RELEVANCE AND FIRST-DEGREE ENTAILMENTS

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§0. *Introduction*. A striking feature of the Ackermann-Anderson-Belnap relevant logics E and R is that they do not conform to the Boolean Order of Things – according to which nothing is more true than a theorem of logic nor more false than the denial of a theorem of logic. More precisely, E, R and kindred systems do not satisfy the principle of *conformity* (to the Boolean Order), that if A is a theorem, then so is $\sim A \rightarrow A$.(1) (1), for example, is a theorem of R, though (2) is not:

(1) $(p \rightarrow p)\ \&\ (q \rightarrow q)$
(2) $\sim((p \rightarrow p)\ \&\ (q \rightarrow q)) \rightarrow (p \rightarrow p)\ \&\ (q \rightarrow q)$.

The models for R (see [16]) provide a devilish way to give greater credence to the antecedent of (2) (the denial of a theorem of logic) than is given to its consequent (a theorem of logic), which makes mincemeat of Boolean priorities.(2) (1), however, is verified at each consistent point in the models and is therefore valid.

It is no accident that R is non-conforming. The strategy that renders R *relevant* precludes conformity. For

(3) $A \rightarrow A \vee B$ (addition)
(4) $A \,\&\, B \rightarrow B$ (simplification)

are of course theorem (schemata) of R; whence by De Morgan principles and transitivity (of $\rightarrow$), (2) entails

(5) $\sim(p \rightarrow p) \rightarrow .\ q \rightarrow q$;

(1) The notation throughout this paper is that of [6] which is essentially the same as that of [1].

(2) Altough I have characterized conformity in terms of negation and entailment (or implication), the non-conformist nature of E and R is ultimately due solely to the properties of entailment and implication in these systems. The formula $A \rightarrow (A \rightarrow A)$ might be termed a characteristic non-theorem of R; and to paraphrase a remark of Dunn's (cf. [10]), the honest way to reject this formula is to have models in which $A \rightarrow A$ is false and A is true. Such are the models that flout Boolean Law. Conformity and its bearing on relevance is further discussed in [9].

but (5), in the terminology of [1], is blatantly irrelevant. One of the reasons for calling implication in R "relevant" is that whenever $A \to B$ is a theorem of R, A and B share at least one propositional variable (in virtue of which it is said that there is some connection in content or meaning between antecedent and consequent). (5) violates this variable-sharing requirement. (3)

It appears that relevance and conformity are somewhat at odds, and in a sense that is unfortunate. Conformity makes for a certain simplicity of structure, as is apparent in the contrast between R and its conforming neighbor R-Mingle (RM). RM is obtained from R by adding the axioms of the form $A \to (A \to A)$. Since RM is conforming, (4) (2) and hence (5) are theorems of RM. This has induced Meyer in [1] to call RM a "quasi-relevant logic." It is true that RM is "paradox-free" in the sense of Sugihara [17]. That is, there are no formulas B and C of RM such that $\vdash_{RM} A \to B$ and $\vdash_{RM} C \to A$, for every formula A of RM; but it fails the crucial variable-sharing test, thereby, as Meyer says, "undermining the *raison d'être* of the enterprise."

Nevertheless, RM has a lucid Kripke-style modelling framed in terms of a binary linear accessibility relation. In contrast, R requires models with a ternary accessibility relation governed by various postulates. The algebraic models of RM are simple (Sugihara) chains with the truer points up above and the more false points down below – just as it should be, as it were, according to Boolean Law. In contrast, the algebraic models of R are the more general, and hence somewhat more mysterious, De Morgan monoids. (5) Meyer remarks in [1] that RM is more easily visualized than R; and indeed RM is much more well-understood than R. (6)

It would be nice to be able to reconcile relevance with the simplicity that conformity tends to induce. In what follows I shall explore this possibility by first (§ 1) developing the rationale – in terms of rele-

(3) (5) also of course fails to satisfy the "use-criterion" of relevance; see [1] § 3.

(4) See [1] § 29.3 for the relevant facts about RM.

(5) The algebraic modelling of R is due to Dunn (see [1]).

(6) Urquhart's recent discovery ([19] and [20]) that a large class of relevant logics, including R and E, are undecidable at once illuminates and deepens the mystery of these systems. Basic questions remain unanswered. For example, there is as yet no relational semantics for RQ and EQ, the quantified extensions of R and E. (But see [15].)

vance and conformity – for a well-behaved family of conforming relevant logics, and then in § 2 and § 3 setting out some of the basic axiomatic and semantic facts about the first-degree entailment fragment of these systems.(7) I shall argue that the system of first-degree entailments (entailments among truth-functions) I shall propose is every bit as relevant in every relevant respect as the system of "tautological entailments" that forms the first-degree entailment fragment of R and E. The conforming relevant logics are in some respects similar to RM, but unlike RM they preserve key relevance properties – variable-sharing for one – and they are in certain ways more reasonable than RM.(8)

§ 1. *Relevance and conformity*. The argument given above that conformity leads R into irrelevance turns on (3) and (4). The approach I shall adopt here is to retain (4) and the spirit of (3) though not its letter. Specifically, I propose to replace (3) by the principle that you are entitled to affirm a disjunction A ∨ B provided that you affirm one disjunct and either affirm or deny the other; merely affirming one disjunct while withholding judgment on the other is not sufficient:

(6) $(A \& B) \vee (\sim A \& B) \vee (A \& \sim B) \rightarrow (A \vee B)$.(9)

Perhaps in the spirit of [2] one could view (6) as an explicitly correct version of the enthymematic (3). However, I do not think that (3) is enthymematic, or otherwise incorrect.(10) Rather, I am proposing to replace (3) by (principles underlying) (6) simply because the latter can

(7) The full systems are studied in [6], [7], [8], and [9].

(8) See the discussion of interpolation below.

(9) I am not suggesting that (6) replace (3) either as an axiom or, in rule form, as a primitive rule of inference. See below p. 7ff.

(10) In [14] Parry argues that we do not in practice make use of the full liberties of (3). He claims that when we have occasion to infer A ∨ B from A, that is always on the basis of some premise containing B as a sub-sentence: "If I wish to infer from the fact that I am over 65, that I am over 65 or blind, it is because I know already that if I am over 65 or blind, I get an extra income-tax exemption." Even if Parry is right, however (and perhaps he is), there are other forms of inference, frequently exploited in "actual" argumentation, from which (3) can be derived. For example, from the premise that groups that are either abelian or of order less than 60 have normal subgroups, one might infer that abelian groups have normal subgroups. This inference has the form: $A \vee B \rightarrow C$; hence $A \rightarrow C$. (3) follows from this by taking C to be $A \vee B$.

be made to do the work of (3) in a way that preserves relevance.

The converse of (6), *viz*

(7) $A \vee B \rightarrow . (A \& B) \vee (A \& \sim B) \vee (\sim A \& B)$,

does not hold in either R or RM. (3) and (7) yield

(8) $A \rightarrow B \vee \sim B$

which is an anathema to both R and RM. But having taken the sting out of (3) as above, (7) seems a pretty good idea: You are entitled to affirm a disjunction only if you affirm one disjunct and either affirm or deny the other.

It is true that (7) entails

(9) $A \vee B \rightarrow B \vee \sim B$,

which is likely to gall the strict relevantist who believes that (9) commits one to (8). Yet in the absence of an equivalent of (3), (9) does not entail (8). In other words, if what one means to do in affirming $A \vee B$ is to affirm one disjunct and either affirm or deny the other, then (9) should be tolerable even to those with the highest standards of relevance. Accordingly, I shall take (9) to be true without fear of reprisal by the forces of relevance. After all, on my account it is not the independent modal status of $B \vee \sim B$ that explains the truth of (9); it is rather that given a certain (not implausible) understanding of $A \vee B$, namely, that given by (6) and (7) (to which one might adhere for the sake of conformity), $B \vee \sim B$ really does follow *from* $A \vee B$. For according to (7), $A \vee B$ entails one or other of $A \& B$, $A \& \sim B$, $\sim A \& B$; and from each of these $B \vee \sim B$ follows relevantly.

Something of course has to give; in particular, *contraposition*.

(10) $A \rightarrow B \rightarrow . \sim B \rightarrow \sim A$.

The point of the preceding remark that, rightly understood, $A \vee B$ does relevantly entail $B \vee \sim B$ depends on the presence of (4) to guarantee that each disjunct of the consequent of (7) entails B or else entails $\sim B$. But (10) and (4) combine to reestablish (3). Worse, (9) and (10) (by (4) and De Morgan) give

(11) $(B \& \sim B) \rightarrow A$

(which in turn also reestablishes (8)).

Now (10) corresponds to an intuitive and useful form of proof, and accordingly it holds in E, R, RM *et al.* To jettison it may seem too much to ask, and perhaps the proper conclusion is that relevance and conformity just don't mix. I think, however, that it is worth observing that there is another familiar route to (11) (other than by (9) and (10)) and that is by (3) and

(12) $\sim A \,\&\, (A \vee B) \rightarrow B$ (disjunctive syllogism).

(12), too, corresponds to an intuitive and useful form of proof, ([11]) but it (notoriously) does not hold in the relevant logics. (12) takes the following weakened form in R and E

(13) $\sim A \,\&\, (A \vee B) \rightarrow .\, B \vee (A \,\&\, \sim A)$

which according to [3] says that *ordinarily* B follows from ~A and A ∨ B. Thus, with relevance and conformity in mind, I propose the following in place of (10):

(14) $A \rightarrow B \rightarrow .\, \sim B \,\&\, (A \vee \sim A) \rightarrow \sim A \vee (B \,\&\, \sim B)$.

(14) seems to say that *ordinarily*, if B follows from A, then ~A follows from ~B. ([12])

As mentioned, (12) and (3) lead to (11). Similarly, (12) and (6) imply

(15) $(A \,\&\, \sim A) \,\&\, B \rightarrow B \,\&\, \sim B$,

which, from the viewpoint of R-style relevance, is just as bad as (11) itself. It is true that (15), unlike (11), satisfies the variable-sharing criterion of relevance; but variable sharing is not everything. In addition, in the relevant logics the deductive effects of inconsistency are minimized. Unlike standard logic or the Parry-like logics, ([13]) the relevant logics are able to distinguish between those troubled theories

([11]) In [4] Burgess gives some instructive examples that illustrate just how useful (12) can be, but he does not seem to notice that the rejection of (13) in the relevant logics is not based solely on the problematic argument of [1] § 25.1. There are other considerations as well. For example, there is the fact that (13) cannot be added conservatively to $R_{\rightarrow}$, the pure implicational fragment of R, which in turn is motivated by concerns somewhat different from those Burgess attacks.

([12]) But pure contraposition can in any case be retained in rule form. See [6].

([13]) See [13]. For further references see [6].

that are *simply inconsistent* (i.e., trivial) and those that are *negation inconsistent* (i.e., that contain a contradiction). This is surely an important distinction to be able to draw at the level of logic. The concepts of simple consistency and negation consistency are clearly different; and the principles of logic that express the extensional equivalence of the two concepts might be viewed as simplifying assumptions rather than as indispensable forms of inference. (Admittedly, to view (12) in this way may take some practice; it helps to start with (11) and to bear (13) in mind.)

So it seems that in order to preserve as much R-style relevance as possible within the context of conformity, we must settle for (13) in place of (12).

Do these maneuvers combine to yield a coherent theory of conforming entailment? I shall try to show that they do – especially when entailment is represented as a relation between propositions, rather than as an operation capable of iteration. This is not to say that conforming entailment cannot be coherently represented as an operation; it is represented as such in [6]. As the preceding discussion suggests, however, the issues that arise in the attempt to reconcile relevance and conformity have to do primarily with principles of inference among truth functions (such as (3) and (8)), and so the restriction to first-degree entailment (no nesting of '$\rightarrow$') is not inappropriate.

§2. S_{fde}. Accordingly, the required language contains two kinds of formulas: *O-degree formulas* (zdfs) and *first-degree entailments* (fdes). The former are formed from propositional variables, p, q, r, etc., the logical constants &, $\vee$, ~, and punctuation symbols, as usual. If A and B are zdfs, then $A \rightarrow B$ is a *first-degree entailment*.

What is wanted is a system of axioms and rules of inference, and an associated intuitive semantics, that reflect the rationale of § 1. The following set of schemata and rules (which might as well be called S_{fde}) should be compared with those of E_{fde} given in [1]. In what follows, t_A abbreviates $A \vee \sim A$. The notation $\leftrightarrows$ in A8 indicates that both $\sim(A \& B) \rightarrow \sim A \vee \sim B$ and $\sim A \vee \sim B \rightarrow \sim(A \& B)$ are axiom schemata, and similarly for A9.

Conjunction

A1 $A \& B \to B$
A2 $A \& B \to A$
R1 $A \to B$, $A \to C$, infer $A \to B \& C$

Disjunction

A3 $A \& t_B \to A \vee B$
A4 $B \& t_A \to A \vee B$
R2 $A \to C$, $B \to C$, infer $A \vee B \to C$

Excluded Middle

A5 $B \to t_A$, provided every propositional variable occurring in A occurs also in B.

Negation

A6 $A \to {\sim}{\sim}A$
A7 ${\sim}{\sim}A \to A$
A8 ${\sim}(A \& B) \leftrightarrows {\sim}A \vee {\sim}B$
A9 ${\sim}(A \vee B) \leftrightarrows {\sim}A \& {\sim}B$

Distribution

A10 $A \& (B \vee C) \to (A \& B) \vee (A \& C)$

Transitivity

R3 $A \to B$, $B \to C$, infer $A \to C$

A proof in S_{fde} is a finite sequence of entailments ($\gamma_1, \ldots \gamma_n$) where as usual each γ_i (i = 1 ... n) is either an axiom or results from prior elements of the sequence by applications of R1, R2, or R3.

The axioms on conjunction require no comment. Those on disjunction reflect the decision to adopt (6) in place of (3). A5 appears to be a generalization of (9), but in fact the two are equivalent. With the help of A5, (7) is easily proved. ([14]) A8 and A9 are symptomatic of the fact

([14]) Putting (9) in place of A5 yields a more elegant axiomatization, though A5 is somewhat easier to work with.

that a full-fledged principle of contraposition is not available in S_{fde}.

Notice that every propositional variable that occurs in the consequent of an axiom also occurs in its antecedent, and it is clear that the rules preserve this property. Thus every theorem of S_{fde} satisfies this strong variable-sharing property. In this respect, S_{fde} resembles the systems of Parry [13] and Dunn [11]. But these latter systems validate (12) (disjunctive syllogism), and hence validate (15) which is hardly consistent with the goal of minimizing the deductive effects of inconsistency. ([15])

For the semantics I shall invoke many values; specifically the four values $\{t\}$, $\{f\}$, $\{t, f\}$, and $\emptyset$ – the empty set. These are labeled, respectively, *affirmed, denied, both* (affirmed and denied) and *neither* (affirmed nor denied). That seems as thoroughgoing a set of semantic values as one could desire. We have both truth value gaps and truth value gluts; and as will become clear, the gaps and gluts are quite different conditions. The gluts (gaps) are not, unlike the gaps and gluts of Dunn's three-valued semantics for RM, merely reinterpreted gaps (gluts). ([16]) Moreover, affirmation and denial exhibit the very best Boolean behavior; and all four values combine in comely conformity to Boolean Law.

A valuation is a mapping of zdfs into the power set of Boolean values $\{t, f\}$ that respects the following conditions.

We say that v is *defined at* (a variable) p_i if $v(p_i) \neq \emptyset$, and v is *defined at* (a zdf) A if v is defined at each immediate proper subformula of A. Naturally then, (i) $v(A) = \emptyset$ iff v is not defined at A.

Now assume that v is defined at A. A takes one of the forms $\sim B$, $B \& C$ or $B \vee C$, for zdfs B and C. Accordingly, (ii) $t \in v(\sim B)$ iff $f \in v(B)$, and $f \in v(\sim B)$ iff $t \in v(B)$; (iii) $t \in v(B \& C)$ iff $t \in v(B)$ and $t \in v(C)$, and $f \in v(B \& C)$ iff $f \in v(B)$ or $f \in v(C)$. Finally, (iv) $t \in v(B \vee C)$ iff $t \in v(B)$ or $t \in v(C)$, and $f \in v(B \vee C)$ iff $f \in v(B)$ and $f \in v(C)$.

An entailment is *valid* (in symbols $\models A \rightarrow B$) iff for each valuation v, $t \in v(A)$ only if $t \in v(B)$.

([15]) "Parry-entailment" is virtually the same as classical or Boolean entailment. Let X be a set of zdfs containing all tautologies. Then the closure of X under Parry-entailment is the same as the closure of X under Boolean-entailment. (Cf. [7], p. 6.)

([16]) See [10].

A proof of completeness is in order, but before proceeding to that argument we ought to check that the semantics does what it should do according to § 1.

As mentioned above, the theorems of S_{fde} satisfy a strong variable-sharing condition. In the semantics, if B contains a variable p not found in A, then $A \to B$ will be falsified by the valuation that assigns the value *both* to each variable other than p and is undefined at p.

It is worth emphasizing that from the viewpoint of § 1 such "analytic" variable-sharing is not an end in itself but merely a by-product of the "explicitly truth-functional" treatment of disjunction in S_{fde}. (17) If we mean by $A \vee B$ what any good truth-functionalist would have us mean – namely, that at least one of A and B is true *and* if only one of A and B is true, the other is false – we really ought to abandon *addition* ((3)) in favor of A3 and A4, for we ought to say what we mean. And if that is what we mean by $A \vee B$, then any good relevantist should grant us A5, for he ought to mean what he says (about proof from hypotheses). For if, as mentioned, A5 and (9) are equivalent, and if disjunction is interpreted as explicitly truth-functional, then neither (9) nor A5 violates the "use-criterion" imposed by relevantists on the concept of proof from hypotheses. Thus relevance *qua* variable-sharing is secured by what I have called an "explicitly truth-functional" treatment of disjunction; and this analysis of disjunction also guarantees that the theory of inference contained in S_{fde} is compatible with the relevantist's favored concept of proof from hypothesis: If A entails B in S_{fde}, then B really does follow *from* A, A5 and (9) notwithstanding.

It is easy to see that the semantics falsifies (12) (disjunctive syllogism), and hence that, as desired, S_{fde} is "paraconsistent". Let v be a valuation such that $v(p) = \{t, f\}$ and $v(q) = \{f\}$, for variables p and q; then $v({\sim}p \,\&\, (p \vee q)) = \{t, f\}$, which falsifies ${\sim}p \,\&\, (p \vee q) \to q$. The closest one can come in S_{fde} to a deductive "breakdown" in the presence of a contradiction is this: Let A be a conjunction of atoms (variables or their negations) such that if a variable occurs in A then so does its negation; then $A \to B$ is valid if and only if B is a truth function

(17) This view of "analytic" variable-sharing is thus quite different from Parry's in [14].

of the variables in A. That is, one can come no closer to such a breakdown in S_{fde} than one can come in E_{fde}.

It is almost as easy to see that conformity is forthcoming from the semantics. In the present context conformity means that if B is a classical i.e., Boolean tautology, then $\sim B \rightarrow B$ is valid. In fact, the converse is true as well. So we have the following modest

Theorem (conformity). B is a Boolean tautology iff $\sim B \rightarrow B$ is valid.

In this form the conformity theorem says that S_{fde} contains classical propositional calculus in a certain direct way. In contrast, E_{fde} does not contain classical propositional calculus in this way.(18) E_{fde} is non-conforming; for as one would expect (given the remarks at the outset of § 0),

$$(16)\quad \sim((p \vee \sim p) \,\&\, (q \vee \sim q)) \rightarrow (p \vee \sim p) \,\&\, (q \vee \sim q)$$

is not a theorem of E_{fde}.

As a corollary to the conformity theorem we have

Corollary. B is inconsistent iff $B \rightarrow \sim B$ is valid.

Hence, the semantics is able to sort out the inconsistent and tautologous zdfs from the rest. Unlike the semantics for E_{fde}, the semantics for S_{fde} does not allow that it may be appropriate in some context ("possible world") to affirm a contradiction or deny a tautology. A contradiction may either be ignored ($\emptyset$), denied ($\{f\}$), or both affirmed and denied ($\{t, f\}$), but never simply affirmed ($\{t\}$). According to the corresponding semantics for E_{fde} (see [1]), however, if p is both affirmed and denied, and q is neither affirmed nor denied, then $(p \vee \sim p) \,\&\, (q \vee \sim q)$ is simply false.

For the proof of the conformity theorem, we define a *Boolean valuation* to be a mapping of zdfs into the set $\{\{t\}, \{f\}\}$ that respects the standard Boolean matrices for &, $\vee$ and $\sim$. Oviously any Boolean valuation is a valuation since the semantical rules are entirely Boolean

(18) Meyer has shown, however, that E contains S4 on translation; and [7] contains a proof that S4 is contained in S (the many-degree extension of S_{fde} first studied in [6]) on the very same translation.

for affirmation and denial. So if B is not a tautology, v(B) = {f}, for some valuation v; whence v affirms ~B (v(~B) = {t}), and hence ~B→B is not valid. That proves the "if" part of the theorem.

Next, observe that if a valuation v either affirms or denies B, then some Boolean valuation v either affirms or denies B, then some Boolean valuation v′ affirms B or else denies B, according as v does so. (This fact is easily proved using induction on the construction of B. For details see [7].) It follows that if B is a tautology, no valuation denies B, and hence ~B→B is valid, which completes the proof.

The reader may have noticed that the "only if" part of the conformity theorem follows directly from the semantical validity of A5, and that conversely, the validity of A5 is established on the basis of the fact stated in the first sentence of the preceding paragraph. Conformity and A5 rest on the same semantical or logical precept; namely, that while it may be useful and interesting to allow for the possibility of reasoning cogently even with inconsistent information, we need not at the same time allow that there may be circumstances in which it would be reasonable to simply affirm what logically cannot be true, or to simply deny what logically cannot be false; for to do so is to unduly blur the distinction between truth and logical truth.

It is sometimes said (e.g., by Meyer in [1] and Dunn in [10]) that the semantics of E and R incorporate inconsistent and/or incomplete "set-ups", worlds, situations, theories, etc., whereas the semantics of RM incorporates only inconsistent set-ups and not incomplete ones. It might appear that in virtue of A5 (or equivalently, in virtue of conformity) the foregoing semantics does not allow for incomplete set-ups. But that is not true. Any valuation on which some formula is undefined will thereby be "incomplete". The special sort of incompleteness represented by a valuation that is somewhere undefined has a number of intuitive interpretations and applications. Most of our theories do not purport to give us the truth about utterly everything under the sun, but only about this or that topic or set of topics – some topics being totally irrelevant to the import of a given theory.
A valuation that is undefined here and there can be thought of as a semantic representation of the notion of truth-about-this-or-that-topic-or-(smallish)-set-of-topics. This idea usually receives a partly syntactic, partly semantic, analysis. If we wish to know the arithmetic truths, for example, we look to models (particularly, the "standard"

model) of the *language* of arithmetic. But for certain purposes a purely semantic sorting of topics may be preferable, e.g., if we wish to develop a theory of models for natural language; for the vocabulary of a natural language encompasses a potentially limitless range of topics.

§3. *Completeness*. I shall be brief. Soundness is proved as usual by establishing that every axiom of S_{fde} is valid and that the rules preserve validity. The proof is here omitted (for the details, see [7]).

Theorem. (soundness) If $\vdash A \rightarrow B$, then $\models A \rightarrow B$.

Completeness is proved by a simple "reduction to normal form" argument. For this the following definitions are required. A *coentails* B if and only if $\vdash A \leftrightarrows B$. Coentailment is an equivalence relation, but it lacks the replacement property. (It is not a congruence relation.) For example, although both (6) and (7) are provable in S_{fde},

$$\sim(p \vee q) \leftrightarrows \sim((p \,\&\, q) \vee (p \,\&\, \sim q) \vee (\sim p \,\&\, q))$$

is not. Nevertheless, a useful replacement theorem for S_{fde} is available. Let $\vdash A \approx B$ abbreviate the statement that $\vdash A \leftrightarrows B$ and $\vdash \sim A \leftrightarrows \sim B$. Equivalence in the sense of $\approx$ does possess the replacement property.

Theorem. (replacement) Let C_A be a zdf of which A is a well-formed part, and let C_B result from C_A by replacing zero or more occurrences of A in C_A by B. If $\vdash A \approx B$, then $\vdash C_A \approx C_B$.
The proof of the replacement theorem is by a standard argument, using the fact that if $\vdash A \approx B$, then $\vdash \sim A \approx \sim B$, $\vdash A \,\&\, C \approx B \,\&\, C$, and $\vdash A \vee C \approx B \vee C$.

Following the usage of [1], a *primitive conjunction (primitive disjunction)* is a conjunction (disjunction) of atoms. A zdf is in *conjunctive normal form* (cnf) if it is a conjunction each conjunct of which is a primitive disjunction, and dually for a zdf in *disjunctive normal form* (dnf). The replacement theorem guarantees that each zdf is equivalent (in the sense of $\approx$) to one in cnf and to one in dnf. (19) If A is a zdf, $\phi(A)$ is to be the zdf $(p_1 \vee \sim p_1) \,\&\, (p_2 \vee \sim p_2) \,\&\, \ldots \,\&\, (p_n \vee \sim p_n)$

(19) If A is equivalent to B in dnf (cnf), then B is a *dnf (cnf) of* A.

where $p_1, \ldots, p_n$ lists the variables occurring in A in alphabetical order (and without repetitions).

Theorem (reduction). $\vdash A \rightarrow B$ if and only if there is a dnf $A_1 \vee \ldots \vee A_m$ of A, and a cnf $B_1 \& \ldots \& B_k$ of B such that $\vdash \phi(A) \& A_i \rightarrow B_j$ for each i and j $(1 \leq i \leq m,\ 1 \leq j \leq k)$.

Proof. Since $\vdash A \approx (A_1 \vee \ldots \vee A_m)$ and $\vdash B \approx (B_1 \& \ldots \& B_k)$, $\vdash A \rightarrow B$ iff $\vdash A_1 \vee \ldots \vee A_m \rightarrow B_1 \& \& B_k$. Also, $\vdash \phi(A) \& A_i \rightarrow A_i$, and moreover every variable in $A_1 \vee \ldots \vee A_m$ is in $\phi(A) \& A_i$; hence $\vdash \phi(A) \rightarrow A_i \& A_i \vee (A_1 \vee \ldots \vee A_m)$. Thus $\vdash \phi(A) \& A_i \rightarrow A_1 \vee \ldots \vee A_m$. Therefore, if $\vdash A_1 \vee \ldots \vee A_m \rightarrow B_1 \& \ldots \& B_k$, then $\vdash \phi(A) \& A_i \rightarrow B_1 \& \ldots \& B_k$, and so $\vdash \phi(A) \& A_i \rightarrow B_j$. Conversely, suppose that $\vdash \phi(A) \& A_i \rightarrow B_j$, for every i and j. Then $\vdash \phi(A) \& A_i \rightarrow B_1 \& \ldots \& B_k$, for every i. Then $\vdash (\phi(A) \& A_1) \vee \ldots \vee (\phi(A) \& A_m) \rightarrow B_1 \& \ldots \& B_k$; and by a principle of generalized distribution easily proved in S_{fde}, $\vdash \phi(A) \& A_1 \vee \ldots \vee A_m \rightarrow B_1 \& \ldots \& B_k$. Since $\vdash A_1 \vee \ldots \vee A_m \rightarrow \phi(A)$, it follows that $\vdash A_1 \vee \ldots \vee A_m \rightarrow B_1 \& \ldots \& B_k$.

By means of the reduction theorem the question of the provability of an entailment is reduced to the same question for *primitive entailments*, entailments having the form $\phi(A) \& C \rightarrow D$, where C is a primitive conjunction and D is a primitive disjunction. Completeness is now forthcoming.

Theorem (completeness). If $\models A \rightarrow B$, then $\vdash A \rightarrow B$.

Proof. Suppose it is not true that $\vdash A \rightarrow B$. By the reduction theorem there is a dnf $A_1 \vee \ldots \vee A_n$ of A and a cnf $B_1 \& \ldots \& B_k$ of B such that it is not true that $\vdash \phi(A) \& A_i \rightarrow B_j$, for some i and j. Consequently, either (i) some variable in B_j is not in $\phi(A) \& A_i$ or (ii) A_i and B_j fail to share an atom. In each case it follows that $\phi(A) \& A_i \rightarrow B_j$ is not valid. For if (i) is true, there is a valuation that is undefined at B_j and over-defined at $\phi(A) \& A_i$ (i.e. the latter receives the value *both*.) Assume then that every variable in B_j is in $\phi(A) \& A_i$, and that A_i and B_j do not share an atom. Thus B_j cannot be a tautology; for otherwise it would contain both p and ~p as disjuncts, for some variable p, and in that event A5 implies that $\vdash \phi(A) \& A_i \rightarrow B_j$, contrary to hypothesis.

Hence, there is a valuation v such that $v(B_j) = \{f\}$. It is now easy to construct a valuation v′ such that $t \in v'(\phi(A) \& A_i)$ and $t \notin v'(B_j)$. For any variable p, if p is in B_j, then $v'(p) = v(p)$; otherwise, (1) if both p and ~p occur as conjuncts of A_i, then $v'(p) = \{t,f\}$. (2) If p (but not ~p) occurs as a conjunct of A_i, then $v'(p) = \{t\}$. (3) If ~p (but not p) occurs as a conjunct of A_i, then $v'(p) = \{f\}$. And finally, (4) if neither p nor ~p occurs as a conjunct of A_i, then $v'(p) = \{t,f\}$. Thus $v'(B_j) = v(B_j) = \{f\}$; and since $v'(q) \neq \phi$, $t \in v'(q \vee {\sim} q)$ for each variable q, and so $t \in v'(\phi(A))$. Since $A_i = \alpha_1 \& \ldots \& \alpha_r$, for atoms α_i, it remains to prove that $t \in v'(\alpha_i)$, $1 \leq i \leq r$. There are two cases to consider. First, suppose that α_i is a variable occurring in B_j. Then ${\sim}\alpha_i$ but not α_i (no sharing) occurs as a disjunct of B_j. Hence $v'({\sim}\alpha_i) = \{f\}$ and $v'(\alpha_i) = \{t\}$. Similarly, if (2) or (3) applies, $t \in v'(\alpha_i)$ as desired. Secondly, suppose that $\alpha_1 = {\sim} p$ for some variable p. If p occurs in B_j it occurs as a disjunct since ~p does not. Whence $v'(p) = \{f\}$ and $v'(\alpha_i) = \{t\}$. Similarly, if (2) or (4) applies, $t \in v'(\alpha_i)$, as desired. Thus as predicted, $\phi(A) \& A_i \to B_j$ is not valid. It follows (using the fact S_{fde} is sound – together with previously noted equivalences) that $A \to B$ isn't valid either, which completes the proof.

I claimed above that S_{fde} is more reasonable than RM. It is time to make good on that claim. As Meyer has observed, RM is "unreasonable in the sence of Craig" since the interpolation theorem fails for RM. In the spirit of [1] § 15.2, I shall show that a near "perfect" interpolation theorem holds for S_{fde}. The theorem is of the Lyndon variety rather than the Craig variety since the latter holds trivially for S_{fde}. ([20])

A variable p occurs *positively (negatively)* in B if some occurrence of p in B lies within the scope of an even (odd) number of negation signs. B is said to be *negation-reduced* if it is constructed from atoms by means of & and ∨. Every zdf is equivalent (in the sense of ≈) to one that is negation-reduced. If A is negation-reduced, then p is *positive (negative) in A* if not all positive (negative) occurrences of p in A have the form (p ∨ ~p). Lastly, C is an interpolant of an entailment $A \to B$ if $\vdash A \to C$ and $\vdash C \to B$, and every positive (negative) variable in C occurs positively (negatively) in both A and B.

([20]) See [5] for a proof of Lyndon's theorem.

Theorem (interpolation). An entailment is provable in S_{fde} if and only if it has an interpolant.

Proof. The "if" part follows directly from the rule R2. For the rest, assume that $\vdash A \to B$, and put $A \to B$ into normal form:

$$A' = A_1 \vee \ldots \vee A_m \to B_1 \& \ldots \& B_k = B'.$$

For each $i \leq m$, let A_i^* be the least conjunction of which $\phi(B)$ is a subconjunction and which, additionally, contains as conjuncts any atoms that A_i shares with any B_j, for $j \leq k$. Note that A_i^* is always a non-null subconjunction of $(\phi(A) \& A_i)$. Let C be $A_1^* \vee \ldots \vee A_m^*$. Assume that p is positive in C. Then p is positive in some A_i^*. The latter, therefore, has the form $\phi(B) \& A_i'$, where A_i' is a subconjunction of A_i and p is an atom of A_i'. Thus p is also an atom of some B_j. But any positive atom of A' (B') must occur positively in A (B); hence p occurs positively in both. The proof is similar in case p is negative in C.

Note that $\vdash \phi(B) \& A_i \to A_i^*$, for every i. Therefore, $\vdash (\phi(B) \& A_1) \vee \ldots \vee (\phi(B) \& A_m) \to C$, and hence $\vdash A \to C$. Finally, by the reduction theorem, $\vdash \phi(A) \& A_i \to B_j$; whence by completeness, $\models \phi(A) \& A_i \to B_j$. Thus, either B_j is a tautology or A_i and B_j share an atom – for otherwise it is possible to construct an invalidating valuation for $\phi(A) \& A_i \to B_j$ (as in the proof of the completeness theorem). If B_j is a tautology $\models A_i^* \to B_j$, and so $\vdash A_i^* \to B_j$. And if A_i and B_j share an atom, then again $\vdash A_i^* \to B_j$. Thus, $\vdash A_i^* \to B_j$, for each i and j, and hence $\vdash C \to B$.

Despite this difference in degree of reasonableness, S_{fde} and RM are closely related. The first-degree entailment fragment of RM is characterized by the three element Sugihara chain RM3 (see [12]). Our semantics for S_{fde} contains a copy of this chain, and there is the following simple correspondence between the two systems: $A \to B$ is valid iff (i) each variable in B is in A, and (ii) $A \& t_B \to B$ is valid in RM3. Equivalently, $A \to B$ is valid iff (i) holds and (ii)′ $A \models_{\overline{RM}} B$. (See [7] for proof of these facts.) Thus entailment in S_{fde} corresponds to "analytic" logical consequence in RM (as applied to 0-degree formulas only).

A final fact. S_{fde} is a Hilbert style formalism. But the proof of the interpolation theorem suggests another sort of syntactic representation of the class of valid entailments.

Let us say that an entailment is *relevantly analytic* if and only if it satisfies the following two conditions: (i) every variable in B is in A, and (ii) $A \rightarrow B$ has a normal form $A_1 \vee \ldots \vee A_m \rightarrow B_1 \& \ldots \& B_k$ such that for each pair of formulas A_i and B_j, either B_j contains p and ~p, for some variable p, or else A_i and B_j share an atom. Using the reduction theorem it is easy to show that an entailment is relevantly analytic if and only if it is provable in S_{fde}.

§4. *Summary*. I have argued that S_{fde} provides a simple theory of relevant entailment between truth functions that rivals that of E_{fde}. By rendering disjunction explicitly truth functional while at the same time liberating negation in a way common to the relevant logics, one obtains a form of first-degree entailment that might satisfy the whims of relevantist (for relevance) and classicist (for conformity) alike. It is at least instructive to see how far one can go in avoiding the "paradoxes" of implication and entailment without doing utter violence to Boolean Law.

Of course E_{fde} is only a fragment of a system founded on certain key intuitions regarding conditional proof or proof from hypotheses. It is perhaps best to regard E_{fde} as a result of conservatively extending the insights concerning conditional proof represented in $R_{\rightarrow}$ – the pure implicational fragment of R – to inference among truth functions. ([21]) From this viewpoint some of the apparent "excesses" of E_{fde} gain in plausibility. In response to the clamor (e.g., in [4]) over the rejection of the disjunctive syllogism (DS) in E_{fde}, for example, the relevantist can point out that DS is simply incompatible with the (otherwise?) reasonable and intuitive concept of conditional proof contained in $R_{\rightarrow}$; for DS cannot be added conservatively to $R_{\rightarrow}$.

There is little reason to suppose, however, that the theory of truth functional inference contained in R([22]) is the only viable theory that preserves the pure implicational insights of $R_{\rightarrow}$; nor is there good reason to believe that $R_{\rightarrow}$ is the only viable realization of a relevant

([21]) See [1] §3 for an account of $R_{\rightarrow}$ and [18] for an associated general theory of implication.

([22]) I am referring here to the first-degree *formulas* provable in R. See [1] § 19.

concept of conditional proof. Since S_{fde} is not internally incompatible with the "use-criterion" of relevance that motivates $R_{\rightarrow}$, it is possible that S_{fde} could be combined with an $R_{\rightarrow}$-like theory of conditional proof to yield entirely credible theories of entailment and relevant implication. ([23])

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