ON THREE-VALUED MOISIL ALGEBRAS

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In this paper we investigate the properties of the family I(A) of all intervals of the form $[x, \sim x]$ with $x \leq \sim x$, in a De Morgan algebra A, and we obtain a necessary and sufficient condition for a complete De Morgan algebra to be a Kleene algebra in terms of I(A) (Lemma 11). We prove that I(A) is a complete Boolean algebra if A is a complete Moisil algebra (Lemma 12).

Let A be a distributive lattice which is bounded, that is, has a largest element 1 and a smallest element 0. We use $x \vee y$ for the join, and $x \wedge y$ for the meet of two elements x, y in A. If $\sim$ is a unary operation defined on A satisfying $\sim\sim x = x$ and $\sim(x \vee y) = \sim x \wedge \sim y$, A is called a *De Morgan algebra*. The set of all complemented elements of A is noted by B(A). A *Kleene algebra* is a De Morgan algebra satisfying: $x \wedge \sim x \leq y \vee \sim y$ [5]. A *three-valued Lukasiewicz algebra* is a system $(L, 1, \sim, \wedge, \vee, \nabla)$ such that $(L, 1, \sim, \wedge, \vee)$ is a De Morgan algebra and ∇ is a unary operation (called *possibility operation*) defined on A with the properties: $\sim x \vee \nabla x = 1$, $x \wedge \sim x = \sim x \wedge \nabla x$ and $\nabla(x \wedge y) = \nabla x \wedge \nabla y$, [6], [7], [8], [10], [11], [14].

The *necessity operation* Δ is defined by $\Delta x = \sim \nabla \sim x$. L is called a centered three-valued Lukasiewicz algebra, or a three-valued Post algebra, if it has a *center*, that is, an element c of L such that $\sim c = c$. The center of L (if it exists) is unique [6], [7], [8], [12], [15].

An *axis* of a three-valued Lukasiewicz algebra L is an element e of L with the properties: $\Delta e = 0$ and $\nabla x \leq \Delta x \vee \nabla e$ for all x of L. Again, if the axis of L exists, it is unique. Following A. Monteiro, L is called a *three-valued Moisil algebra* if it has an axis. For the basic properties of three valued Moisil algebras we refer to [12], [13], [15].

Let E be a three-valued Moisil algebra, and $e \in E$ the axis of E. In [12] L. Monteiro proved that:

(1) $x = (\Delta x \vee e) \wedge \nabla x$, for all $x \in E$, and
(2) $x = (\Delta x \vee \sim e) \wedge \nabla x$, for all $x \in E$.

Let us introduce the following set:

$$S = \{t \in E: x = (\Delta x \vee t) \wedge \nabla x, \text{ for all } x \in E\}.$$

From (1) and (2) it is clear that e and $\sim e$ belong to S. On the other hand, we know that $\Delta e = 0$ and this is equivalent to say that $e \leqslant \sim e$, thus the interval $[e, \sim e] = \{y \in E: e \leqslant y \leqslant \sim e\}$ can be considered.

LEMMA 1. $S = [e, \sim e]$.

Proof. If $y \in [e, \sim e]$, $e \leqslant y \leqslant \sim e$, then we have $x = (\Delta x \vee e) \wedge \nabla e \leqslant (\Delta x \vee y) \wedge \nabla x \leqslant (\Delta x \vee \sim e) \wedge \nabla x = x$ for all $x \in E$. Therefore $x = (\Delta x \vee y) \wedge \nabla x$ for all $x \in E$, whence $y \in S$.
If $s \in S$, then $x = (\Delta x \vee s) \wedge \nabla x$ for all $x \in E$. Then $e \leqslant s$, for $e = (\Delta e \vee s) \wedge \nabla e = (0 \vee s) \wedge \nabla e = s \wedge \nabla e \leqslant s$, and $s \leqslant \sim e$, for $\sim e = (\Delta \sim e \vee s) \wedge \nabla \sim e = (\Delta \sim e \vee s) \wedge 1 = \Delta \sim e \vee s \geqslant s$.

It is well known that $S = [e, \sim e]$ is a distributive lattice, the elements e and $\sim e$ being the least and greastest elements of S respectively [4]. It is not difficult to see that $(S, \sim e, \sim, \wedge, \vee)$ is a Kleene algebra. Furthemore, S is a Boolean algebra. For, if $s \in S$, then $\sim e \in S$ and $t = s \wedge \sim s \in S$, then (i) $x = (\Delta x \vee (s \wedge \sim s)) \wedge \nabla x$ for all $x \in E$. Since (ii) $\Delta t = \Delta(s \wedge \sim s) = \Delta s \wedge \Delta \sim s = 0$, from (i) and (ii) from definition and uniqueness of the axis, we have (iii) $s \wedge \sim s = e$, whence (iv) $s \vee \sim s = \sim e$. So S is a Boolean algebra.

Now we define the following mapping $f: E \to [e, \sim e]$,

$$f(x) = (\sim e \wedge x) \vee e, \text{ for each } x \in E.$$

LEMMA 2. *The mapping f has the following properties.*

F1) $f(1) = \sim e$; F2) $f(0) = e$; F3) $f(x \wedge y) = f(x) \wedge f(y)$; F4) $f(\sim x) = \sim f(x)$; F5) $f(x \vee y) = f(x) \vee f(y)$;
F6) $f(s) = s$ *if and only if* $s \in [e, \sim e]$.

From F1 to F4, it follows that f is an M-homomorphism (homomorphism between Morgan algebras). From F6, it follows that f is an M-epimorphism. L. Monteiro [12] proved that if E is an algebra with axis e, then E is isomorphic to the direct product of a Boolean algebra B and a centered three-valued Lukasiewicz algebra C. More precisely, B is the principal ideal generated by $\sim \nabla e$, $I(\sim \nabla e) = \{x \in E: x \leqslant \sim \nabla e\}$, and C is $I(\nabla e) = \{x \in E: x \leqslant \nabla e\}$.

LEMMA 3. $I(\sim \nabla e)$ *and* S *are isomorphic Boolean algebras.*

Proof. The correspondence $h(x) = x \wedge \sim \nabla e$ sets up an M-epimorphism from E onto $I(\sim \nabla e)$. It is an easy verification that Ker f = Ker h, which completes the proof.

LEMMA 4. *A three-valued Lukasiewicz algebra* L *is a Moisil algebra if* $S = \{s \in L: x = (\Delta x \vee s) \wedge \nabla x$, for all $x \in L\}$, *is a nonvoid set.*

Proof. It is clear that S is closed under $\wedge$ and $\vee$. If $s \in S$, then $\sim x = (\Delta \sim x \vee s) \wedge \nabla \sim x$, for all $x \in L$, hence $x = (\nabla x \wedge \sim s) \vee \Delta x$ for all $x \in L$ and then $\sim s \in S$. From this, $s \wedge \sim s \in S$ for $s \in S$. But $\Delta(s \wedge \sim s) = 0$ and $x = (\Delta x \vee (s \wedge \sim s)) \wedge \nabla x$ for all $x \in L$, therefore $e = s \wedge \sim s$ is the axis of algebra L. From lemma 1, $S = [e, \sim e]$.

We remark that L has a center c if and only if $S = \{c\}$. For if L has a center c, we know that c is also an axis [12] and then $S = [c, \sim c] = [c, c] = \{c\}$. Conversely, if S has only one element c, by lemma 4, c is an axis of L and moreover $\sim c \in S = \{c\}$, thus $\sim c = c$ and so c is the center of L.

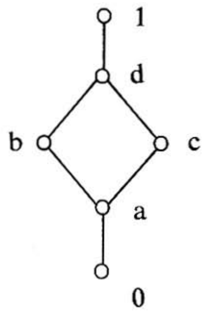
We now deal with the dual concept of ideal. Recall that a filter is a nonvoid subset F of a lattice L with the properties: $a \in F$, $x \in L$, $a \leqslant x$ imply $x \in F$, and $a \in F$, $b \in F$ imply $a \wedge b \in F$. Given an element a in any lattice L, the set $F(a) = \{x \in L : a \leqslant x\}$ is evidently a filter; it is called a principal filter of L.

LEMMA 5. *Let* E *be a three-valued Moisil algebra*, e *its axis. Then the ordered sets* $F(e)$, $I(\sim e)$ *and* $B(E)$ *are isomorphic.*

Proof. Define $H: F(e) \to B(E)$ by $H(x) = \Delta x$, and $G: F(e) \to I(\sim e)$ by $G(x) = \sim x$.

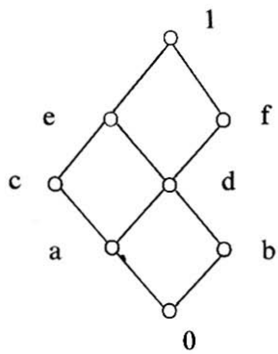
DEFINITION 6. *Let* $(A, 1, \sim, \wedge, \vee)$ *be any De Morgan algebra,* x_0, y_0 *elements of* A *such that* $x_0 \leqslant y_0$. *The interval* $S = [x_0, y_0]$ *is said to be Boolean if the system* $(A, y_0, \sim, \wedge, \vee)$ *is a Boolean algebra.*

EXAMPLES.



x	$\sim x$
0	1
a	d
b	c

Fig. 1



x	$\sim x$
0	1
a	e
b	f
c	c
d	d

Fig. 2

In the algebra of fig. 1, [a, d] is Boolean, and [0, 1] is not Boolean. In the algebra of fig. 2, [a, e] with its natural ordering is a Boolean algebra, but it is not a Boolean interval because $\sim c = c \neq d$. The only Boolean intervals are [c, c] and [d, d].

REMARK. If $S = [x_0, y_0]$ is a Boolean interval, then 1) $\sim x_0 = y_0$; 2) $x \wedge \sim x = x_0$ for all $x \in S$. Furthemore, the condition 1 implies 3) $\sim S \subseteq S$, and conversely, if $\sim [x_0, y_0] \subseteq [x_0, y_0]$, then $\sim x_0$ and $\sim y_0$

belong to $[x_0, y_0]$, so $x_0 \leqslant \sim y_0$ and $\sim x_0 \leqslant y_0$, whence $x_0 = \sim y_0$. Thus, every Boolean interval can be written as $[x_0, \sim x_0]$ with $x_0 \leqslant \sim x_0$. However, an interval of this kind need not be Boolean as we can see for [0, 1] of the examples above.

It should pointed out that a De Morgan algebra need not have Boolean intervals, as the following example shows.

EXAMPLE. Let $T = \{0, c, 1\}$, with $0 < c < 1$, and $\sim 0 = 1$, $\sim c = c$, $\sim 1 = 0$. T is a Kleene algebra, and if N is the set of all positive integers, it is readily verified that $A = T^N$ is a Kleene algebra with pointwise operations.

Let K_1 be the set of all $f \in T^N$ such that $f(x) \in \{0, 1\}$ for all $x \in N$, and K_2 the set of all $f \in T^N$ such that there exists a nonvoid finite subset F of N with $f(x) = c$ for all $x \in F$, and $f(x) \in \{0, 1\}$ if $x \notin F$. Clearly $K_1 \cap K_2 = \emptyset$. Consider $K = K_1 \cup K_2$. It is easy to see that K is a proper Kleene subalgebra of A. Suppose $[f, \sim f]$ is a Boolean interval of K. Since $f \leqslant \sim f$, then $f(x) \in \{0, c\}$ for all $x \in N$, so there exists a finite subset F of N such that $f(x) = c$ for all $x \in F$ and $f(y) = 0$ if $y \notin F$. Let i_0 be a positive integer such that $i_0 \notin F$, and define $b \in K : b(x) = c$, if $x \in F \cup \{i_0\}$, $b(x) = 0$, otherwise. Then $b \in [f, \sim f]$, and $b \leqslant \sim b$, so $b \wedge \sim b = b \neq f$, which contradicts that $[f, \sim f]$ is Boolean.

LEMMA 7. *If A is a Kleene algebra and there exists a Boolean interval in A, it is unique.*

Proof. Let $S_1 = [e, \sim e]$ and $S_2 = [f, \sim f]$ Boolean intervals.
Since A is a Kleene algebra: (1) $e = e \wedge \sim e \leqslant f \vee \sim f = \sim f$. Then $e \vee f \leqslant \sim f \vee f = \sim f$, hence $f \leqslant e \vee f \leqslant \sim f$, so $e \vee f \in S_2$, then (2) $(e \vee f) \wedge \sim(e \vee f) = f$. By (1), $f \leqslant \sim e$ then $e \leqslant f \vee e \leqslant \sim e \vee e = \sim e$, hence $e \vee f \in S_1$, then (3) $(e \vee f) \wedge \sim(e \vee f) = e$. From (2) and (3) $e = f$ and $\sim e = \sim f$. Then $S_1 = S_2$.

Denote the set of intervals of a De Morgan algebra A of the form $[x, \sim x]$ with $x \leqslant \sim x$ by I(A). $I(A) \neq \emptyset$ seeing that $A = [0, 1] \in I(A)$. Notice that $I(A) = \{[x \wedge \sim x, x \vee \sim x], x \in A\}$.

LEMMA 8. *Any minimal element of the ordered set* $(I(A), \subseteq)$, *whenever it exists, is a Boolean interval.*

Proof. Let $S = [x_0, \sim x_0]$ be minimal element of I(A). Clearly $x_0 \leq t \wedge \sim t \leq \sim x_0$ for every t of S. Then $S_t = [t \wedge \sim t, t \vee \sim t] \in I(A)$ and $S_t \subseteq S$ for every $t \in S$. But since S is minimal, $S_t = S$ and therefore $t \wedge \sim t = x_0$ for every $t \in S$. Thus S is Boolean.

LEMMA 9. *If A is a complete De Morgan algebra, then the ordered set* I(A) *has the Zorn's property for descending chains.*

Proof. Let $C = \{S_j : j \in J\}$ be any chain of sets in I(A), $S_j = [x_j, \sim x_j]$. Since A is complete, there exists the element $x = \bigvee_{j \in J} x_j$. We next show that $\bigvee_{j \in J} x_j \leq \sim \bigvee_{j \in J} x_j = \bigwedge_{j \in J} \sim x_j$. We have either $S_k \subseteq S_j$ or $S_j \subseteq S_k$, then $x_j \leq x_k$ or $x_k \leq x_j$. But $x_j \leq \sim x_j$, so $x_j \leq x_k$ or $x_k \leq \sim x_j$, then $x_j \leq x_k$ or $x_j \leq \sim x_k$. Hence, $x_j \leq x_k \vee \sim x_k = \sim x_k$ for every j, k. Then $\bigvee_{j \in J} x_j \leq \sim x_k$ for every k and therefore $\bigvee_{j \in J} x_j \leq \bigwedge_{k \in J} \sim x_k = \sim \bigvee_{k \in J} x_k$. Whence $S = [\bigvee_{j \in J} x_j, \sim \bigvee_{j \in J} x_j] \in I(A)$. Let us see that $S = \bigcap_{j \in J} S_j$. If $t \in S$ then $x_j \leq \bigvee_{j \in J} x_j \leq t \leq \bigwedge_{j \in J} \sim x_j \leq \sim x_j$ for all j, and $t \in \bigcap_{j \in J} S_j$. Conversely, $t \in \bigcap_{j \in J} S_j$ implies that $x_j \leq t \leq \sim x_j$ for every j and then $\bigvee_{j \in J} x_j \leq t \leq \bigwedge_{j \in J} \sim x_j$, that is, $t \in S$.

COROLLARY 10. *Any complete De Morgan algebra has Boolean intervals.*

LEMMA 11. *A complete De Morgan algebra* A *is a Kleene algebra if and only if* I(A) *has least element.*

Proof. By the above lemmas, if A is a Kleene algebra, it has a unique Boolean interval, which is the least element in I(A).

For the converse let $P = [b, \sim b]$ be the least element in I(A). Then, $P \subseteq [x \wedge \sim x, x \vee \sim x]$ for all $x \in A$. Since $b \in P$, then $x \wedge \sim x \leq b$ for all $x \in A$ and $b \leq y \vee \sim y$ for $y \in A$. Therefore $x \wedge \sim x \leq y \vee \sim y$ for x, $y \in A$, that is, A is a Kleene algebra. (Note that in the converse, the completeness of A has not been used).

Let A be Kleene algebra. Define the following binary operations on I(A):

$$[x, \sim x] \vee [y, \sim y] = [x \wedge y, \sim x \vee \sim y]$$
$$[x, \sim x] \wedge [y, \sim y] = [x \vee y, \sim x \wedge \sim y].$$

From Kleene condition, it follows at once that $[x \wedge y, \sim x \vee \sim y]$ and $[x \vee y, \sim x \wedge \sim y]$ belong to I(A), if $x \leq \sim x$ and $y \leq \sim y$.

Moreover, it is easy to check that $(I(A), A, \wedge, \vee)$ is a distributive lattice with greatest element A.

In fact, $[x, \sim x] \wedge [y, \sim y] = [x, \sim x]$ if and only if $[x, \sim x] \subseteq [y, \sim y]$ for $[x, \sim x]$ and $[y, \sim y] \in I(A)$.

Observe that if A is a complete three-valued Moisil algebra, then I(A) has least element $[e, \sim e]$.

Indeed, we know that $[x, \sim x] \leqslant [e, \sim e]$ if and only if $x \in [e, \sim e]$ and $\sim x \in [e, \sim e]$, but then $x \wedge \sim x = e$ and $x \vee \sim x = \sim e$. Then $[x, \sim x] = [e, \sim e]$. Therefore $[e, \sim e]$ is minimal in I(A). Since every minimal element of I(A) is Boolean, and since $[e, \sim e]$ is the unique Boolean interval of A, thus $[e, \sim e]$ is the unique minimal interval of A. Then $[e, \sim e]$ is the least element of I(A).

LEMMA 12. *If* A *is a complete Moisil algebra, then* I(A) *is a complete Boolean algebra.*

Proof. To prove that for all $[x, \sim x] \in I(A)$ there exists $[y, \sim y] \in I(A)$ such that $[x, \sim x] \wedge [y, \sim y] = [e, \sim e]$ and $[x, \sim x] \vee [y, \sim y] = [0, 1]$ it is equivalent to prove that for all $x \leqslant e$ there exists $y \leqslant e$ such that $x \vee y = e$ and $x \wedge y = 0$.

Let us put $y = \Delta \sim x \wedge e$. Then

(i) $x \wedge (\Delta \sim x \wedge e) = 0$ because $x \wedge \Delta \sim x = 0$.

(ii) $x \vee (\Delta \sim x \wedge e) = e$. Indeed, $\Delta(x \vee (\Delta \sim x \wedge e)) = \Delta x \vee (\Delta \sim x \wedge \Delta e) = \Delta x = 0 = \Delta e$ and $\nabla(x \vee (\Delta \sim x \wedge e)) = \nabla x \vee (\Delta \sim x \wedge \nabla e) = (\nabla x \vee \Delta \sim x) \wedge (\nabla x \vee \nabla e) = (\nabla x \vee \Delta \sim x) \wedge \nabla e = 1 \wedge \nabla e = \nabla e$, and by determination principle (see [12] and [14]) (ii) holds.

The completeness of I(A) follows from $\bigwedge_{i \in I} [x_i, \sim x_i] = [\bigvee_{i \in I} x_i, \bigwedge_{i \in I} \sim x_i]$, $\bigvee_{i \in I} [x_i, \sim x_i] = [\bigwedge_{i \in I} x_i, \bigvee_{i \in I} \sim x_i]$, since A is complete.

Observe that the mapping $H: F(\sim e) \to I(A)$ defined by $H(\sim y) = [y, \sim y]$, for all element $\sim y \in F(\sim e)$, is an order isomorphism. Indeed, H is clearly a bijection and $\sim y \leqslant \sim y'$ if and only if $H(\sim y) \leqslant H(\sim y')$. Thus we can state

COROLLARY 13. *If* A *is a three-valued Post algebra such that* I(A) *is a complete Boolean algebra, then* A *is complete.*

Proof. By the above remark, if I(A) is a complete Boolean algebra, then F(~e) is a complete Boolean algebra. But in a three-valued Post algebra, e = ~e, thus F(~e) = F(e), which is isomorphic (Lemma 5) to the set B(A) of all Boolean elements of A. Then B(A) is complete, and this is sufficient (see [12]) to assure that A is complete.

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