NATURAL DEDUCTION SYSTEMS FOR SOME QUANTIFIED RELEVANT LOGICS

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On the whole, natural deduction systems enable proofs of most theorems to be obtained far more easily than in a corresponding Hilbert-style axiomatization. So, it is worthwhile determining natural deduction systems for sentential and quantified relevant logics for which there is some point in their study, e.g. those systems appearing in the literature and those that are assuming importance as logics that avoid the set-theoretic paradoxes. (See [3] and [5] for relevant logics that enable naive set theory to be shown to be non-trivial and simply consistent, respectively.)

Anderson in [1] was the first to establish a natural deduction system for a relevant logic. He showed that axiomatic systems for E and EQ have the same sets of theorems as the respective natural deduction systems E* and EQ*, which are now called FE and FEQ after Fitch. The method of proof used by Anderson is applied in Anderson and Belnap, [2], to yield the natural deduction systems FR, FT, FRM and FEM for the logics R, T, RM and EM, respectively.

In this paper, I will employ Anderson's method to obtain natural deduction systems for the sentential and quantified relevant logics axiomatized in § 1. The method would also be extendible to include many additional systems not listed in § 1, for which interest may be found.

§ 1. *Hilbert-style Axiomatizations.*

All the sentential logics will have primitives, $\sim$, &, $\vee$, $\rightarrow$, and sentential variables, and are composed from the following axioms and rules as indicated.

Axioms.

A1. $A \rightarrow A$.

A2. $A \& B \to A$.
A3. $A \& B \to B$.
A4. $(A \to B) \& (A \to C) \to .\ A \to B \& C$.
A5. $A \to A \vee B$.
A6. $B \to A \vee B$.
A7. $(A \to C) \& (B \to C) \to .\ A \vee B \to C$.
A8. $A \& (B \vee C) \to (A \& B) \vee (A \& C)$.
A9. $\sim\sim A \to A$.
A10. $A \to \sim B \to .\ B \to \sim A$.
A11. $(A \to B) \& (B \to C) \to .\ A \to C$.
A12. $A \vee \sim A$.
A13. $A \to \sim A \to \sim A$.
A14. $A \to B \to .\ B \to C \to .\ A \to C$.
A15. $A \to B \to .\ C \to A \to .\ C \to B$.
A16. $(A \to .\ A \to B) \to .\ A \to B$.
A17. $A \to .\ A \to B \to B$.
A18. $A \to B \to .\ A \& C \to B \& C$.
A19. $A \to B \to .\ A \vee C \to B \vee C$.

Rules.
R1. $A, A \to B \Rightarrow B$.
R2. $A, B \Rightarrow A \& B$.
R3. $A \to B, C \to D \Rightarrow B \to C \to .\ A \to D$.
R4. $A \to \sim B \Rightarrow B \to \sim A$.
R5. $C \vee A, C \vee (A \to B) \Rightarrow C \vee B$.
R6. $E \vee (A \to B), E \vee (C \to D) \Rightarrow E \vee (B \to C \to .\ A \to D)$.
R7. $C \vee (A \to \sim B) \Rightarrow C \vee (B \to \sim A)$.

Sentential Logics.
B = A1-9, R1-4.
DW = B + A10 − R4.
DJ = DW + A11.
DK = DJ + A12.
DL = DJ + A13 − A12.
TW = DW + A14 + A15 − R3.
TJ = TW + A11.
TK = TJ + A12.
T = TK + A13 + A16 − A11 − A12.

RW = TW + A17 − A15.
R = T + A17 − A13 − A15.
= RW + A16.
RF = R + A18 + A19.

Disjunctive Rules.
Some of these logics can be extended by using disjunctive rules (R5-7) where appropriate. A superscript '^{d}' will be used to indicate such an extension in the following logics:
B^d = B + R5 + R6 + R7 − R3 − R4.
DW^d = DW + R5 + R6 − R3.
DJ^d = DJ + R5 + R6 − R3.
DK^d = DK + R5 + R6 − R3.
DL^d = DL + R5 + R6 − R3.
TW^d = TW + R5.
TJ^d = TJ + R5.
TK^d = TK + R5.
RW^d = RW + R5.
The subtraction of axioms and rules in the above is to eliminate redundancies.

Quantificational Extensions.
The quantificational extension XQ of any of the above sentential logics X without disjunctive rules can be obtained by adding the primitives, ∀, ∃, and individual and predicate variables, and the following axioms and rule:
QA1. $(\forall x)A \to A^{y}/x$, where y is free for x in A.
QA2. $(\forall x)(A \to B) \to. A \to (\forall x)B$, where x is not free in A.
QA3. $(\forall x)(A \vee B) \to A \vee (\forall x)B$, where x is not free in A.
QA4. $A^{y}/x \to (\exists x)A$, where y is free for x in A.
QA5. $(\forall x)(A \to B) \to. (\exists x)A \to B$, where x is not free in B.
QA6. $A \& (\exists x)B \to (\exists x)(A \& B)$, where x is not free in A.
QR1. $A \Rightarrow (\forall x)A$. (1)

(1) On p. 212 of [1], Anderson uses the clause 'if A is an axiom scheme then (∀x)A is an axiom scheme' instead of the Generalization Rule. Here, I prefer the rule, as it is more customary and there is no problem relating such an axiomatization to my corresponding natural deduction system.

It is also possible to introduce these logics with '$\vee$' and '$\exists$' defined in the usual way, by omitting A5, A6, and A7 in each of the above sentential systems and by omitting as well QA4, QA5 and QA6 in each of the quantified systems, except that A7 is retained in B and B^d and QA5 is retained as well in BQ and B^dQ.

The quantificational extension X^dQ of any of the sentential logics X^d with disjunctive rules can be obtained by adding the appropriate symbols, the axioms QA1-6 and the rule QR1, as above, by deleting R5 and replacing R6 and R7 by R3 and R4, respectively, and by adding the following two meta-rules:

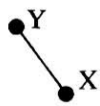
MR1. If $A \Rightarrow B$ then $C \vee A \Rightarrow C \vee B$

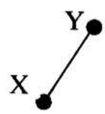
QMR1. If $A \Rightarrow B$ then $(\exists x)A \Rightarrow (\exists x)B$

Both of these meta-rules carry the proviso that, in the derivation of B from A, QR1 does not generalize on any variable free in A.

These meta-rules enable the right hand derivations, $C \vee A \Rightarrow C \vee B$ and $(\exists x)A \Rightarrow (\exists x)B$, respectively, to be included in a proof, provided the left hand derivations, $A \Rightarrow B$, subject to the proviso, can be carried through. This derivation $A \Rightarrow B$ can be set out as a subproof of the main proof. Also, the need for these meta-rules arises when combining the disjunctive rules with the quantificational extension because there is a need to allow arbitrary interactions between the derived rules established by these two meta-rules, and these interactions are difficult to represent using transformation rules.

On this table, 

, 

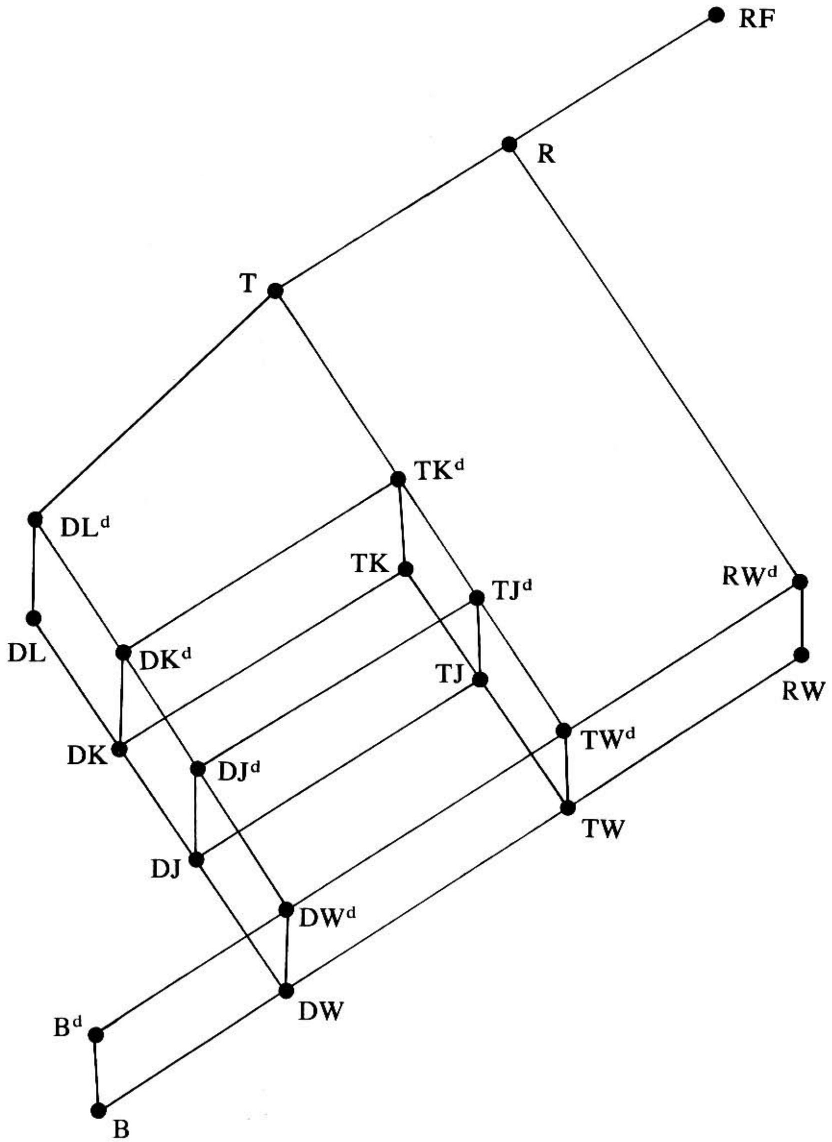
and 

signify that X is is theorem-wise and rule-wise contained in Y, though it may be the case that some of the pairs X and X^d have the same theorems.

Motivation for the Logics.

Of course, T and R are studied in [2], as well as in other sources, but I do offer a slightly simpler natural deduction system for T than that given in [2], pp. 346-7. RF is studied in [11] as an interesting extension of R, which, though not a relevant logic, falls short of being classical. I have included it because it has a rather interesting natural deduction system. The logic B is the weakest system which is given a Routley-Meyer semantics in [11].

Table of Sentential Logics.



The interest in the remaining logics is essentially in their ability to produce a simply consistent or a non-trivial naive set theory. White, in [12], has shown that the Lukasiewicz infinitively-valued logic L_∞ yields a simply consistent naive comprehension axiom. The logic RW^d, and hence RW, is contained in L_∞ and so the result apllies to it also. In my paper, [5], I showed that TW, and also TW^d, yields a simply consistent naive set theory consisting of a comprehension axiom and an extensionality rule. Also, Kron, in [7], establishes decision procedures for RW_+ and TW_+, the positive fragments of RW and TW. It is hoped that RW and TW are decidable also, and this provides additional interest in these logics.

In [3], I showed that DK, and also DK^d, yields a non-trivial naive set theory, i.e. the set-theoretic paradoxes enable contradictions to be proved, but not all formulae are provable. [9] and [10] can be consulted for the motivation behind this approach. DL was introduced in the earlier work, [8], as a candidate for a logic which would yield such a set theory. I have also shown in [4] that DK^d and indeed all logics X contained in it satisfy the Depth Relevance Condition: For all formulae A and B, if $\vdash_X A \rightarrow B$ then A and B share a variable at the same depth.

I have shown (2) that the simple consistency result of [5] holds with TW^d extended to TJ^d, i.e. with A11 added. I have also shown (3) that the non-triviality result of [3] holds with DK^d extended to TK^d, i.e. with A14 and A15 added. The two following remarks will then fill the gaps remaining in the table of logics. Firstly, DW is included as a logic which may not have much intrinsic interest but is a stepping stone between B and DK, and between B and TW. I believe that DJ and DJ^d are not just stepping stones, but will have substantial interest as

(2) This can be fairly easily shown by replacing the Lukasiewicz 3-valued logic L_3, used in [5], by the following 3-valued matrix logic C_3:

~	
*1	0
½	½
0	1

&	1	½	0
*1	1	½	0
½	½	½	0
0	0	0	0

$\vec{C}$	1	½	0
*1	1	0	0
½	1	1	0
0	1	1	1

(3) This can also be easily shown by replacing L_3 in [5] by the matrix logic D_3, which is the same as C_3 above, except that '½' is also designated.

paradox-avoiding logics. (4) Secondly, the reason for the interest in the logics with disjunctive rules, i.e. with superscript '^{d}', is that such logics have a simpler semantics, called a reduced modelling, the principle feature being that such disjunctive rules preserve truth in the base set-up T of each model structure. Details of this can be found in [11].

§ 2. *The Natural Deduction Systems.*

The natural deduction systems FX and FXQ for the above sentential logics X will take the form of the Fitch-style systems expressed in Anderson and Belnap [2], pp. 346-7, with the quantificational extensions taken from Anderson, [1]. Restrictions will have to be placed on the →E, ~E and ∨E rules which are appropriate for the logic involved, except for the stronger logics R and RF, where no restrictions are needed. A rule form of ∨E will be introduced for those logics with disjunctive rules. Also ~I will be deleted or weakened for those logics without $A \to \sim A \to \sim A$. Further, B and B^d require a weakening of the ~E rule, and RF requires a strengthening of the rules, &I and ∨E.

The sentential natural deduction systems have as primitives, sentential variables and connectives, ~, &, ∨, and →. Their quantificational extensions have in addition individual and predicate variables and quantifiers, ∀ and ∃. We start by presenting FB in full, and then indicate the additions and alterations for the other sentential systems.

FB.

Hyp. A formula A may be introduced as the hypothesis of a new subproof, with a subscript {k}, where k is the rank of the new subproof. (*Rank* is defined on p. 70 of [2] as the number of vertical lines to the left of the formula A. One could also call this the depth of the subproof.)

Rep. A_a may be repeated in the same subproof, retaining its index set a.

(4) I am writing a book, "Universal Logic", in which I will be arguing in favour of DJ^d as a general relevant entailment, which in particular yields a consistent naive set theory and also solves the semantic paradoxes in a similar way.

Reit. A_a may be reiterated into any subproof inside the subproof containing A_a, retaining the index set a.

→I. From a proof of B_a on the hypothesis $A_{\{k\}}$, to infer $A \rightarrow B_{a-\{k\}}$ in the next outer subproof, provided $k \in a$. (Such a subproof has a rank, $k-1$.)

→E. From A_a and $A \rightarrow B_b$, to infer $B_{a \cup b}$, provided the B-restriction holds (see below).

~E. From $\sim B_a$ and $A \rightarrow B_\phi$, to infer $\sim A_a$.

~~I. From A_a, to infer $\sim\sim A_a$.

~~E. From $\sim\sim A_a$, to infer A_a.

&I. From A_a and B_a, to infer $A \& B_a$.

&E. From $A \& B_a$, to infer A_a.
From $A \& B_a$, to infer B_a.

∨I. From A_a, to infer $A \vee B_a$.
From B_a, to infer $A \vee B_a$.

∨E. From $A \vee B_a$, $A \rightarrow C_b$ and $B \rightarrow C_b$, to infer $C_{a \cup b}$, provided the B-restriction holds (see below).

&∨. From $A \& (B \vee C)_a$, to infer $(A \& B) \vee (A \& C)_a$.

B-restriction: If $b \neq \phi$ then a is a singleton set $\{m\}$ such that $\max(b) < m$.

FDW. As for FB, except for ~E, which is as follows:
~E. From $\sim B_a$ and $A \rightarrow B_b$, to infer $\sim A_{a \cup b}$, provided the B-restriction holds.

FDJ. As for FDW, except for →E, ~E and ∨E, where the B-restriction is replaced by the *DJ-restriction*: If $b \neq \phi$ then (i) $a \neq \phi$, (ii) $\max(b) < \max(a)$, and (iii) $a - \{\max(a)\} = b$ or ϕ.
Note that the B-restriction can be obtained by replacing (iii) by $a - \{\max(a)\} = \phi$, or by adding (iv) $a \cap b = \phi$.

FDK. As for FDJ, with the addition of the following ~I rule:
~I. From $A \rightarrow \sim A_\phi$, to infer $\sim A_\phi$.

FDL. As for FDK, but with the stronger ~I rule:
~I. From $A \rightarrow \sim A_a$, to infer $\sim A_a$.

FTW. As for FDW, except for →E, ~E and ∨E, where the B-restriction is replaced by the *TW-restriction*: If $b \neq \phi$ then (i) $a \neq \phi$, (ii) $\max(b) < \max(a)$, and (iii) $a \cap b = \phi$.

FTJ. As for FTW, except that the TW-restriction is replaced by the *TJ-restriction*: If $b \neq \phi$ then (i) $a \neq \phi$, (ii) $\max(b) < \max(a)$, and

(iii) for all natural numbers m and n, if $n \in a \cap b$ and $m < n$ then, $m \in a$ iff $m \in b$. [5]

FTK. As for FTJ, with the addition of the following ~I rule:
~I. From $A \rightarrow \sim A_\phi$, to infer $\sim A_\phi$.

FT. As for FTK, but with ~I strengthened to:
~I. From $A \rightarrow \sim A_a$, to infer $\sim A_a$, and with the TJ-restriction replaced by the *T-restriction*: If $b \neq \phi$ then (i) $a \neq \phi$ and (ii) $\max(b) \leq \max(a)$.
Note that there is no restriction on the Reit. rule, as appears in [2], p. 347, and that the T-restriction is more precisely put than in [2].

FRW. As for FTW, with the TW-restriction replaced by the *RW-restriction*: $a \cap b = \phi$.

FR. As for FT, with the rules →E, ~E and ∨E without restriction. This is exactly the system appearing in [2].

FRF. As for FR, but with &I and ∨E strengthened to the following:
&I. From A_a and B_b, to infer $A \,\&\, B_{a \cup b}$, provided $a = b = \phi$ or ($a \neq \phi$ and $b \neq \phi$ and $\max(a) = \max(b)$).
∨E. From $A \vee B_a$, $A \rightarrow C_b$ and $B \rightarrow C_c$, to infer $C_{a \cup b \cup c}$.
Each of the above natural deduction systems FX for X without disjunctive rules can be converted into the system FX^d by adding the rule form of ∨E:

∨E^r. From $A \vee B_\phi$, $A_\phi \Rightarrow C_\phi$, and $B_\phi \Rightarrow C_\phi$, to infer C_ϕ, where
(i) the rule is applied only to proofs and not to subproofs of proofs,
(ii) $A_\phi \Rightarrow C_\phi$ means that C_ϕ is derivable using natural deduction rules from the assumption A_ϕ made in the proof, i.e. A_ϕ is not a hypothesis of a subproof, and
(iii) the formula $A \vee B_\phi$, the derivations $A_\phi \Rightarrow C_\phi$ and $B_\phi \Rightarrow C_\phi$ and the conclusion C_ϕ are all subject to the same sequence of assumptions (if any).

[5] (iii) of the TJ-restriction can be replaced by the following:
(iii)′ $a - \{\max(a)\} = b$ or $a \cap b = \phi$.
Any pair (a, b) satisfying (iii)′ satisfies (iii), but the converse fails. This makes (iii) superior to (iii)′ as any proof of a theorem requiring the satisfaction of (iii) and not (iii)′ would have to be replaced by some alternative proof if (iii)′ was adopted, and such an alternative proof may be hard to find.

Note that the term 'assumption' is used for the hypothesis of a derivation, i.e. A_ϕ and B_ϕ in $\vee E^r$.

Each of the above sentential natural deduction systems FX for X without disjunctive rules can be given a quantificational extension FXQ by adding the following rules:

∀I. From A_a, to infer $(\forall x)A_a$, provided x is not free in any hypothesis $H_{\{k\}}$ with $k \in a$.

∀E. From $(\forall x)A_a$, to infer A^y/x_a, where y is free for x in A.

∀∨. From $(\forall x)(A \vee B)_a$, to infer $A \vee (\forall x)B_a$, where x is not free in A.

∃I. From A^y/x_a, to infer $(\exists x)A_a$, where y is free for x in A.

∃E. From $(\exists x)A_a$ and $(\forall x)(A \rightarrow B)_b$, to infer $B_{a \cup b}$, where x is not free in B, and provided a and b satisfy the same restriction as for the rules →E and ∨E, as is appropriate for the logic X involved.

∃&. From $A \,\&\, (\exists x)B_a$, to infer $(\exists x)(A \,\&\, B)_a$, where x is not free in A.(6)

Each of the sentential natural deduction systems FX^d can be given a quantificational extension FX^dQ by adding the above quantificational rules, ∀I, ∀E, ∀∨, ∃I, ∃E and ∃&, by putting a proviso on $\vee E^r$, as given below, and by adding the rule $\exists E^r$ below.

$\vee E^r$. From $A \vee B_\phi$, $A_\phi \Rightarrow C_\phi$, and $B_\phi \Rightarrow C_\phi$, to infer C_ϕ, where this rule is carried out in the same manner as indicated for FX^d, but with the proviso that in the derivations, $A_\phi \Rightarrow C_\phi$ and $B_\phi \Rightarrow C_\phi$, ∀I is not used to generalize on any free variable of A and B, respectively.

$\exists E^r$. From $(\exists x)A_\phi$ and $A_\phi \Rightarrow B_\phi$, to infer B_ϕ, where x is not free in B and where the rule is carried out in the same manner as for $\vee E^r$ and has the proviso that in the derivation, $A_\phi \Rightarrow B_\phi$, ∀I is not used to generalize on any free variable of A.

§3. *The Equivalence between the Sentential Logics X and the Systems FX.*

(6) Anderson entirely follows Fitch, [6], in setting up his natural deduction system, whereas I have not differentiated hypothetical and categorical subproofs for the quantificational extensions. However, the form of my quantificational rules reflects Anderson's in [1], pp. 212-3, except that I do not place any restriction on the Reit rule. This lack of restriction is better as it gives more flexibility in proofs.

We first show that B and FB are equivalent, i.e. have the same theorems, and then indicate the modifications to this equivalence proof that are required for the remaining sentential logics.

Theorem 1.
For all formulae A, if $\vdash_{\overline{B}}$ A then $\vdash_{\overline{FB}}$ A.

Proof. As can easily be checked by the reader, all the axioms of B are theorems of FB and all the rules of B preserve theoremhood in FB.

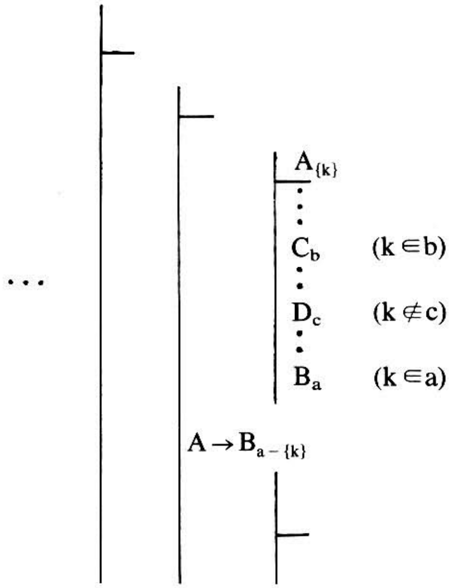
Theorem 2.
For all formulae A, if $\vdash_{\overline{FB}}$ A then $\vdash_{\overline{B}}$ A.

Proof. The proof proceeds in the manner of Anderson [1], pp. 206-210, and of Anderson and Belnap [2], pp. 24-6. We begin with the definition of a quasi-proof.

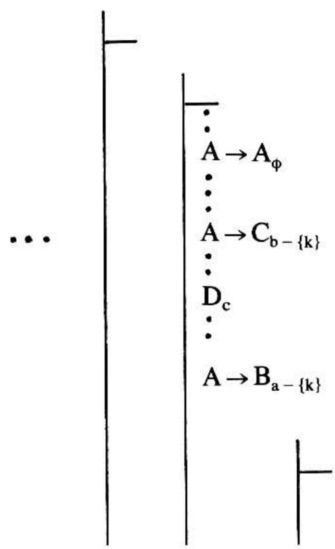
A *quasi-proof* in FB is a natural deduction proof in FB except that the following 4 extra rules are allowed:
Thm. Any theorem A_ϕ of B may be inserted at any point in a proof.
Pref. From $A \to B_\phi$, to infer $C \to A \to . \ C \to B_\phi$.
Suff. From $A \to B_\phi$, to infer $B \to C \to . \ A \to C_\phi$.
Contrap. From $A \to B_\phi$, to infer $\sim B \to \sim A_\phi$.

We take a proof in FB of a formula A′ and, noting that it is also a quasi-proof, replace each innermost subproof by corresponding steps in its next innermost subproof to form a new quasi-proof. By successive applications of this procedure a quasi-proof is obtained without any subproofs at all. Such a quasi-proof can then be seen to yield a proof of the formula A′ in B.

Let us consider an innermost subproof Q of a quasi-proof P of A′ in FB, diagrammed as follows:



This innermost subproof is replaced by steps in the next innermost subproof of P to form the sequence Q′, as follows:



I.e. if C_b has $k \in b$ put $A \to C_{b-\{k\}}$, and if D_c has $k \notin c$ put D_c.

Then we need to show that the quasi-proof P with the subproof Q replaced by the sequence Q′ forms a quasi-proof P′ of the formula A′. To do this, we consider in turn each of the rules that can be used in Q.

Hyp. $A_{\{k\}}$ of Q is replaced by $A \rightarrow A_{\phi}$ in Q′, which satisfies the rule Thm.

Rep. If C_a is replaced in Q with $k \in a$, then $A \rightarrow C_{a-\{k\}}$ can be repeated in Q′. If C_a is repeated in Q with $k \notin a$, then it can also be repeated in Q′.

Reit. If C_a is reiterated into Q then $k \notin a$ and C_a can be reiterated into Q′ or repeated in Q′.

→I. This can only be applied to the conclusion B_a of Q, yielding $A \rightarrow B_{a-\{k\}}$, which is what this proof establishes as a concluding step for Q′. Thus Rep. can be used to join up Q′ to the rest of the quasi-proof P.

→E. Let $D_{a \cup b}$ be derived from C_a and $C \rightarrow D_b$ in Q, with the B-restriction holding. There are 4 cases:

(a) $k \notin a$ and $k \notin b$. The same application of →E is then made within the sequence Q′.

(b) $k \notin a$ and $k \in b$. Then $k = \max(b)$ and by the B-restriction $a = \{m\}$, for some m, and $k < m$, which is impossible, Q being innermost.

(c) $k \in a$ and $k \notin b$. Then $k = \max(a)$.

(i) Let $b = \phi$. Let $A \rightarrow C_{a-\{k\}}$ and $C \rightarrow D_{\phi}$ of Q′ be derivable. By Pref and →E, $A \rightarrow D_{a-\{k\}}$ is derivable.

(ii) Let $b \neq \phi$. Then, by the B-restriction, $a = \{k\}$. Let $A \rightarrow C_{\phi}$ and $C \rightarrow D_b$ of Q′ be derivable. By Suff and →E, $A \rightarrow D_b$ is also derivable.

Note that $(a \cup b) - \{k\} = b$, making b the required index set.

(d) $k \in a$ and $k \in b$. Then $k = \max(a) = \max(b)$. Since $b \neq \phi$, by the B-restriction, $\max(b) < \max(a)$, which is a contradiction.

~E. Let $\sim C_a$ be derived from $C \rightarrow D_{\phi}$ and $\sim D_a$ in Q.

(a) Let $k \in a$. Let $C \rightarrow D_{\phi}$ and $A \rightarrow \sim D_{a-\{k\}}$ of Q′ be derivable. By Contrap, Pref and →E, $A \rightarrow \sim C_{a-\{k\}}$ is derivable.

(b) Let $k \notin a$. Then $\sim$E can be applied within Q′.

∼∼I. Let $\sim\sim C_a$ be derived from C_a in Q.

(a) Let $k \in a$. $A \to \sim\sim C_{a-\{k\}}$ can be derived from $A \to C_{a-\{k\}}$ in Q′ by inserting the theorem $A \to C \to .\ A \to \sim\sim C_\phi$ and using →E.

(b) For $k \notin a$, apply ∼∼I within Q′.

∼∼E, &I, &E, ∨I and &∨ are all similar to ∼∼I with the following respective theorems of B being inserted: $A \to \sim\sim C \to .\ A \to C$, $(A \to C) \& (A \to D) \to .\ A \to C \& D$, $A \to C \& D \to .\ A \to C$, $A \to C \& D \to .\ A \to D$, $A \to C \to .\ A \to C \vee D$, $A \to D \to .\ A \to C \vee D$, $A \to C \& (D \vee E) \to .\ A \to (C \& D) \vee (C \& E)$.

∨E. Let $E_{a \cup b}$ be derived from $C \vee D_a$, $C \to E_b$ and $D \to E_b$ in Q, where the B-restriction holds. There are 4 cases:

(a) $k \notin a$ and $k \notin b$. ∨E is applied within Q′.

(b) $k \notin a$ and $k \in b$. This is impossible, as for case (b) of →E above.

(c) $k \in a$ and $k \notin b$. Then $k = \max(a)$.

(i) Let $b = \phi$. Let $A \to C \vee D_{a-\{k\}}$, $C \to E_\phi$ and $D \to E_\phi$ of Q′ all be derivable. By &I, inserting $(C \to E) \& (D \to E) \to .\ C \vee D \to E_\phi$ and →E, $C \vee D \to E_\phi$ is derivable. Further, by Pref and →E, $A \to E_{a-\{k\}}$ is established, as required.

(ii) Let $b \neq \phi$. By the B-restriction, $a = \{k\}$. Let $A \to C \vee D_\phi$, $C \to E_b$ and $D \to E_b$ of Q′ be derivable. By &I, inserting $(C \to E) \& (D \to E) \to .\ C \vee D \to E_\phi$, →E, Suff and →E again, we obtain $A \to E_b$, as required.

(d) $k \in a$ and $k \in b$. As for case (d) of →E, this is impossible.

It remains to examine each of the rules of a quasi-proof of A′ with no subproofs to see that they are derived rules of B and thus to show that A′ is provable in B. This is easily shown and is left to the reader.

Theorem 3.

For all formulae A, if $\vdash_{\mathrm{X}}$ A then $\vdash_{\mathrm{FX}}$ A, for each sentential logic X.

Proof. It is a straight-forward procedure to check the derivability in FX of each axiom of X and to check the preservation of theoremhood in FX of each rule of X.

Theorem 4.
For all formulae A, if $\vdash_{\overline{FX}}$ A then $\vdash_{\overline{X}}$ A, for each sentential logic X.

Proof. For each logic X, other than B, the differences will be indicated between the proof of the theorem for it and the proof given for B in Theorem 2. Generally, each Thm rule of a quasi-proof of FX is modified so that theorems of X are inserted.

DW. The rule Contrap is removed as a rule in quasi-proofs. The case ~E is as follows:

~E. Let $\sim C_{a \cup b}$ be derived from $\sim D_a$ and $C \rightarrow D_b$ in Q, with the B-restriction holding. We need only consider $k \in a$ and $k \notin b$, for reasons given in the →E case for system B.

(i) Let $b = \phi$. Let $A \rightarrow \sim D_{a-\{k\}}$ and $C \rightarrow D_\phi$ of Q′ be derivable. By inserting $C \rightarrow D \rightarrow . \sim D \rightarrow \sim C_\phi$, and by →E, Pref and →E again, $A \rightarrow \sim C_{a-\{k\}}$ is derivable.

(ii) Let $b \neq \phi$. Let $A \rightarrow \sim D_\phi$ and $C \rightarrow D_b$ of Q′ be derivable. Again, by inserting $C \rightarrow D \rightarrow . \sim D \rightarrow \sim C_\phi$, $A \rightarrow \sim C_b$ is derivable.

DJ. As for DW except for →E, ~E and ∨E, where the DJ-restriction applies.

→E. Let $D_{a \cup b}$ be derived from C_a and $C \rightarrow D_b$ in Q, with the DJ-restriction holding. We only need consider $k \in a$ and $k \notin b$ for the same reasons as for system B.

(i) Let $b \neq \phi$. This case proceeds as for system B.

(ii) Let $b \neq \phi$ and $a = \{\max(a)\} = \{k\}$. This case also proceeds as for B.

(iii) Let $b \neq \phi$ and $a - \{\max(a)\} = b$. Let $A \rightarrow C_{a-\{k\}}$ and $C \rightarrow D_b$ of Q′ be derivable. By &I, $(A \rightarrow C) \,\&\, (C \rightarrow D)_b$ and, by inserting $(A \rightarrow C) \,\&\, (C \rightarrow D) \rightarrow . A \rightarrow D_\phi$, $A \rightarrow D_b$ is derivable, as required.

~E. This is similar to →E, but with insertion of $C \rightarrow D \rightarrow . \sim D \rightarrow \sim C_\phi$ (see ~E above for DW.).

∨E. Again, this is similar to →E, with insertion of $(C \rightarrow E) \,\&\, (D \rightarrow E) \rightarrow . C \vee D \rightarrow E_\phi$ (see ∨E for B.).

DK. As for DJ with the addition of the case ~I (from $A \rightarrow \sim A_\phi$, to infer $\sim A_\phi$). Since $k \notin \phi$, we need only the proof of $A \rightarrow \sim A \Rightarrow \sim A$, as a derived rule of DK. However, this is a standard proof using $A \vee \sim A$.

DL. As for DJ with addition of the case ~I (From $A \to {\sim}A_a$, to infer A_a).

~I. Let $\sim C_a$ be derived from $C \to {\sim}C_a$ in Q. If $k \notin a$, then ~I can be applied in Q′. If $k \in a$, let $A \to. C \to {\sim}C_{a-\{k\}}$ of Q′ be derived. By inserting the theorem $(A \to. C \to {\sim}C) \to. A \to {\sim}C_\phi$ and applying →E, $A \to {\sim}C_{a-\{k\}}$ is obtained.

TW. As for DW with the removal of the rules Pref and Suff, as well as Contrap, from the list of extra rules allowable for quasi-proofs, and with the following changes in the cases →E, ~E and ∨E to allow for the TW-restriction.

→E. Let $D_{a \cup b}$ be derived in Q from C_a and $C \to D_b$, with the TW-restriction holding. The TW-restriction narrows the cases for k down to $k \in a$ and $k \notin b$.

(i) Let $b = \phi$. Let $A \to C_{a-\{k\}}$ and $C \to D_\phi$ of Q′ be derivable. By inserting $C \to D \to. A \to C \to. A \to D_\phi$ and by applying →E twice, $A \to D_{a-\{k\}}$ is derivable.

(ii) Let $b \neq \phi$. Let $A \to C_{a-\{k\}}$ and $C \to D_b$ of Q′ be derivable. By the TW-restriction, $a \cap b = \phi$ and hence $(a - \{k\}) \cap b = \phi$.

Three subcases emerge:

(I) $a = \{k\}$. Let $A \to C_\phi$ and $C \to D_b$ of Q′ be derivable. By inserting $A \to C \to. C \to D \to. A \to D_\phi$ and by applying →E twice, $A \to D_b$ is derivable.

(II) $a - \{k\} \neq \phi$ and $\max(a - \{k\}) < \max(b)$. Let $A \to C_{a-\{k\}}$ and $C \to D_b$ of Q′ be derivable. By inserting $A \to C \to. C \to D \to. A \to D_\phi$ and applying →E, we obtain $C \to D \to. A \to D_{a-\{k\}}$. Since the TW-restriction is satisfied, we can apply →E to yield $A \to D_{(a \cup b)-\{k\}}$.

(III) $a - \{k\} \neq \phi$ and $\max(b) < \max(a - \{k\})$. As for (II), but with $C \to D \to. A \to C \to. A \to D_\phi$ inserted.

~E and ∨E are dealt with in the manner of →E, but $C \to D \to. {\sim}D \to {\sim}C_\phi$ and $(C \to E) \,\&\, (D \to E) \to. C \vee D \to E_\phi$, respectively, need inserting.

TJ. As for TW, but noting that (iii) of the TJ-restriction holds for the pair $(a - \{k\}, b)$, thus enabling (II) and (III) to go through, and noting that the following subcase (IV) needs to be added:

(IV) $a - \{k\} \neq \phi$ and $\max(a - \{k\}) = \max(b)$. Since (iii) of the TJ-restriction holds for $(a - \{k\}, b)$, $a - \{k\} = b$.

Thus, given $A \to C_{a-\{k\}}$ and $C \to D_b$ of Q′, $(A \to C) \& (C \to D)_b$ and, by inserting $(A \to C) \& (C \to D) \to . A \to D_\phi$, $A \to D_b$ are derivable.

TK. As for TJ, with the case ~I added. This case is dealt with as for DK.

T. As for TW, with the case ~I added (from $A \to {\sim}A_a$, to infer ${\sim}A_a$), and with changes in →E, ~E and ∨E corresponding to the T-restriction. We need consider only ~I and →E.

~I. For $k \in a$, $A \to {\sim}C_{a-\{k\}}$ is inferred from $A \to . C \to {\sim}C_{a-\{k\}}$ using the theorem, $(A \to . C \to {\sim}C) \to (A \to {\sim}C)_\phi$.

→E. For $k \in a$ and $k \notin b$, there are 4 subcases: (i) $b = \phi$, (ii) $b \neq \phi$ and $a = \{k\}$, (iii) $b \neq \phi$, $a - \{k\} \neq \phi$ and $\max(a - \{k\}) \leq \max(b)$, and (iv) $b \neq \phi$, $a - \{k\} \neq \phi$ and $\max(b) \leq \max(a - \{k\})$. These are dealt with similarly to the corresponding subcases for TW. However, there is a new case to be considered for T:

$k \in a$ and $k \in b$. Here, $k = \max(a) = \max(b)$. Let $A \to C_{a-\{k\}}$ and $A \to . C \to D_{b-\{k\}}$ of Q′ be derivable. There are 4 subcases: (i) $b = \{k\}$, (ii) $b \neq \{k\}$, $a = \{k\}$, (iii) $b \neq \{k\}$, $a \neq \{k\}$, $\max(a - \{k\} \leq \max(b - \{k\})$, and (iv) $b \neq \{k\}$, $a \neq \{k\}$, $\max(b - \{k\}) \leq \max(a - \{k\})$.

These subcases can be dealt with similarly to subcases (i) – (iv) of the above case for k, with the insertion of $A \to C \to . (A \to . C \to D) \to . A \to D_\phi$ or of $(A \to . C \to D) \to . A \to C \to . A \to D_\phi$ yielding $A \to D_{(a \cup b)-\{k\}}$ for each subcase.

Note that the unrestricted Reit rule is used, which is preferable to the Anderson and Belnap version in [2], as it allows more flexibility in the natural deduction proofs.

RW. As for TW, with →E and hence ~E and ∨E set out as follows:

→E. The case, $k \in a$ and $k \notin b$, can be simplified into one subcase since $(a - \{k\}) \cap b = \phi$, by the RW-restriction. There is an additional case:

$k \notin a$ and $k \in b$. Let C_a and $A \to . C \to D_{b-\{k\}}$ of Q′ be derivable. Insert $C \to . C \to D \to D_\phi$ and hence derive $C \to D \to D_a$.

Then, insert $(A \to . C \to D) \to . (C \to D \to D) \to . A \to D_\phi$ and obtain $A \to D_{a \cup (b-\{k\})}$, since $a \cap (b - \{k\}) = \phi$ by the RW-restriction.

Note also that $a \cup (b - \{k\}) = (a \cup b) - \{k\}$.

R. As for T, with →E and hence ~E and ∨E set out as follows:

→E. The cases, ($k \in a$ and $k \notin b$) and ($k \in a$ and $k \in b$) can be taken as appears for T, or they can be simplified into single subcases due to the lack of restriction on →E in FR.
There is an additional case:
$k \notin a$ and $k \in b$. This is dealt with as for RW above, without reference to any restrictions on →E.

RF. As for R, with &I and ∨E changed as follows:

&I. Let $C \& D_{a \cup b}$ be derived in Q from C_a and D_b, subject to ($a = \phi$ and $b = \phi$) or ($a \neq \phi$ and $b \neq \phi$ and $\max(a) = \max(b)$).

(i) $k \notin a$ and $k \notin b$. Apply &I in Q′.

(ii) $k \notin a$ and $k \in b$. Here, $\max(a) < \max(b)$, which is impossible.

(iii) $k \in a$ and $k \notin b$. $\max(b) < \max(a)$, which is also impossible.

(iv) $k \in a$ and $k \in b$. Hence, $k = \max(a) = \max(b)$. Let $A \to C_{a - \{k\}}$ and $A \to D_{b - \{k\}}$ of Q′ be derivable.
By inserting theorems, $A \to C \to. A \to C \& A_\phi$ and $A \to D \to. C \& A \to C \& D_\phi$, we obtain $A \to C \& A_{a - \{k\}}$ and $C \& A \to C \& D_{b - \{k\}}$. Hence, $A \to C \& D_{(a - \{k\}) \cup (b - \{k\})}$ by standard moves from FR. Note that $(a - \{k\}) \cup (b - \{k\}) = (a \cup b) - \{k\}$, as required.

∨E. Let $E_{a \cup b \cup c}$ be derived in Q from $C \vee D_a$, $C \to E_b$ and $D \to E_c$. There are 8 cases to consider.

(i) $k \notin a$, $k \notin b$ and $k \notin c$. Apply ∨E in Q′.

(ii) $k \in a$, $k \notin b$ and $k \notin c$. Let $A \to C \vee D_{a - \{k\}}$, $C \to E_b$ and $D \to E_c$ of Q′ be derivable. By inserting $C \to E \to. C \vee D \to E \vee D_\phi$ and $D \to E \to. E \vee D \to E_\phi$, we obtain $A \to E_{(a \cup b \cup c) - \{k\}}$.

(iii) $k \notin a$, $k \in b$ and $k \notin c$. Let $C \vee D_a$, $A \to. C \to E_{b - \{k\}}$ and $D \to E_c$ of Q′ be derivable. By insertion of $C \to E \to.$ $C \vee D \to E \vee D_\phi$, we obtain $A \to. C \vee D \to E \vee D_{b - \{k\}}$.
Further, by inserting $D \to E \to. E \vee D \to E_\phi$,
$C \vee D \to E \vee D \to. C \vee D \to E_c$ and hence $A \to.$
$C \vee D \to E_{(b \cup c) - \{k\}}$ is derivable. Hence, $C \vee D \to.$
$A \to E_{(b \cup c) - \{k\}}$ and $A \to E_{(a \cup b \cup c) - \{k\}}$.

(iv) $k \in a$, $k \in b$ and $k \notin c$. Let $A \to C \vee D_{a - \{k\}}$, $A \to.$ $C \to E_{b - \{k\}}$ and $D \to E_c$ of Q′ be derivable. By inserting

$C \to E \to . C \vee D \to E \vee D_{\phi}$, we obtain $A \to .$
$C \vee D \to E \vee D_{b-\{k\}}$ and hence $A \to E \vee D_{(a \cup b)-\{k\}}$. Further, by inserting $D \to E \to .\ E \vee D \to E_{\phi}$, we obtain $E \vee D \to E_c$ and hence $A \to E_{(a \cup b \cup c)-\{k\}}$.

(v) $k \notin a$, $k \notin b$ and $k \in c$. Similar to (iii).

(vi) $k \in a$, $k \notin b$ and $k \in c$. Similar to (iv).

(vii) $k \notin a$, $k \in b$ and $k \in c$. Let $C \vee D_a$, $A \to .\ C \to E_{b-\{k\}}$ and $A \to .\ D \to E_{c-\{k\}}$ of Q′ be derivable. First, note that $(A \to .\ C \to E) \to .\ (A \to .\ D \to E) \to .$
$A \to (C \to E)\ \&\ (D \to E)_{\phi}$ is a theorem of RF. Hence, $A \to (C \to E)\ \&\ (D \to E)_{(b \cup c)-\{k\}}$ and then $A \to .$ $C \vee D \to E_{(b \cup c)-\{k\}}$ are derivable.
Then since $C \vee D \to .\ A \to E_{(b \cup c)-\{k\}}$, $A \to E_{(a \cup b \cup c)-\{k\}}$ follows.

(viii) $k \in a$, $k \in b$ and $k \in c$. Let $A \to C \vee D_{a-\{k\}}$, $A \to .$ $C \to E_{b-\{k\}}$ and $A \to .\ D \to E_{c-\{k\}}$ of Q′ be derivable. Again, $A \to .\ C \vee D \to E_{(b \cup c)-\{k\}}$ is derivable, and this yields $A \to E_{(a \cup b \cup c)-\{k\}}$, as required.

We next consider the systems FX^d all of which have the rule $\vee E^r$ added. Consider first the system FB^d with $\vee E^r$ applied, not within the scope of any assumption, and with no applications of $\vee E^r$ used in establishing its rules, $A_{\phi} \Rightarrow C_{\phi}$ and $B_{\phi} \Rightarrow C_{\phi}$. We show that $\vee E^r$ preserves theoremhood in B^d. That is, we assume that $A \vee B$ is a theorem of B^d and that $A \Rightarrow C$ and $B \Rightarrow C$ are derived rules of B (also B^d).

To prove that C is a theorem of B^d, we show that each step D in the derivation of C from A can be replaced by $B \vee D$, and each step E in the derivation of C from B can be replaced by $C \vee E$. Thus, a proof of C in B^d can be obtained by showing that each step $B \vee D$ and $C \vee E$, above, is a theorem of B^d. Any theorem F used in the proof of C from A or from B, can be replaced by the theorem $B \vee F$ or $C \vee F$, respectively. It remains to check the rules of B.

(i) An application A', $A' \to B' \Rightarrow B'$ of R1, can be replaced by $B \vee A'$, $B \vee (A' \to B') \Rightarrow B \vee B'$ due to R5 of B^d.

(ii) An application A', $B' \Rightarrow A'\ \&\ B'$ of R2, can be replaced by $B \vee A'$, $B \vee B' \Rightarrow B \vee (A'\ \&\ B')$, as it is a form of distribution derivable using A8.

(iii) An application $A' \to B', C' \to D' \Rightarrow B' \to C' \to. A' \to D'$ of R3, can be replaced by $B \vee (A' \to B'),\ B \vee (C' \to D') \Rightarrow B \vee (B' \to C' \to. A' \to D')$ due to R6 of B^d.

(iv) An application $A' \to \sim B' \Rightarrow B' \to \sim A'$ of R4 is replacable by $B \vee (A' \to \sim B') \Rightarrow B \vee (B' \to \sim A')$ due to R7.

Similarly, '$C \vee$' can replace '$B \vee$' in (i) – (iv).

We next move to the general case where assumptions can lie inside other assumptions. In the same manner as above, each assumption A_i has a disjunct D_i which is added to the left of it and D_i is added to each step up to the conclusion C_i of A_i. So a step E inside the scope of assumptions $A_1, \ldots, A_n$ is replaced by a step $D_1 \vee \ldots \vee D_n \vee E$, where D_i is the disjunct added to the left of assumption A_i, for each i. So, if the rule: from $A \vee B$, $A \Rightarrow C$ and $B \Rightarrow C$, to infer C, is applied inside assumptions, $A_1, \ldots, A_n$, with corresponding disjuncts, $D_1, \ldots, D_n$, then it is replaced by the steps, $D_1 \vee \ldots \vee D_n \vee A \vee B$, $D_1 \vee \ldots \vee D_n \vee B \vee A$, ..., $D_1 \vee \ldots \vee D_n \vee B \vee C$, $D_1 \vee \ldots \vee D_n \vee C \vee B$, ..., $D_1 \vee \ldots \vee D_n \vee C \vee C$, and finally, $D_1 \vee \ldots \vee D_n \vee C$.

As each pair of assumptions are eliminated, so are their corresponding disjuncts, until finally all the disjuncts are removed. That theoremhood in B^d is maintained throughout such a proof can be seen from our simpler case above, where a single disjunct is replaced by a string of disjuncts. This completes the proof of Theorem 4 of B^d.

The remainder of the systems FV^d can be dealt with in the same way as FB^d with appropriate removal of rules R6 and R7 from the sentential logics X^d.

§ 4. *The Equivalence between the Quantificational Logics XQ and the Systems FXQ.*

We show that XQ and FXQ are equivalent, with X a sentential logic without disjunctive rules. Recall that the quantificational axioms and rule of XQ are the same for all such X, and that the quantificational natural deduction rules of FXQ are the same for all such X.

Theorem 5.
For all formulae A, if $\vdash_{XQ}$ A then $\vdash_{FXQ}$ A.

Proof. This is simply a matter of proving QA1 – 6 in FBQ and showing that QR1 is a derived rule of FBQ.

Theorem 6.
For all formulae A, if $\vdash_{FXQ}$ A then $\vdash_{XQ}$ A.

Proof. We consider each of the quantificational rules of FXQ, applied in an innermost subproof Q, to see if, when modified as described for Q′ of the proof of Theorem 2, they are derivable in FXQ, with the help of the additional rules for quasi-proofs.

$\forall I$. Let $(\forall x)C_a$ be derived from C_a in Q, with x not free in any hypothesis $H_{\{i\}}$, where $i \in a$. If $k \notin a$, ∀I can be applied in Q′. Let $k \in a$ and $A \to C_{a-\{k\}}$ of Q′ be derivable. x is not free in any hypothesis $H_{\{i\}}$, where $i \in a - \{k\}$. Hence, by ∀I, $(\forall x)(A \to C)_{a-\{k\}}$ is derivable. Note that x is not free in A since $A_{\{k\}}$ is a hypothesis and $k \in a$. Then we can insert the theorem $(\forall x)(A \to C) \to .\ A \to (\forall x)C_\phi$, and obtain $A \to (\forall x)C_{a-\{k\}}$, as required.

$\forall E$. Let C^y/x_a be derived from $(\forall x)C_a$ in Q, where y is free for x in C. If $k \notin a$ then ∀E can be applied in Q′. Let $k \in a$ and $A \to (\forall x)C_{a-\{k\}}$ of Q′ be derivable. Then insert the theorem $A \to (\forall x)C \to .\ A \to C^y/x_\phi$ and obtain $A \to C^y/x_{a-\{k\}}$, as required.

$\forall\vee$. Similar to ∀E, using the theorem, $A \to (\forall x)(C \vee D) \to .\ A \to C \vee (\forall x)D$, where x is not free in C.

$\exists I$. Similar to ∀E, using the theorem, $A \to C^y/x \to .\ A \to (\exists x)C$, where y is free for x in C.

$\exists E$. Let $D_{a \cup b}$ be derived from $(\exists x)C_a$ and $(\forall x)(C \to D)_b$ in Q, where x is not free in D and (a, b) satisfies the same restriction as for →E and ∨E in the system FX.

(a) $k \notin a$ and $k \notin b$. Apply ∃E in Q′.

(b) $k \notin a$ and $k \in b$. Let $(\exists x)C_a$ and $A \to (\forall x)(C \to D)_{b-\{k\}}$ of Q′ be derivable. This case only arises for systems FRWQ, FRW^{d}Q, FRQ and FRFQ. In which case, we have the theorem $(\exists x)C \to .\ (\exists x)C \to D \to D_\phi$, and hence $(\exists x)C \to D \to D_a$ is derivable. Insert $A \to (\forall x)(C \to D) \to .\ A \to .\ (\exists x)C \to D_\phi$ and obtain $A \to .\ (\exists x)C \to D_{b-\{k\}}$. For FRWQ and FRW^{d}Q, $a \cap b = \phi$ and hence $a \cap (b - \{k\}) = \phi$. So →E can be applied to yield $A \to D_{(a \cup b)-\{k\}}$.

(c) $k \in a$ and $k \notin b$. Let $A \to (\exists x)C_{a-\{k\}}$ and $(\forall x)(C \to D)_b$ of Q′ be derivable. Insert $(\forall x)(C \to D) \to .\ (\exists x)C \to D_\phi$, and obtain $(\exists x)C \to D_b$. Then $A \to D_{(a \cup b)-\{k\}}$ is derivable by following the procedure given in the $\to$E case for each of the natural deduction systems.

(d) $k \in a$ and $k \in b$. Let $A \to (\exists x)C_{a-\{k\}}$ and $A \to (\forall x)(C \to D)_{b-\{k\}}$ of Q′ be derivable. This case only appears for systems FTQ, FRQ and FRFQ. In which case we can insert the theorems, $(A \to .\ (\exists x)C \to D) \to .\ A \to (\exists x)C \to .\ A \to D_\phi$ or $A \to (\exists x)C \to .\ (A \to .\ (\exists x)C \to D) \to .\ A \to D_\phi$, as required to obtain $A \to D_{(a \cup b)-\{k\}}$. The procedure is given under $\to$E for FT, a simplification of which applies for FR and FRF.

$\exists\&$. Similar to $\forall$E, using the theorem, $A \to C\ \&\ (\exists x)D \to .\ A \to (\exists x)(C\ \&\ D)$, where x is not free in C.

It is easy to check that each of the quantificational natural deduction rules are derived rules of BQ, when considered outside the scope of any subproofs.

We now show that X^dQ and FX^dQ are equivalent for any sentential logic X^d with disjunctive rules.

Theorem 7.

For all formulae A, if $\vdash_{\overline{X}}{}^d{}_Q A$ then $\vdash_{\overline{FX}}{}^d{}_Q A$.

Proof. We essentially need to show that any derived rule produced by a meta-rule is a derived rule of FB^dQ. For derived rules produced by MR1, this is shown using $\vee E^r$, and for derived rules produced by QMR1, we use $\exists E^r$.

Theorem 8.

For all formulae A, if $\vdash_{\overline{FX}}{}^d{}_Q A$ then $\vdash_{\overline{X}}{}^d{}_Q A$.

Proof. We need just consider the rules $\vee E^r$ and $\exists E^r$ to see that they are derivable in B^dQ. This is less complex than in the proof of Theorem 6 because the employment of MR1 and QMR1 in proofs in B^dQ is similar to the employment of $\vee E^r$ and $\exists E^r$ in FX^dQ, and also the provisos on both types of rules are essentially the same. In fact, we use MR1 to derive $\vee E^r$ in B^dQ, given that $A_\phi \Rightarrow C_\phi$ and $B_\phi \Rightarrow C_\phi$

are expressible as derived rules of B^dQ with no generalization on any free variable of A or B, respectively, Also, QMR1 is similarly used to derive $\exists E^r$ in B^dQ.

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