

THE NOT-SO-STRANGE MODAL LOGIC OF INDETERMINACY

by
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P.F. Gibbins closes his article ("The Strange Modal Logic of Indeterminacy" *Logique et Analyse* #100:443-446) with

But indeterminacy generates a strange modal logic. The semantical business of there being classes of indeterminate worlds accessible to no worlds not even to themselves is strange and not intuitively attractive.

I wish to suggest that the logic of indeterminacy is not so strange as that! While I agree Gibbins' final conclusion

...that the modal logic of indeterminacy, construed as an extension of maximally determinate classical logic, affords a poor model for the deep idea of vagueness *de re*

my reasons have rather to do with the idea that vagueness *de re* – that is, vagueness inhering in an object – is not plausibly construed by *any* operator on sentences. To say that an *object* is vague is to say at least that some predicate neither applies nor doesn't apply to it; and this seems to call for some construal of sentences like *Fa* in a manner opposed to treating it first as meaningful and then prefixing an indeterminacy operator to it. But this is not the point of the present note. Rather, I content myself with showing that Gibbins' argument about the "strangeness" of the semantics of indeterminacy is ill-founded.⁽¹⁾

⁽¹⁾ I also ignore his direct reconstruction of Evans' argument, preferring not to comment on his use of "the expected Indeterminacy thesis $A \rightarrow \Delta A$ ", which thesis seems to me to be completely implausible as a truth about indeterminacy. (Evans, G. "Can There Be Vague Objects?" *Analysis* 38:208).

Following Gibbins,⁽²⁾ we use ∇ as a sentential operator meaning "it is indeterminate whether" and introduce Δ as its dual "it is determinate that". ∇ and Δ are genuine duals, that is

$$[\text{Def } \nabla] \quad \nabla A \leftrightarrow \neg \Delta \neg A$$

It is also plausible to suppose, along with Gibbins

$$[\text{RE}] \quad \text{if } \vdash (A \leftrightarrow B) \text{ then } \vdash (\Delta A \leftrightarrow \Delta B)$$

$$[\text{RN}] \quad \text{if } \vdash A \text{ then } \vdash \Delta A$$

That is, if the equivalence of A and B is provable, so is the equivalence of whether they are definite; and if a formula is provable, then it is definite. Some further theorems not mentioned by Gibbins, but seemingly plausible for "determinateness" are⁽³⁾

$$[\text{C}] \quad \vdash (\Delta A \ \& \ \Delta B) \rightarrow \Delta (A \ \& \ B)$$

$$[\text{I}] \quad \vdash (\Delta A \leftrightarrow \neg \nabla A)$$

$$[\text{I}'] \quad \vdash (\Delta A \leftrightarrow \Delta \neg A)$$

$$[\text{I}'] \quad \vdash (\nabla A \leftrightarrow \nabla \neg A)$$

(Theorems [I], [I'] and [I''] are equivalent in the presence of [Def ∇]). Since this logic has [Def ∇] and [RE] it is *classical* in the sense of Segerberg,⁽⁴⁾ and therefore can be given an analysis by "possible worlds". Gibbins mentions some principles that fail in this logic, such as

$$[\text{T}] \quad \Delta A \rightarrow A$$

$$[\text{P}] \quad A \rightarrow \nabla A$$

Other principles, not mentioned by Gibbins, that fail in this logic are

$$[\text{D}] \quad \Delta A \rightarrow \nabla A$$

$$[\text{M}] \quad \Delta (A \ \& \ B) \rightarrow (\Delta A \ \& \ \Delta B)$$

(Principle [D] obviously fails in the intended understanding of Δ and ∇ . [M] fails because, for example, $(p \ \& \ \neg p)$ is definite (definitely

(2) Who follows Evans in this. A similar use of Δ and ∇ as "determinate" and "contingent" operators can be found in a series of papers from the late 1960's in *Logique et Analyse* by R. Routley and G. Montgomery.

(3) I use the names of the rules of inference and axioms found in B. Chellas *Modal Logic*, Cambridge U.P. 1980. The "I'" is new and stands for "indeterminacy".

(4) K. SEGERBERG *An Essay in Classical Modal Logic* (1971), Uppsala.

false, that is, but definite nonetheless) while neither p nor $\neg p$ are definite. Strangely, Gibbins thinks that

[K] $\Delta(A \rightarrow B) \rightarrow (\Delta A \rightarrow \Delta B)$

should be a theorem of the logic, thereby making the logic be *normal*. But it obviously should not be a theorem: let $A = (p \& \neg p)$ and $B = q$; $\Delta((p \& \neg p) \rightarrow q)$ is true and $\Delta(p \& \neg p)$ is true, but Δq needn't be.⁽⁵⁾

Since [K] is not in the logic, it is not normal and hence there is no normal, relational possible world semantics for the logic. But there can nonetheless be a possible world semantics, done by the "Montague-Scott" (or "neighborhood" or "minimal model") method. We first consider the logic as axiomatized by the propositional logic, [C], [I] with the rules of inference [RE] and [RN]. (In Chellas' notation it would be the logic ECNI.) Arguably there are more principles that should be valid in a logic of indeterminacy. We shall shortly consider them. For now we concentrate on just these few.

A model is a triple $M = \langle W, N, P \rangle$ such that W is a set of indices ("worlds"), P is mapping from natural numbers to subsets of W (i.e., $P(n) \subseteq W$ for each natural number n - telling us for each atomic proposition $P(n)$ which subset of W it is true in), and N is a mapping from W to sets of subsets of W (i.e., $N\alpha \subseteq \mathcal{P}(W)$ for every world $\alpha \in W$ - that is, what propositions (subsets of worlds) are necessary at α). Define ΔA to be true at an index α in M iff the set of indices at which A is true, $|A|$, is a member of $N\alpha$, and ∇A to be true at α iff $(W - |A|) \notin N\alpha$. It is well known that propositional logic, [RE] and [Def ∇] are valid in any class of such models. It remains only to find that subclass determined by [RN], [C] and [I]. It is again well known that [RN] holds when M contains the unit, i.e.,

(n) $W \in N\alpha$

for every $\alpha \in M$ (anything true at all worlds is an element of the necessitation of any world); and that [C] holds if M is *closed under intersections*, i.e.,

(c) if $X \in N\alpha$ and $Y \in N\alpha$ then $(X \cap Y) \in N\alpha$

⁽⁵⁾ Besides, if [K] were in the system then [M] would be also, which we have already seen to be wrong.

for every $\alpha \in M$ and all sets of indices X and Y . I dub the property which validates [I] as *contrariety* (if something is necessary so is its opposite)

(i) $X \in N\alpha$ iff $[(W-X) \in N\alpha]$

Standard methods (cf. Chellas *op cit* ch. 7) would clearly suffice to show that ECNI is determined by the class of contrary models that are closed under intersections and contain the unit. It is also obvious that principles [K], [M], [T] and [P] are not universally valid in this class of models.

In the scheme of modal logics we find ECNI located

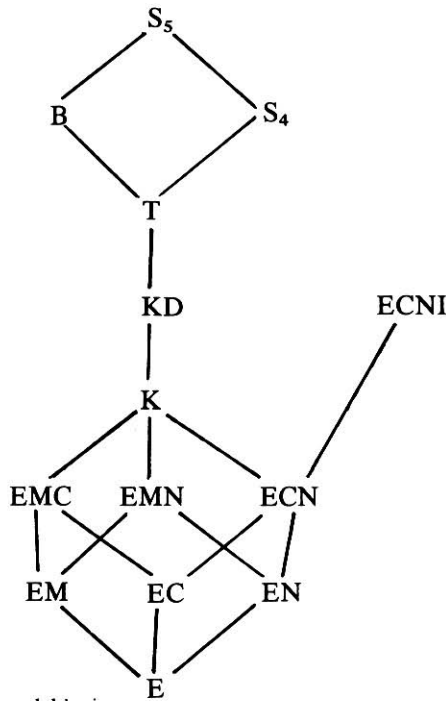


FIG. 1: A map of some modal logics

Far from being “at least as strong as S_5 ” (as Evans said), it is seen to be independent of it. And far from being “trivial” (as Gibbins says), there are no thesis of the form ΔA unless A is a propositional theorem or derived by repeated applications of rules [RE] and [RN] from

propositional theorems. For example, if p is an atomic proposition, then Δp is not a theorem. ECNI is not so strange - it does just about what one would expect of a logic of indeterminacy.

What other theses might one suggest for a logic of indeterminacy than [RE], [RN], [Def ∇], [C] and [I]? We've seen that principle [M] does not hold, but a closely related one does seem to be valid, namely

$$[M^*] \vdash (\Delta(A \& B) \& (A \& B)) \rightarrow (\Delta A \& \Delta B)$$

which says that if a conjunction is not only definite but also true then each conjunct has to be definite. I say "seems to be valid" when one understands ' Δ ' as "definitely", but I would not insist on it. The (semantic) idea behind this feeling is that the antecedent of $[M^*]$ claims that $\Delta(A \& B)$ is true. Hence by our understanding of ' Δ ', $(A \& B)$ must either be (universally) true or else (universally) false. But the antecedent also claims that $(A \& B)$ is (actually) true; so it must be universally true. The question then becomes: how can a conjunction be universally true? Certainly one way is if each conjunct is universally true, which would entail $(\Delta A \& \Delta B)$. There may be other ways, however, depending on other aspects one might think of for "definiteness"; but I can think of none. So it (tentatively) seems to me that the antecedent of $[M^*]$ should be allowed to imply its consequent when ' Δ ' is understood as "definitely". The semantic condition corresponding to $[M]$ is called *supplementation*

(m) if $x \in N\alpha$ and $X \subseteq Y$ then $Y \in N\alpha$

For our weaker $[M^*]$, I recommend the name *partial supplementation*

(m*) if $X \in N\alpha$ and $\alpha \in X$ and $X \subseteq Y$ then $Y \in N\alpha$

Since the logic $ECNM^*$ (without the I) is a sublogic of K and a superlogic of ECN, it falls on the line between K and ECN in Fig. 1. Our logic for vagueness is now $ECNM^*I$, still independent of K (due to the presence of [I]), but a superlogic of $ECNM^*$.

Other principles that might be thought of, seem to me to have considerably less plausibility as truths about vagueness.

$$[4] \quad \Delta A \rightarrow \Delta \Delta A$$

$$[B] \quad A \rightarrow \Delta \nabla A$$

$$[G] \quad \nabla \Delta A \rightarrow \Delta \nabla A$$

$$[5] \quad \nabla A \rightarrow \Delta \nabla A$$

[U] $\Delta(\Delta A \rightarrow A)$

(For some reason Gibbins thinks [4] and [5] are obviously valid principles of a logic of indeterminacy - but then he also thinks that $A \rightarrow \Delta A$ is too, so who knows why he thinks anything.) If one takes the view that all indices are "accessible" to any other, so that Δ means "is either true at all indices or false at all indices" and ∇A means "is true at some index and false at some index", then one will have the principles (all of which are equivalent, given [I] and [Def ∇])

[V₁] $\vdash \Delta \Delta A$

[V₂] $\vdash \neg \nabla \Delta A$

[V₃] $\vdash \Delta \nabla A$

[V₄] $\vdash \neg \nabla \nabla A$

i.e., whether A is definite or vague is itself always definite. For, if A is definite then A is either true at all indices or false at all, and hence ΔA is true at all indices - i.e., $\Delta \Delta A$. On the other hand if A is vague then it is true at each index that A is true at some index and false at some index, i.e., $\Delta \nabla A$. Given then one of the principles [V₁] - [V₄] we can see why the other principles ([4], [B], [G], [5], and [U]) hold - the [V₁] - [V₄] principles are the consequents of those conditionals.⁽⁶⁾ This is the logic that Evans and Gibbins apparently wish to employ for "indeterminacy". The sense in which it is "at least as strong as S_5 " is that, given S_5 we can define this system's Δ operator:

$$\Delta A = \text{df } (\Box A \vee \Box \neg A)$$

And given the present logic we can define the $S_5 \Box$:

$$\Box A = \text{df } (\Delta A \ \& \ A)$$

On the other hand, of course, the present logic of indeterminacy is independent of S_5 , since for example it does not have [K].⁽⁷⁾

⁽⁶⁾ Actually, it takes a bit of an argument to show that [U] holds. Perhaps the following informal argument will suffice. Either (1) A is true at each index, or (2) A is false at each index, or (3) A is true at some and false at another index. In case (3), ΔA is false at each index, so $(\Delta A \rightarrow A)$ is true at each index and hence [U]. In case (1), since A is true at each index $(\Delta A \rightarrow A)$ is always true. Hence [U]. In case (2), since A is false in each index ΔA is true and A is false, so $(\Delta A \rightarrow A)$ is false at each index, hence [U].

⁽⁷⁾ For further discussion on how these systems can both be and not be the same system, see my "Six Problems in Translational Equivalence" (*Logique et Analyse* 108, pp. 423-434).

This logic is axiomatized by $\text{ECNM}^* \text{IV}_1$. The semantic condition on models for V_1 is *all-pervasiveness* (the necessity of each proposition is necessary)

$$(v1) \quad \{\beta : X \in N\beta\} \in N\alpha$$

And this logic for indeterminacy is determined by the class of all-pervasive, contrary, partially supplemented models that are closed under intersections and contain the unit. Again, not the trivial logic. One can withhold $[V_1]$, adding instead $[4]$ to $\text{ECNM}^* \text{I}$, and get a logic equivalent to S_4 – with the above definitions. Or, rather than $[4]$, one could add

$$[B'] \quad A \rightarrow \Delta(\nabla A \vee A)$$

and have a logic equivalent to B , under the above definitions. The logic ECNM^* is in fact just logic T in disguise, under the definitions given above, as can be seen by the following argument. Substitute $(\Box A \vee \Box \neg A)$ for ΔA in the axioms of $\text{ECNM}^* \text{I}$, and the result will be theorems of T ; substitute $(\Delta A \& A)$ for $\Box A$ in the axioms of T and the result will be theorems of $\text{ECNM}^* \text{I}$. The same substitution in the inference rules of one system will yield derivable rules of the other. Here are a few examples to support these claims. Consider

$$\vdash \Box A \rightarrow A$$

(the t -axiom). After substitution we get

$$(\Delta A \& A) \rightarrow A$$

which is a theorem of $\text{ECNM}^* \text{I}$ (a propositional theorem). Now consider

$$\vdash (\Delta A \leftrightarrow \Delta \neg A)$$

(the i -axiom). After substitution we get

$$((\Box A \vee \Box \neg A) \leftrightarrow (\Box \neg A \vee \Box \neg \neg A))$$

which is a theorem of T . The final requirements on equivalence are that “double substitutions” in any formula of one system are provably equivalent to the original formula of that system. For example, starting with $\Box A$, replacing this by $(\Delta A \& A)$, and then substituting $(\Box A \vee \Box \neg A)$ for the ΔA therein, is equivalent to our original $\Box A$. I.e.,

$$\vdash \Box A \leftrightarrow ((\Box A \vee \Box \neg A) \& A)$$

– obviously a theorem of T. And similarly with a “double substitution” for ΔA

$$\vdash \Delta A \leftrightarrow ((\Delta A \& A) \vee (\Delta \neg A \& \neg A))$$

we get a theorem of ECNM^{*}I (shown by the following argument)

$\vdash \Delta A \leftrightarrow \Delta A$	– prop. logic
$\vdash \Delta A \leftrightarrow (\Delta A \& (A \vee \neg A))$	– prop. equivalence
$\vdash \Delta A \leftrightarrow ((\Delta A \& A) \vee (\Delta A \& \neg A))$	– prop. logic distribution
$\vdash \Delta A \leftrightarrow ((\Delta A \& A) \vee (\Delta \neg A \& \neg A))$	– by [I]

Further details of these equivalence proofs can be found in the paper mentioned in the previous footnote.

I said before that none of [4], [B], [G], [5], [U], [V₁] - [V₄], etc., seem plausible candidates for a logic of indeterminacy. This is because of “higher order indeterminacy”. It seems to me that a proposition might be definite, but not definitely so. Thus

$$\Delta A \& \neg \Delta \Delta A$$

seems possible, as does

$$\nabla A \& \neg \Delta \nabla A$$

and so on, for any number of iterations of the operators Δ and ∇ . If we allow that all of these can happen, we shall want no reduction laws of the sort mentioned in [4], [5], etc.

I therefore recommend ECNM^{*}I as a logic for indeterminacy. Is this a strange modal logic? No - at least no stranger than system T is. Is it appropriate for vagueness *de re*? Probably not, as indicated earlier - but that has nothing to do with whether it is strange.⁽⁸⁾

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