"KRIPKE SEMANTICS" = ALGEBRA + POETRY

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Résumé

L'équivalence (bien connue) entre la sémantique des calculs propositionnels non-classiques qui est basée sur des interprétations philosophiques et celle, plus sobre et plus ancienne, qui ne fait intervenir que les structures algébriques sous-jacentes, est explicitée et expliquée par la spécification d'une application démunissante qui réduit tout "modèle de Kripke" à un sous-modèle du "modèle" des filtres premiers de l'algèbre (pseudo-) Booléenne des classes de formules interdéductibles.

The tenor of the argument should be sufficiently clear from the title. The thesis is basically that the algebra is already quite well known, while the additional poetry only obscures and unnecessarily complicates the essential aspects of the situation as they will be revealed to those incisive enough to forego the imaginative sugarcoating.

The relevant algebraic analysis often enough appears in the literature (more often for the modal logics, where the facts are closer to the surface, than for the intuitionistic) still enmeshed in the symbolism of the initially presented formal systems and doused more or less liberally with "possible world semantics" sauce; there are even treatments which show explicitly that algebraic validation comes to the same as that using "Kripke models"; what has apparently not been carried out is a structural, in contrast to a functional, comparison: one which relates the individual constructs in the two approaches to each other. This is realized below by means of a canonically defined map which sends the "possible worlds" of any "Kripke model" onto a special kind of subset in the Stone space of the Lindenbaum algebra of the logic: a subset of filters which mirrors set theoretically the formal validation procedure in the model (and thus explains the functional adequacy of the algebra). The details are summarized under the rubric "Das Kripke-Modell an sich" at the end of the discussions which take up in order the types of systems treated in the three cited Kripke references. Only propositional logics have been investigated for lack of time; it may be possible to return to the predicate logics on another occasion.

Some brief comments on this literature before getting down to work. Already in his JSL review of the first Kripke paper, D. Kaplan pointed out that in place of the plethora of "models" one can make do with a single structure: the "complete, deductively consistent subsets" of the formal system (algebraists read: ultrafilters of the Boolean algebra i.e.: Stone space) equipped with the appropriate relation. This "canonical model" has been popular in subsequent expositions (Lemmon, [HC chap. 17], Segerberg, Cresswell); it is presented in Boolean form by Ursini; its naturalness may be guaged by its having been recently reconstituted independently by Loparic; and, as Lemmon p. 42ff has pointed out, the advantages (such as decidability) that the finite models may offer are to be had by reducing it with a suitable equivalence (a technique which has since taken on the rather awesome title "filtration"). As for explicitly reducing modal "Kripke models" to such relation-equipped subsets of the Stone space, constituents of the map to be presented below are partially foreshadowed in [HC pp. 326-328] for the special case of the T-system and (a more essential restriction) for finite models and algebras – but marred by the apparent confusion of construing the range of the map as the algebra; in Segerberg p. 30 where just the kernel of the map puts in an appearance as that of an inessential contraction of models – but note that the congruence assertion is in default (specifically neither the "It should be clear" on p. 327 of the former nor the first "It is clear" on p. 30 of the latter should be clear to the reader); and most completely in Lemmon's JSL articles where Theorem 15 of I goes part of the way by associating to every "model structure" an operator-equipped Boolean algebra of its subsets: then by combining this with his proof of Theorem 32 of II (which is only used to show that every algebra is a subalgebra of one of these) one would effect the reduction as far as a relation-equipped subset in the dual of *some* Boolean algebra.

This is also as far as the reduction is carried by the intuitionistic literature. The adequacy of the algebraic analysis for intuitionistic logic goes back to Stone 1937 (a textbook treatment is in [RS]). In developing the "Equivalence of algebraic and Kripke validity" (i.e. that the same formulae are universally validated by both semantics) Fitting, ch. 1 § 6 pp. 23-27, produces for every "Kripke model" a homomorphism to a pseudo-Boolean algebra which effects validation of the same formulae (following a suggested exercise in Beth's book; see below for a more streamlined proof) and for every such homomorphism a like - validating "Kripke model" on the prime filters of the image (by combining it with Stone's 1937 representation whose proof he retraces) – the same correspondences are developed by Hermes § 29 pp. 170-172. The composition of the maps between underlying sets induced by these correspondences is almost the canonical map below which demystifies intuitionistic "Kripke models": the final link, which ties down the image as a subset of the "canonical" intuitionistic model furnished by Stone, is provided by composing further with the dual of the homomorphism from the algebra of (interdeducible) formulae.

Closest (certainly in spirit) to the present enterprise is Anderson's development of the "contracting" of intuitionistic "Kripke models". Formally he presents this as a reduction to a submodel with retention of the original order, but it could equally be construed as a retraction, in line with what is done here; however, he restricts himself to tree models and whereas his treatment is optimal for these, admitting more general partially ordered sets can result in further "contracting", as will be seen below. The latter part of the section on intuitionistic logic analyses Anderson's work within the present framework.

Intuitionistic propositional logic:

A "Kripke model" for this logic is taken to be a set W (the so-called "possible worlds") equipped with an internal binary relation R (of "possible succession") which is to be reflexive and transitive (i.e. a preorder) and with an external relation V, between its elements $w \in W$ and the usual propositional formulae ϕ, of "valuation" or "validation" satisfying

(P0) If $V(w, \phi)$ then $V(w', \phi)$ for every $w'Rw$

(P1) $V(w, \phi \wedge \psi)$ just when both $V(w, \phi)$ and $V(w, \psi)$

(P2) $V(w, \phi \vee \psi)$ just when at least one of $V(w, \phi)$ and $V(w, \psi)$
(P3) $V(w, \sim\phi)$ just when not $V(w, \phi)$ and (P0) is not violated
(P4) $V(w, \phi \to \psi)$ just when either not $V(w, \phi)$ or $V(w, \psi)$, and (P0) is not violated.

This is just the list on p.20 of Fitting (as taken directly from Kripke) except that (P0) has been strengthened by including explicitly the consequence Theorem 4.4 proved on p. 22, while (P3) and (P4) have been reformulated by virtue of the so strengthened (P0).

For every formula ϕ let $S(\phi) \subset W$ be the subset of all $w \in W$ for which $V(w, \phi)$ – the subset of those "worlds" in which ϕ is valid. The content of (P0) – that with any $w \in S(\phi)$ every $w'Rw$ is also in $S(\phi)$ – will be abbreviated by saying that $S(\phi)$ is an *upper subset*. Observe that the upper subsets of W are closed for arbitrary (ordinary set-theoretic) union and intersection i.e. form a complete sublattice of the lattice of subsets of W, hence a complete, completely join distributive lattice and thus in particular (e.g. Birkhoff p. 128 Theorem 24) a bounded relatively pseudocomplemented lattice (a concept found under various aliases e.g. as "pseudo-Boolean algebra" in [RS]). Now P1-P4 come to: $S(\phi \wedge \psi) = S(\phi) \cap S(\psi)$, $S(\phi \vee \psi) = S(\phi) \cup S(\psi)$, $S(\sim \phi)$ is the largest upper subset disjoint from $S(\phi)$, and $S(\phi \to \psi)$ is the largest upper subset contained in $[W\text{-}S(\phi)] \cup S(\psi)$ – i.e. they come to that the mapping S from the propositional formulae to the upper subsets is a homomorphism when the formulae are construed as an algebra for the operations $\wedge, \vee, \sim, \to$ (it is in fact the absolutely free algebra for these operations on the propositional variables, or atomic formulae, as generators) to the upper subsets of W, under the corresponding operations of intersection, union, pseudo- and relative pseudo-complement. But since the image is a pseudo-Boolean algebra, this homomorphism factors into the quotient map modulo the pseudo-Boolean identities (i.e. the intuitionistic propositional tautologies [RS] p. 382) whose image is the largest quotient pseudo-Boolean algebra (in fact the free one on the propositional variables as generators) followed by a pseudo-Boolean homomorphism to the upper subsets of W. It is even possible to pin down the "possible worlds" as sublunary entities: indeed since pseudo-Boolean (i.e. intuitionistically) equivalent formulae are mapped by S on the same upper subset, a fortiori $V(w, \ )$ for any individual w will not distinguish such formulae. Now for fixed w, the totality of (classes of equivalent) formulae for which $V(w, \phi)$ holds is, according to (P1), a filter which according to (P2) is prime and to (P3) proper. Conversely if V is defined to hold at every w just for the formulae (in the inverse image under a homomorphism to a pseudo-Boolean algebra) of a proper prime filter, then it verifies (P1) and (P2). As for (P0), it just expresses the order preserving character of the map from the preordered set W into the inclusion ordered proper prime filters of the pseudo-Boolean algebra of intuitionistically equivalent classes of formulae. To have also (P3) and (P4) it must be the case that a $\sim\phi$ (or a $\phi \to \psi$) fails to be in the image of a w only if ϕ is (and ψ – or equivalently $\phi \to \psi$ – fails to be) in the image of some w′Rw. Since V is order preserving, its image is therefore a set of prime filters such that $\sim\phi$, resp. $\phi \to \psi$, fails to belong to one of them only when ϕ belongs resp. and ψ does not, to a larger one in the set. Calling for convenience "$\to$-justifying" any subset (in the dual of the free pseudo-Boolean algebra) which satisfies this, we arrive at last at:

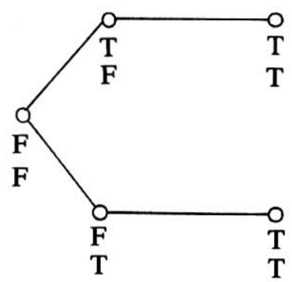
Das Kripke – Modell an sich:

A Kripke model (for intuitionistic propositional logic) is a preordered set equipped with an order preserving map onto an $\to$-justifying set of proper prime filters in the pseudo-Boolean algebra of equivalence classes of formulae. Its preorder, besides being contained in the inverse image of the inclusion order, must like it provide for every $\sim\phi$ (or $\phi \to \psi$) not in the image of a w, a w′Rw whose image contains ϕ (and excludes ψ).

The auxiliary preordered set W of "worlds" is thus a dispensable elaboration in that the same formulae would be validated directly by the inclusion ordered set of the prime filters designated by, i.e. which occur in the image of, the validation map V. In fact, the set of *all* proper prime filters is $\to$-justifying: if $\phi \to \psi$ does not belong to some filter F i.e. dominates no element of it, then the filter generated by F and ϕ does not contain ψ and so is contained in a filter maximal for excluding ψ, hence prime. Thus since the intersection of the prime filters in a bounded distributive lattice is its unit – e.g. [RS] I9.2 p. 49 – every non-tautology ψ will be excluded by some prime filter, whence the subset of all its prime overfilters will be a "Kripke counter-model" for ψ. This can be made "minimal ψ-rejecting" (Anderson p. 264) by starting with a prime filter maximal for excluding ψ.

But to exhibit ψ as a non-tautology it is not necessary to take all the prime overfilters of a prime filter excluding it – any $\rightarrow$-justifying set would do. It is a fact – [RS] IV9.3 p. 141 – that every finite subset of a pseudo-Boolean algebra can be completed to a finite algebra; hence – [RS] IX3.2 p. 386 – that any non-tautology can be mapped, along with its non-tautologous subformulae, homomorphically into the non-units of a finite pseudo-Boolean algebra; and this can be extended by freeness to a homomorphism from all the formulae, whence the dual map will order inject the finite number of proper prime filters of the image onto a *finite* Kripke counter-model. (A detailed syntactic proof of the special case takes up Fitting's chapter 2.)

Kripke has shown how every (finite) model can formally be construed as a (finite) tree model with essentially the same validation map V. For tree models Anderson has furnished a contraction process which produces an uncontractable normal form *tree*. This process does not get down to the proper prime filters in the image of V: here is an Anderson tree whose endpoints are identified by V



Such a subtree must occur in any tree counter-model for the two-variable formula $[(\sim\sim x) \wedge (\sim\sim y)] \rightarrow [(x \rightarrow y) \vee (y \rightarrow x)]$: this formula could not have a *tree* counter-model in the poset of prime filters.

An "n-assignment Kripke model" is a finite poset K with smallest element (actually in Anderson p. 262 more particularly a tree with root) g, with the further designation of an n-tuple of upper subsets in it; two such are "equivalent", loc. cit. p. 266, if whenever an n-tuple of propositional variables is validated with the respectively designated n-tuples of upper subsets, the same formulae in these variables become valid at the respective g. Here the validation assignment for n-variable formulae at nodes of K is made by recursion on the (absolutely) free generation of formulae starting from the n propositional variables using (P1) to (P4). It follows as above that this is a homomorphism from the n-variable formulae to the pseudo-Boolean algebra of the upper subsets of K generated by the designated n upper subsets, which thus factors via the free pseudo-Boolean algebra on n generators.

The dual map sends the nodes of K in order-preserving fashion on a finite(1) set of proper prime filters of the free pseudo-Boolean algebra on n generators (each node going to the classes of equivalent formulae valid at it) and the image is, by the Stone representation theorem, a model "canonical" for this n-generated presentation of the algebra of upper subsets. But a priori, the same prime filter of classes valid at g could be embedded in different $\rightarrow$-justifying sets of filters in the dual of the free n-generated algebra, and this would furnish distinct such canonical models "equivalent" in the above sense. However, if we define a stronger form of equivalence, between (not necessarily finite or tree) models, whereby for every submodel (as defined on p. 263) of the one there is a submodel of the other which is equivalent to it in the preceding sense – then every model again becomes equivalent to one built with an $\rightarrow$-justifying subset of the proper prime filters in the free pseudo-Boolean algebra on n generators; but now the models for distinct such subsets are never equivalent and this yields a normal form – alternatively one may show that "contracted" models equivalent in this sense are isomorphic.

Modal propositional logic

A modal propositional logic in the most general sense is obtained by augmenting the classical propositional connectives with an additional unary operator $\Box$, which is used along with the classical ones to generate the modal formulae. The most common versions include the classical propositional tautologies and rules, whence on the quotient of these formulae modulo interdeducibility the classical connectives still impose a Boolean structure; if one wishes to take account also of

(1) Recall that the free pseudo-Boolean algebra on even one generator is infinite, whence must have infinitely many prime filters: the homomorphism to the upper subsets of the finite K is certainly not injective.

$\Box$ at this quotient Boolean level, one will want it to be well defined as a unary operator on this algebra of equivalence classes of formulae. ([2]) The latter comes to having $p \leftrightarrow q$ yield $\Box p \leftrightarrow \Box q$ back in the logic. This is the "rule RE (replacement of material equivalents)" whose presence singles out the systems called "classical" (Lemmon p. 18, Segerberg p. 6) and ensures the substitutivy of interdeducibility $\leftrightarrow$ in the modal (as well as in each propositional) connective. The resulting quotient structure is a Boolean algebra equipped with a unary operator (which it will not harm to denote with the same symbol $\Box$ as the logical operator which induces it).

Although they are not Kripke-style, we pause to dispose of the "semantics" for this type of system. Every function $\Box$ from A′ to A may be described by the family $\Box^{-1}(a)$ of inverse images (whose non-void members partition A′). If the A's are Boolean algebras they may be represented as algebras of subsets on their respective spaces of ultrafilters and this description then becomes an assignment $\Box^{-1}$ to every clopen subset C in U of a set of clopens in U′. This may in turn be specified as an assignment to each $u \varepsilon U$ of the various sets of clopens in U′ assigned to the different clopens C containing u: $\Box^{-1}C$ may be recovered as the set common to all those assigned to the $u \varepsilon C$. With $A' = A$, this is the "Fundamental Theorem for Neighborhood Semantics": the $\Box^{-1}C$ assigned to u are (mis)called the "neighborhoods" of u, although it is the C which are its neighborhoods in the Stone topology.

The above rule RE, whose content is that $\Box$ is well defined as a unary operator on the Boolean algebra (of classes of interdeducible formulae), follows from having $p \rightarrow q$ yeld $\Box p \rightarrow \Box q$, whose content is the stronger requirement that it be well defined as an order-preserving unary operator. Lemmon calls this the "rule RM", Segerberg "the rule RR (regularity)" but requires for a system to be "regular" that it also provide the axiom (scheme) $\Box p \wedge \Box q \rightarrow \Box(p \wedge q)$ whose (additio-

([2]) This is not the only way Boolean algebra can be used however. There is an ingenious proof for the decidability of the modal system S2, which does not satisfy this requirement, by McKinsey: he divides out not by the equivalence of full interdeducibility, but by the weaker one of strict interdeducibility, which leaves him with a Boolean algebra in which the tautologies still constitute a proper filter; and then performs a reduction to a finite subsystem of this algebraic-logical hybird in which a given non-tautology is still outside the filter.

nal) content is that $\Box$ preserve not only order but $\wedge$. This entails $\Box p \wedge \Box(p \to q) = \Box[p \wedge (p \to q)] \leq \Box q$ i.e. $\Box(p \to q) \leq \Box p \to \Box q$, the Boolean translation of Axiom A6 in [HC] and A4 in Lemmon; conversely an order-preserving $\Box$ satisfying this preserves $\wedge$: since $p \leq p \wedge q \to q$ one has $\Box p \leq \Box(p \wedge q) \to \Box q$ (also proved as T3 in [HC] p. 34).

The missing order-preservation becomes available when this axiom is combined with the "rule RN of necessitation" which authorizes the inference from p to $\Box p$; its content, that $\Box$ preserve tautologies (by RE it would suffice for it to send a single tautology, and by RM even any formula, on a tautology) becomes, in the Boolean quotient algebra, preservation of the unit 1.

These "normal" systems are commonly axiomatized by using just this A6 and necessitation as the modal supplement to the standard classical formalism, rather than by requiring explicitly that $\Box$ act on the Boolean equivalence classes as an $\wedge$-monoid endomorphism. One then obtains the most widely studied systems by further specialization (Cf [HC]): system T by adding the requirement that $\Box$ be decreasing (more formally by subjoining axiom A5: $\Box p \to p$); system S4 that it also be idempotent (more formally, the further axiom A7: $\Box p \to \Box\Box p$) or equivalently, induce the identity on its image; and system S5 that also this image be closed under negation (more formally axiom A8: $\sim\Box p \to \Box \sim \Box p$).

A "Kripke model" (for a normal system) is again a set W equipped with an internal binary relation R and an external relation V, between its elements and now the modal propositional formulae, satisfying

(P1) $V(w, \varphi \wedge \psi)$ just when both $V(w, \varphi)$ and $V(w, \psi)$
(P3) $V(w, \sim\varphi)$ just when not $V(w, \varphi)$
(P5) $V(w, \Box\varphi)$ just when $V(w', \varphi)$ for every $w'Rw$.

The analysis here is even easier than it was before. (P1) and (P3) just say that V assigns to every $w \in W$ a homomorphism, as regards only the combination of formulae by these classical connectives, to the two-element Boolean algebra: thus an ultrafilter in the quotient algebra of formulae modulo the Boolean identities. (Observe that (P5) in conjunction with (P1) shows that V also does not distinguish $\Box(\varphi \wedge \psi)$ from $\Box\varphi \wedge \Box\psi$, and since 1 belongs to all ultrafilters this modelling could only be appropriate for normal systems.) Thus, without changing the sets of formulae validated at elements of the model (which will continue to be those in the corresponding ultrafilters) we may again substitute for the possible worlds W their image in the Stone space of ultrafilters; and then need only equip this image with a binary relation satisfying (P5) in order to make it a model. The strongest possibility would be to relate u′ to u just when $\varphi \varepsilon u'$ whenever $\Box\varphi \varepsilon u$: thus would be the relation $u' \supset \Box^{-1}u$. This makes the map from worlds to ultrafilters a relational homomorphism – w′Rw implies $u' \supset \Box^{-1}u$ for the image ultrafilters – and under any such, $\bigcap\{u' : w'Rw\}$ remains at least as small, whence its coincidence with $\Box^{-1}u$, which is the content of (P5), will be maintained: therefore this does make the image a model. Calling for convenience "$\Box$-justifying" any subset of the Stone space for which every $\Box^{-1}$ of one of its u is the intersection of ultrafilters from the subset, we have identified *Das Kripke-Modell an sich:*

A Kripke model (for a normal modal propositional logic) is a set W equipped both with a map onto a $\Box$-justifying subset of the Stone space of ultrafilters and with any relation contained in the inverse image of $u' \supset \Box^{-1}u$ which (like it) includes, for every $\Box\varphi$ not in the image of a w, a pair (w′, w) with φ not in the image of w′.

It remains to remark that the Stone space of all ultrafilters is $\Box$-justifying(3): for since $\Box$ is $\wedge$-preserving, $\Box^{-1}u$ is a filter and is therefore the intersection of the ultrafilters containing it.

In general this map from the a priori given model of "worlds" to the canonically determined Stone space of ultrafilters can only be expected to be a relational morphism; the more restrictive "p -" or "frame morphisms" referred to e.g. Segerberg p. 37, Goldblatt p. 53 – i.e. those relational morphisms which also send the section of the domain relation at each element onto (rather than just into) the section of the coldomain relation at its image: thus here which send the w′Rw onto the $u' \supset \Box^{-1}u$ whenever w is sent on u – will be obtained when R is the strongest relation satisfying P5 on W, thus the strongest for which $\Box\phi$ fails to be validated at any w only if ϕ fails to be validated at some w′Rw: i.e. when R fails to relate w only to those w′ at which some ϕ, whose $\Box$ is validated at w, fails to be validated; and also when every filterbase of validation subsets of W has non-void intersection. Indeed, letting f be the map from worlds to the Stone space, from

(3) This is the first Remark of Ursini.

$\Box^{-1}fw \subset u'$ follows $f^{-1}\Box^{-1}fw \subset f^{-1}u' \subset$ some principal filter w' by the latter condition, which inclusion entails $w'Rw$ by the former; and $f^{-1}u' \subset w$ ensures $u' = fw'$ by separatedness of the Stone space. This is proved by Goldblatt in the course of his 13.5 Theorem under superfluous supplements: that the space W be separated by the validation subsets (Axiom I p. 64) and that the map is to the Stone subspace dual to the quotient algebra obtained by reducing modulo the universally validated formulae – this has the effect of making f a surjection.

These "descriptive frames" of Goldblatt are of course nothing other than Stone represented modal algebras with the above "canonical" relation on the $\Box$-justifying set of all ultrafilters. Van Benthem has noted that every relation-equipped set W is embedded in relation-preserving fashion as the principal ultrafilters in the Stone representation space of all ultrafilters on the power set of W with the modal structure derived from the initial relation.

We pass quickly through the more specialized normal systems. If $\Box p \leq p$ then $u \supset \Box^{-1}u$ so that R may be taken reflexive; if $\Box\Box p = \Box p$ then $u' \supset \Box^{-1}u$ is transitive whence this may be imposed on R; if both hold the relation is the same as $\Box^{-1}u' \supset \Box^{-1}u$ or, since ultrafilters are sent by $\sim$ on complements, $\Box^{-1}\sim u' \subset \Box^{-1}\sim u$, and now A8 $\Box^{-1}\sim u \subset \Box^{-1}\sim\Box^{-1}u \subset \Box^{-1}\sim u'$, shows it symmetric.

Finally, we drop the "rule of necessitation" RN but retain the remainder of the axiomatic base, i.e. we treat Segerberg's "regular systems": algebraically, $\Box$ remains an $\wedge$-endomorphism on the Boolean algebra but may now send 1 on some element $\Box 1 \neq 1$. There will in this event be ultrafilters which do not contain $\Box 1$, hence by order-preservation of $\Box$, no $\Box p$: i.e. for which $\Box^{-1}u$ is void – and these cannot be excluded from consideration since the intersection of the ultrafilters in their complement is the non-trivial principal filter generated by $\Box 1$, for no element of which could they produce a counter-model. The Kripke semantics way around this is to exempt these ultrafilters from the requirement (P5) so as to admit "non-normal(4) worlds in which no proposition is necessary" as elements of W: the exclusions of the $\Box\varphi$ from the ultrafilters that these w determine – by virtue of the still valid clauses (P1) and (P3) – will not

(4) Segerberg's term is "singular".

need to be justified by a (P5) appeal to w'Rw. Thus there need not be any w' related to these w; on the other hand, one might need the presence of such an ultrafilter to exclude a φ from a non-void $\Box^{-1}u$ – e.g. if φ were $\Box 1$ – and so one will allow these non-normal worlds to function as w' in w'Rw – although by the above this need be done only for "normal" w. (The further requirement made in [HC] p. 275/6 that every non-normal world does function in this way, while it may be imposed for the system S2 discussed there, need not hold e.g. for even the slightly weaker E2 where for $\Box 1 = 0$ there are no models with normal worlds).

Another possibility would be to allow a single void "world" related just to the ultrafilters not containing $\Box 1$. This would allow retaining (P5) at the cost of modifying (P3) to hold with this one exception.

With what kind of poetic fancy might one imbue such a void "possible world"? It is a world in which neither a formula nor its negation are valid, in which none of our usual distinctions can be drawn: a world of total uniformity. Physics has described such a world for us: the "heat death" or state of maximum entropy towards which the universe is running down. The worlds in which no propositions are necessary, thus in which everything is possible - one might call them "chaotic" or "lawless" - would then be those one step away from extinction.

Das Kripke-Modell an sich may be formulated for these non-normal (regular) logics as above for the normal ones: under the first alternative "$\Box$-justifying", in the ultrafilter space ordered by $u' \supset \Box^{-1}u \neq \phi$, should be taken to apply only to the subset of ultrafilters which meet the image of $\Box$ and the condition on R only to the "normal" worlds w i.e. for which (P5) is in force; under the second, one may take over the previous wording and order after augmenting the ultrafilter space with the new element ϕ and let this element serve as image for any "entropic" element in the model.

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