1] Standard combinators also include S with the property

$$SXYZ = XZ(YZ)$$

B and C can be defined in terms of K and S.

Standard combinators can represent repetitions of terms, BCK-combinators do not. i.e., No term, made up only of the variables $X_1, X_2, \dots, X_n$ and brackets, in which at least one X_i occurs more than once, can be represented by $AX_1 X_2 \dots X_n$, where A is a BCK combinator.

which is what might be expected for arbitrary fcs α and β by (3).

The pfcs for B, C and K translate directly into Axioms B, C and K for BCK logic if each $F\alpha\beta$ is replaced by $\alpha \supset \beta$. Also an application of Rule F has the same effect on the fcs as an application of modus ponens has in BCK logic. (For details on fcs see [2].)

Thus for example to find the pfc of CK, we take the fc $F(F\alpha(F\beta\alpha))(F\beta(F\alpha\alpha))$ of C which is the most general one that will fit with the pfc $F\alpha(F\beta\alpha)$ of K.

Thus by Rule F $\vdash F\beta(F\alpha\alpha)(CK)$.

This corresponds to the application of modus ponens to the major premise

$$\vdash \mathscr{A} \supset (\mathscr{B} \supset \mathscr{A}) \,.\!\supset\!.\, \mathscr{B} \supset (\mathscr{A} \supset \mathscr{A}),$$

an instance of Axiom C, and Axiom K as minor premise, resulting in

$$\vdash \mathscr{B} \supset (\mathscr{A} \supset \mathscr{A})$$

The correspondence between the pfcs of combinators and the theorems of intuitionistic implicational logic was first noticed by Curry (see § 9E of [2].) Further work on this was done, amongst others, by Howard [4].

The stratification theorem of [2] gives sufficient conditions for (standard) combinators to have a functional character. There are standard combinators that have no fc and there are others, including some BCK combinators, that have a fc but that do not satisfy the conditions of the theorem.

The Four Results In this paper we show the following:

Theorem 1 All BCK-combinators have a pfc.

This, in view of the fact that the B, C and K combinators do not give rise to duplications, is not all that surprising, but Theorem 2, the corresponding result in BCK logic, that follows directly from it, is more so.

A BCK algebra is one based on a set T and an operator *. The set T has at least one element 0. Axioms can be written as:

$$((b * c) * (a * c)) * (b * a) = 0$$

$$((c * a) * b) * ((c * b) * a) = 0$$
$$(a * b) * a = 0$$
$$a * 0 = a$$

Substitution of equality is allowed and $a * b = 0$ is usually written as $a \leq b$.

There is an equivalence between a *BCK* logic and a *BCK* algebra (outlined in [1]) which gives, by Theorem 2:

Theorem 3 If in a *BCK* algebra $a \leq b$ and $c = 0$ are theorems, then there is a substitution instance a_1 of a, a corresponding one b_1 of b and once c_1 of c such that $b_1 = c_1$. Thus $a_1 = 0$.

A fourth version of this theorem involves the "Fool's Model" of combinatory logic of [5]. This can be described briefly as follows:

Let S be an arbitrary nonempty set and consider the algebra $<S, \rightarrow, V>$, where V is the closure of S under the binary operation $\rightarrow$.

On $\mathscr{P}V$ now define the operation $\circ$ by:

$$\alpha \circ \beta = \{C \mid (\exists B)(\exists D) B \in \beta \wedge D \in \alpha \wedge C \in \alpha \wedge D = (B \rightarrow C)\}$$

We can then "define" combinators as elements of $\mathscr{P}V$.

$$K = \{A \rightarrow . B \rightarrow A \mid A, B \in V\}$$
$$B = \{(A \rightarrow B) \rightarrow . (C \rightarrow A) \rightarrow (C \rightarrow B) \mid A, B, C \in V\}$$
$$C = \{A \rightarrow (B \rightarrow C) . \rightarrow . B \rightarrow (A \rightarrow C) \mid A, B, C \in V\}$$
$$W = \{A \rightarrow (A \rightarrow B) . \rightarrow . A \rightarrow B \mid A, B \in V\}$$

etc. (Note that these sets correspond to the axiom schemes and to the fcs.)

The operation $\circ$ behaves like composition; for example

$$C \circ K = \{B \rightarrow (A \rightarrow A) \mid A, B \in V\}$$

and

$$(C \circ K) \circ K = \{A \rightarrow A \mid A \in V\}.$$

The latter we call I.

K however works in a slightly restricted sense.

$$K \circ \alpha = \{B \rightarrow A \mid A \in \alpha \wedge B \in \beta\}$$
$$(K \circ \alpha) \circ \beta = \{A \mid A \in \alpha \wedge B \in \beta\}$$

so

$$(K \circ \alpha) \circ \beta = \alpha \text{ if } \beta \neq \emptyset.$$

In the presence of W an "empty combinator" can be defined by $W \circ I$.

so $(K \circ K) \circ (W \circ I) = \emptyset$ rather than K.

If however we consider the subalgebra of the algebra $<\mathscr{P}V, \circ>$ based only on B, C and K, Theorem 1 gives us:

Theorem 4 In the algebra $<\{K,B,C\},\circ,L>$ where L is the closure of $\{K,B,C\}$ under $\circ$, $\emptyset \notin L$.

In this algebra, as it contains no $\emptyset$, K now has its expected property

$$(K \circ \alpha) \circ \beta = \alpha.$$

The Proof of Theorem 1

Before we prove Theorem 1 we need two lemmas and the definition of a *BCK-term*.

B, C, K and the variables $X_1, X_2, \ldots, X_n, \ldots$ are *BCK*-terms.

If X and Y are *BCK* terms so is (XY).

The first lemma is a *BCK* version of the Subject Reduction/Expansion Theorem of [2] (§ 9C2-4) and [3] (§ 14B). The proof of the lemma for *BCK* terms is similar to that of this theorem.

Lemma 1 If X and Y are *BCK* terms and X = Y can be proved by (1), (2) and (3) then X and Y have the same pfc.

Note that in obtaining the pfc of a term with variables the fcs for the variables are chosen so as to make the fc of the term as general as possible.

Our second lemma requires the notion of the level of a term.

A term equal (by (1), (2) and (3)) to one containing no combinators has level 0.

If there is a natural number n such that

$$YX_{i_1}X_{i_2}..X_{i_n} = X_j Y_1 Y_2 .. Y_k,$$

where Y is a term, $X_{i_1}, X_{i_2} \ldots X_{i_n}$ are variables not in Y, X_j is a variable and $Y_1, Y_2, \ldots, Y_k$ are terms of levels $\leq m$, at least one of which has level m, then Y has level $m+1$.

For example, B, BX_1, KX_1 and CX_1X_2 are of level 1, BX_1B and CBB are of level 2 and $BX_1(CBB)$ is of level 3.

Lemma 2 If Y is a term containing no variable more than once, there is a natural number n such that for any β, given an appropriate assignment of fcs to the variables in $YX_{i_1}..X_{i_n}$ ($X_{i_1},\ldots,X_{i_n}$ are variables not in Y), $YX_{i_1}\ldots X_{i_n}$ has fc β.

Proof We first prove the lemma for a term Y without combinators ($n=0$ will do in this case). We prove this by induction on m the number of variables in Y.

$m = 1$ Y is a single variable and so can be assigned the fc β.
Assume the result for $m < p$.

$m = p$ In this case $Y = Y_1 Y_2$, where Y_1 and Y_2 each contain fewer variables than Y (no variable occurs twice). Thus Y_1 and Y_2 can be assigned arbitrary fcs, so we assign Y_1 $F\alpha\beta$ and Y_2 α. Y then has fc β.

Now we prove the lemma by induction on k the level of Y.

$k = 0$ By Lemma 1, Y has the same pfc as a term Z without combinators, so by above this is β.

Assume that result for $k < q$

$k = q$ Now there is a natural number n such that

$$YX_{i_1}\ldots X_{i_n} = X_j Y_1 \ldots Y_k$$

where each Y_t has level $< q$.

Thus for each Y_t there is a natural number r such that $Y_t X_{s_1} \ldots X_{s_r}$ has an arbitrary fc β_t.

If the fcs of $X_{s_1},\ldots,X_{s_r}$ needed to achieve this are $\alpha_{t_1},\ldots,\alpha_{t_r}$, Y_t has fc $\alpha_t = F_q \alpha_{t_1} \ldots \alpha_{t_q} \beta_t$(2).

Note that as no variable appears twice in $YX_{i_1}\ldots X_{i_n}$, none will appear twice in $X_j Y_1 \ldots Y_k$, so the fcs for the variables in each $Y_t X_{s_1} \ldots X_{s_r}$ and for X_j can be chosen without fear of conflict.

(2) $F_1 = F$

$$F_{m+1}\alpha_1\alpha_2\ldots\alpha_{m+2} = F_m\alpha_1\ldots\alpha_m F(\alpha_{m+1}\alpha_{m+2}).$$

Now if $F_k \gamma_1 \ldots \gamma_k \beta$ is chosen as a fc for X_j, $X_j Y_1 \ldots Y_n$ has fc β. Thus the pfc of $YX_{i_1} \ldots X_{i_n}$ is also β by Lemma 1.

Now a combinator Y is obviously a term of finite level (the level will certainly be less than or equal to the number of *B*s, *C*s and *K*s in the combinator), so by Lemma 2, $YX_1 \ldots X_n$ has a pfc β given that $X_1, \ldots, X_n$ are assigned appropriate fcs $\alpha_1, \ldots, \alpha_n$.

Thus Y has fc $F_n \alpha_1 \ldots \alpha_n \beta$.

We have therefore proved Theorem 1 and hence Theorems 2, 3 and 4. We could note finally that the results of this paper apply to all weaker sets of combinators, weaker logics, weaker algebras and weaker models.

For example *BCI*-combinators have the combinator *I* with the property,

$$IX = X$$

instead of *K*. (*I* can be defined as *CKK* using *BCK* combinators.)

BCI logic replaces Axiom *K* with

Axiom I $\mathscr{A} \supset \mathscr{A}$.

and *BCI* algebra replaces the third axiom by:

$$a * a = 0$$

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