# RELEVANT DERIVABILITY AND CLASSICAL DERIVABILITY IN FITCH-STYLE AND AXIOMATIC FORMULATIONS OF RELEVANT LOGICS

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1. It is well-known that Alan Ross Anderson and Nuel D. Belnap, Jr., did not only reject classical logic, but also the classical notion of derivability. Although [1] is in a sense essentially about derivability, very little *direct* attention is paid to the notion of derivability itself in this book. All Fitch-style formulations are devised to prove theorems, not, however, to derive conclusions from sets of premisses, and the same obtains for most other formulations. There is one exception: [1] contains a definition of "a proof in E that $A_1, \ldots, A_n$ entail(s) B" (pp. 277-278). By $A_1, \ldots, A_n \mapsto_E B$ ([1]) – in plain words: B is relevantly derivable from $A_1, \ldots, A_n$ – we shall denote the fact that there is a proof in E that $A_1, \ldots, A_n$ entail(s) B.

Let us briefly paraphrase Anderson and Belnap's definition, which is central to this paper on relevant derivability. In order for B to be relevantly derivable from $A_1, \ldots, A_n$, we first of all need a proof that guarantees classical derivability, i.e., a sequence of formulas $C_1, \ldots, C_m (= B)$ in which each $C_i$ is either one of the $A_j$, or an axiom, or arrived at from previous members by application of one of the rules of E, viz. &I and $\rightarrow$E. Moreover, the derivation of B need to be relevant. To ensure this, Anderson and Belnap introduce the requirement that it be possible to prefix stars to some members of the proof as follows: attach stars to premisses, and not to axioms that are not premisses; if $C_i$ is added to the proof by application of one of the rules to $C_j$ and $C_k$, then $C_i$ is starred iff $C_j$ or $C_k$ is starred, but & I may only be applied if both $C_j$ and $C_k$ are starred (in which case $C_i$ is relevantly derivable from $A_1, \ldots, A_n$) or in case both $C_j$ and $C_k$ are unstarred (in which case $C_i$ is a theorem); as a consequence of this procedure, B should be starred. In the sequel of this paper, we shall treat the stars as integral parts of the proof.

([1]) We reserve '$\alpha \vdash A$' for classical derivability; see section 5.

In the present paper we (i) articulate a Fitch-style system for relevant E-derivability, (ii) illustrate its use, (iii) generalize the result to other relevant logics, (iv) discuss the relation, between relevant and classical derivability, (v) discuss heuristic rules to define relevant derivability, given an axiomatic system from which classical derivability is defined, and (vi) discuss an alternative view on the relation between relevant derivability and axiomatic systems for relevant logics. We start with Fitch-style systems because there is a rather natural way to define relevant derivability with respect to them. Such a definition is interesting also because Fitch-style systems are highly valuable from a pedagogical point of view and because some fragments of relevant logics can only be characterized with respect to such systems provided relevant derivability is defined in them; see section 3 for an example.

2. Let F′ E be the result of adding to the rules of FE (see p. 347 of [1]) the following structural rule (which we formulate somewhat explicitly):

Prem. As the first step or after a formula with rank 0, a step may be introduced with rank 0 receiving the unit class $\{0\}$ of numerical subscripts.

Definition 1. $\alpha \mapsto_{F'E} A$ iff there is an F′ E-proof such that (i) for each $B_{\{0\}}$ introduced by the rule Prem., $B \in \alpha$, and (ii) $A_{\{0\}}$ occurs in the proof with rank zero.

Definition 2. $\vdash_{F'E} A$ iff A occurs with the empty set of numerical subscripts in some F′ E-proof.

Theorem 1. $A_1, \ldots A_n \mapsto_E B$ iff $A_1, \ldots, A_n \mapsto_{F'E} B$.
*Proof*. Transform the F′ E-proof of B from the premisses $A_1, \ldots, A_n$ according to the procedure of § 4.1 of [1]. The result is obviously a proof that $A_1, \ldots, A_n$ entail(s) B (for the stars: replace $A_{\emptyset}$ by A and $A_{\{0\}}$ by *A). The other half of the proof is even more obvious.

Theorem 2. $\vdash_{FE} A$ iff $\vdash_{F'E} A$.
The proof is straightforward.

Corollary 1. $\vdash_E$ A iff $\vdash_{F'E}$ A.

3. Let $F'E_{\rightarrow,\vee}$ consist of Prem, Hyp, Rep, Reit, $\rightarrow$I, $\rightarrow$E, $\vee$I and $\vee$E, i.e. all rules of F′E that do not refer to & or ~. Define theoremhood and (relevant) derivability as for F′E.

To see that $F'E_{\rightarrow,\vee}$ is the Fitch-style formulation of the implication-disjunction fragment of E (as an inferential system), consider the Hilbert-style system $E_{\rightarrow,\vee}$ which consists of the following items of pp. 339-340 of [1]: $\rightarrow$E, A1-4, A8, A9 *plus* (in the absence of &I and A10) the following rule:

$\vee\rightarrow$: from $A \rightarrow C$ and $B \rightarrow C$ to infer $A \vee B \rightarrow C$.

To define $\alpha \mapsto_{E_{\rightarrow,\vee}} A$ we have to introduce the appropriate proviso on stars on ($\rightarrow$E and) $\vee\rightarrow$, viz. that $A \vee B \rightarrow C$ is (un)starred if both $A \rightarrow C$ and $B \rightarrow C$ are (un)starred and that no proofs are permitted in which $A \vee B \rightarrow C$ is derived by $\vee\rightarrow$ from $A \rightarrow C$ and $B \rightarrow C$ in case only one of the latter two is starred. We leave it to the reader to prove that $F'E_{\rightarrow,\vee}$ and $E_{\rightarrow,\vee}$ are coextensive with respect to theoremhood and (relevant) derivability, and also that no system in the style of FE (i.e. without the rule Prem.) is coextensive with the former with respect to derivability.

4. Where L is a relevant logic other than E, how should we define relevant L-derivability? We found it somewhat astonishing that this question is not considered in [1]. Nevertheless we think that the answer we present here for Fitch-style systems agrees with the views defended there.

Let FL be a Fitch-style system which characterizes the set of theorems of L and the structural rules of which are as in [1]. (More specifically, if r is the rank of a hypothesis, the latter receives {r} as its set of numerical subscripts.) We now propose to proceed as follows:

(1) Extend FL into F′L by adding the rule Prem, and define $\alpha \mapsto_{F'L} A$ as in definition 1.

It is quite obvious that (1) is not the only possible way to define relevant L-derivability, and that it cannot be proven under the very weak aforementioned conditions that (1) is the most natural way to define relevant L-derivability.

Nevertheless (1) is in two senses the most natural procedure for the relevant logics that are characterized in terms of Fitch-style systems in [1]. *First*, Fitch-style systems are essentially sets of (a specific form of) rules of inference, and [1] clearly contains the view that premisses should be treated in the same way as hypotheses and formulas derived from hypotheses. It follows that the most natural way to define relevant derivability consists in the generalization, whenever this makes sense, of the rules for subproof formulas to "main proof" formulas. This is exactly the result we obtain by applying (1). *Second*, consider the axiomatic systems presented in [1] for the logics mentioned in this paragraph. We think to do justice to [1] by requiring for *all* these logics that relevant derivability be defined from the axiomatic systems in the same way as it is defined in [1] for E (i.e. "a proof that $A_1, \ldots, A_n$ entail(s) B"), *except in that* applications of $\rightarrow$E in which the minor is unstarred and the major is starred are forbidden. This modification does not make any difference for E and weaker systems (see section 6), but it is essential for T, because, as $\not\mapsto_T A \rightarrow A \rightarrow B \rightarrow B$, the specific properties of T would be violated if we would nevertheless derive B from the premiss $A \rightarrow A \rightarrow B$ and the logical theorem $A \rightarrow A$. Where L is either T, E, R, EM or RM this definition of $\alpha \mapsto_L A$ is clearly the most natural both in general and according to the views of [1]; consequently, the following theorem shows that (1) is the most natural way to proceed for the corresponding Fitch-style systems. Let $\alpha \mapsto_{F'L} A$ be defined by (1).

Theorem 3. $\alpha \mapsto_{F'L} A$ iff $\alpha \mapsto_L A$
The proof is, for each system, wholly analogous to that of theorem 1.[2]

5. We shall now consider the relation between relevant derivability and classical derivability. It not only will turn out that it is harmless to introduce classical derivability into a relevant logic, but even that classical derivability may be defined in terms of relevant derivability

[2] If L is not T, a supplementary step is needed in the transition from the F′L-proof to the L-proof: if *B need to be derived from A and $*A \rightarrow B$, insert a proof of $A \rightarrow B \rightarrow B$ from A before *B and justify *B as derived by $\rightarrow$E from $*A \rightarrow B$ and $A \rightarrow B \rightarrow B$ (remind that $\vdash A \rightarrow B \rightarrow B$ if $\vdash A$).

(see theorem 7 below). To avoid any confusion we start from a Hilbert-style formulation for some relevant logic. In this case we have at least a calculus C in the sense of an axiomatization of a set of "logical truths", i.e. $C = <\omega, R>$, where $\omega$ is a set of formulas (axioms) and R a set of rules, theoremhood ($\vdash_C A$) being defined in the usual way. Whether relevant derivability is defined for C (in terms of restrictions on stars) or not, is immaterial to our present point. We simply introduce classical derivability in the traditional way:

Definition 3. $\alpha \vdash_C A$ iff there is a list of formulas $C_1, \ldots, C_n$ such that $A = C_n$ and any $C_i$ is either a member of $\alpha$ or a member of $\omega$ or a result of the application of a member of R to previous formulas in the list.

It follows immediately that:

Theorem 4. $\vdash_C A$ iff $\emptyset \vdash_C A$

Theorem 5. $\vdash_C A$ iff, for all $\alpha$, $\alpha \vdash_C A$.

Let $\alpha \mapsto_C A$ be defined under the restrictions of [1], viz. that *all* members of R may be used to derive A from $\alpha$,(3) possibly under certain reasonable conditions on stars (see section 6). In this case the following two theorems are equally obvious.

Theorem 6. $\vdash_C A$ iff $\omega \mapsto_C A$.

Theorem 7. $\alpha \vdash_C A$ iff $\alpha \cup \omega \mapsto_C A$

We add some comments on theorem 7. (i) Given a Fitch-style system which contains the rule Prem and in terms of which relevant derivability has been defined, the latter is turned into a definition of classical derivability if we transform Prem into Prem° by stipulating that premisses be introduced with the *empty* set of numerical subscripts. Notice, e.g., with respect to F′E, that

(3) This presupposes that the set of axioms is given in terms of a (finite) set of axiom schemata.

$p \vdash p \& (q \rightarrow q), p \not\mapsto p \& (q \rightarrow q)$ and $\not\vdash p \& (q \rightarrow q)$. In other words, replacing Prem by Prem° results in new items (in the proof) and not only in a change to the sets of numerical subscripts. (ii) Even from a relevantist point of view, the properties of classical derivability are quite sensible: if it is presupposed that any set of premisses implicitly contains all logical axioms or all logical theorems – and this presupposition is quite usual – then $\alpha \vdash A$ and $\alpha \mapsto A$ are coextensive. (iii) Classical derivability is definable in terms of relevant derivability.

As the converse of the last statement does not obtain in general we face the following situation. It is well known that there are multiple ways to extend a calculus (in the above sense) into an inferential system defined in terms of classical derivability. Analogously, there are multiple ways to extend the latter into an inferential system defined in terms of relevant derivability. Arruda and da Costa's well-studied system P (*cf.* [3]) is a relevant logic formulated in terms of classical derivability. It *is* a relevant logic; classical derivability is defined, and hence so is relevant derivability for all sets of premisses that contain all axioms (for all "normal theories", as Meyer and Dunn call them in [1], p. 300 ff.), but relevant derivability is not defined in general. (4) (5)

It seems worthwile to mention that the distinction between classical derivability and relevant derivability does not coincide with Anderson and Belnap's distinction between "derivable *on*" and "derivable *from*" in [2]. (i) In this paper both notions are linked to conditional proof ($\rightarrow$I); "derivable *on*" then leads immediately to irrelevance, viz. $A \rightarrow . B \rightarrow A$. Obviously, the introduction of classical derivability into some relevant logic does not lead to any new theorems. (ii) In [2] "derivable from" is defined in terms of a simple use criterion. In the presence of conjunction or disjunction, both relevant derivability and relevant implication require other restrictions.

6. In the preceding section we mentioned (i) the underdetermination of classical derivability by theoremhood, and (ii) the underdetermina-

(4) Wilhelm ACKERMANN'S "strenge Implikation" system (see [4] or [5]) is also defined in terms of classical derivability, but furthermore contains all PC-inferences.

(5) In a separate paper we use the present results to study the system P and some related systems in terms of Fitch-style proofs.

tion of relevant derivability by classical derivability. However, these two forms of underdetermination differ in degree. Underdetermination (i) is total: theorems *as such* tell us nothing about derivability. On the other hand, classical derivability should reveal something about relevant derivability. For one thing, one should, whenever possible, define relevant derivability in such a way that theorem 7 holds and that $\alpha$ and A share some variable whenever $\alpha \mapsto A$. However, this procedure is heuristically very weak. Something deeper and heuristically more useful may be said. We did so already for Fitch-style proofs, and now shall consider axiomatic proofs. ([6])

Classical derivability does enable us to distinguish between rules which apply to theorems of logic only, and other rules. For the former rules we obviously require that all premisses be unstarred. Suppose now that classical derivability enables us to distinguish between implications and extensional connectives – we disregard logics containing other connectives. Let R be the set of rules on which we still have to define starring conditions. The general requirement is as follows: exclude that the conclusion be starred if all premisses are unstarred and exclude that the conclusion be unstarred if some premisses are starred (only theorems of logic should be unstarred).

*Case 1.* Members of $R_1 \subseteq R$ that are of the form A / B, e.g., A & B / A.
These rules follow the general requirement.
*Case 2.* Members of $R_2 \subseteq R - R_1$ in which majors may be distinguished from minors, e.g., $A \vee B$, $A \to C$, $B \to C$ / C.
Exclude applications in which (i) only some minors are starred, (ii) only some majors are starred, or (iii) majors are starred and minors are not. In section 4 we commented already on (iii) in connection with $\to$E. The same reasoning applies to other rules of $R_2$. One should not, e.g., derive *C from $A \vee B$, $*A \to C$ and $*B \to C$ unless $\vdash A \vee B$ guarantees $\vdash (A \to C) \& (B \to C) \to C$ and A, $B \mapsto A \& B$, or unless $\vdash A \vee B$ guarantees $\vdash A \vee B \to C \to C$ and $A \to C$, $B \to C \mapsto A \vee B \to C$.
*Case 3.* Members of $R_3 \subseteq R - R_1 - R_2$ which concern essentially

([6]) We do not mean here a proof of some theorem from axioms, but a proof of some conclusion from some set of premisses, which proceeds with the help of means provided by an axiomatic system.

some extensional connective, viz. its simple or "hypothetical" introduction or elimination, e.g., A, B / A&B and $A \rightarrow C$, $B \rightarrow C$ / $A \vee B \rightarrow C$.
Exclude applications in which some premisses are starred and some are unstarred.
*Case 4*. Members of $R_4 = R - R_1 - R_2 - R_3$, e.g., $A \rightarrow B$, $B \rightarrow C$ / $A \rightarrow C$.
In this case (and in this case only) several policies may be sensible, depending on the already available theorems and (primitive or derived) rules. E.g., for our example, four approaches may make sense: (i) exclude $A \rightarrow B$ and $^*B \rightarrow C$, (ii) exclude $^*A \rightarrow B$ and $B \rightarrow C$, (iii) exclude both, (iv) exclude neither. In the presence of $\rightarrow$E and $A \rightarrow B \mapsto B \rightarrow C \rightarrow . A \rightarrow C$, the approaches (i) and (iii) are not sensible; in the presence of $\rightarrow$E and $B \rightarrow C \mapsto A \rightarrow B \rightarrow . A \rightarrow C$, the approaches (ii) and (iii) are not sensible. Also, several approaches have sensible interpretations. According to (i) the rule may be seen as a conditional form of $\rightarrow$E, and hence as conforming to the requirements in case 2. Consider another member of $R_4$: $A \rightarrow B$, $\sim A \rightarrow B$ / B. In the presence of $\vdash A \vee \sim A \rightarrow B \rightarrow B$ and $A \rightarrow C$, $B \rightarrow C \mapsto A \vee B \rightarrow C$, one might require that both $A \rightarrow B$ and $\sim A \rightarrow B$ be starred or that both be unstarred. But in the presence of contraposition, transitivity and $\sim A \rightarrow A$ / A, this rule may be seen as dependent on transitivity – the other two rules belong to $R_1$ – and hence the restrictions on stars will go along with transitivity. In another paper we shall relate such approaches to the heuristics of proofs.

7. In the preceding section we implicitly took up the following view on the relation between a calculus $<\omega, R>$ and $\alpha \mapsto A$: a proof of A from $\alpha$ may contain members of $\alpha$ as well as members of $\omega$ as well as applications of members of R, A should occur in the proof, and the proof should fulfil certain starring conditions. However, an alternative view is present in several publications; e.g., it is implicit in a contribution to [1] by Robert K. Meyer and J. Michael Dunn – see pp. 300-314. This view, which we shall call the Meyer-Dunn view, presupposes, in contradistinction to our view, that some connective is identified as an implication and some other as a conjunction. We shall present two formulations of the Meyer-Dunn view. The first presupposes that all theorems of logic are given, or that they are provable in

some independent way. The second formulation presupposes that $\rightarrow$E and &I are members of R.

*First formulation.* $\alpha \mapsto A$ iff there is a sequence $C_1, \ldots, C_n(= A)$ such that each $C_i$ is either a member of $\alpha$, or is derived from two previous members by application of &I, or is derived from some previous member $C_j$ in view of the fact that $C_j \rightarrow C_i$ is a theorem. (No reference to stars is needed.)

*Second formulation.* $\alpha \mapsto A$ iff there is a sequence $C_1, \ldots, C_n(= A)$ such that each $C_i$ is either a theorem of logic or a member of $\alpha$ or derived from previous members by some member of R, where the specific conditions on stars are as follows: (i) &I may not be applied if one premisse is starred and the other is not, (ii) $\rightarrow$E may only be applied if the major is unstarred, and (iii) all other rules may only be applied if all premisses are unstarred.

According to the first formulation, a proof contains only members of $\alpha$ and items relevantly derived from them. According to the second formulation, a proof contains the same items (all starred) and moreover theorems (all unstarred), which either are axioms or are theorems derived from axioms in the proof.

Before comparing both views, we point out that theorems which are not of the form $A \rightarrow B$ are not directly related to relevant derivability on either view. If, e.g., $A \vee \sim A$ is a theorem, this fact might be used to derive some implicative theorem, and the latter will have direct consequences for relevant derivability; however, $A \vee \sim A$ itself will not have any direct bearing on the inference of some conclusion from items that are not theorems of logic. Of course, if the logic is such that $(A \rightarrow B) \rightarrow B$ is a theorem whenever A is a theorem, then the inferential procedure may be adapted in such a way that the minor of an application of $\rightarrow$E may be unstarred. On both views, however, the so qualified rule has to be considered as derived.

Consider the problem of articulating an axiomatic system for some set of expressions of the form $A_1, \ldots, A_n \mapsto B$. The result arrived at on the Meyer-Dunn view will have the same theorems as, or more theorems than, the result arrived at on our view (depending on the aforementioned set), but this in itself is quite unimportant. However, our view may be applied in general, whereas the Meyer-Dunn view may only be applied under suitable conditions on the implication and conjunction.

Next, consider the problem of defining the relevant derivability relation for some axiomatic system with respect to which only classical derivability is defined. On the Meyer-Dunn view, one will define $A_1, \ldots, A_n \mapsto B$ as coextensive with $\vdash A_1 \& \ldots \& A_n \rightarrow B$ or, in the absence of some suitable conjunction, as coextensive with $\vdash A_1 \rightarrow . A_2 \rightarrow \ldots \rightarrow . A_n \rightarrow B$. On our view, one will proceed as explained in section 6. The results will not coincide, e.g., if the axiomatic system contains some rule $A_1, \ldots, A_n$ / B such that neither $A_1 \& \ldots \& A_n \rightarrow B$ nor $A_1 \rightarrow . A_2 \rightarrow \ldots \rightarrow . A_n \rightarrow B$ is a theorem. Moreover, the Meyer-Dunn view will not lead to a relevant derivability relation which conforms to theorem 7. It seems to us that, unless the results arrived at on either view coincide, the result arrived at on our view will be more suitable with respect to the intentions of the persons who devised the axiomatic system and its classical derivability relation. By way of an example, consider the semantics (and more specifically the relevant semantic consequence relation) devised in [6] by Richard Routley and Andréa Loparić for the Arruda-da Costa system P. This semantics reduces the system to a much weaker logic than Arruda and da Costa intended it to be, as may be seen, e.g., from the fact that the regular theories (viz. the theories having all logical truths as theorems) which are studied by Arruda and da Costa are much stronger than they should be on the Routley-Loparić semantics. Arruda and da Costa apply such rules as $A \rightarrow B, B \rightarrow C$ / $A \rightarrow C$ to non-logical theorems, and the semantics fails to validate such applications.

As a final remark we add that we called the alternative approach after Meyer and Dunn, because they implicitly apply it. We do not know, however, the extent to which Meyer and Dunn were or are willing to defend this view as philosophically correct.

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