PARACONSISTENCY, PARACOMPLETENESS, AND VALUATIONS

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1. *Introduction*

A theory $\mathscr{T}$ is called inconsistent if among its theorems there are at least two, one of which is the negation of the other. When this is not the case; $\mathscr{T}$ is said to be consistent. We call $\mathscr{T}$ trivial when all formulas (or all closed formulas) of its language are also theorems of $\mathscr{T}$. If there is at least one formula (or closed formula) of the language of $\mathscr{T}$ that is not a theorem of $\mathscr{T}$, then $\mathscr{T}$ is said to be nontrivial.

A paraconsistent logic is a logic (a logical calculus or simply a calculus) which can be used in the systematization of inconsistent but nontrivial theories. These 'paraconsistent' theories, therefore, may contain inconsistencies (contradictions), i.e., pairs of theorems such that one is the negation of the other, without being trivial. Obviously, most of the extant systems of logic, such as the systems of classical logic, are not paraconsistent(in relation to paraconsistent logic, its applications, and its philosophy, see [1] and [3]).

Every logical system whatever has a two-valued semantics of valuations, which constitutes a generalization of the standard semantics (cf. [8] and the next section of this paper). Taking this fact into account, we can define precisely the notion of paraconsistent logic; a logic is paraconsistent if it can be the underlying logic of theories containing contradictory theorems which are both true. Such theories we call paraconsistent.

Similarly, we define the concept of paracomplete logic: a logical system is paracomplete if it can function as the underlying logic of theories in which there are (closed) formulas such that these formulas and their negations are simultaneously false. We call such theories paracomplete.

As a consequence, paraconsistent theories do not satisfy the principle of contradiction (or of non-contradiction), which can be stated as follows: from two contradictory propositions, i.e., one of which is the negation of the other, one must be false. Moreover, paracomplete theories do not satisfy the principle of the excluded middle, formulated in the following form: from two contradictory propositions, one must be true.

The objective of the present paper is twofold: 1) to develop a propositional system of logic at the same time paraconsistent and paracomplete, which, in a certain sense, contains the classical propositional logic; 2) to illustrate how the method of valuations is convenient for the better understanding of a logical system (some applications of the semantics of valuations may be found in [2], [4], [10] and [16]).

The system studied in this paper may be seen as a kind of logic of vagueness (in the sense of [6] and perhaps also in the sense of [7]; it also constitutes an alternative to the dialectical logic in the sense of the logic DL of da Costa and Wolf (see [5] and [6]), which formalizes some views of McGill and Parry ([12]). Moreover, our system presents some connections with the dialectical logic DK of Routley and Meyer (cfr. [13]). One of the consequences of these parallels is that the main ideas of dialectics seem to be really vague, not being susceptible to unique characterization by a formal system.

Nonetheless, we do not intend here to explore the possible applications of our system, from the philosophical point of view or otherwise. Our aim is merely a technical one: to emphasize the relevance of the semantics of valuations, showing that with the help of such semantics one can, in various cases, obtain decision methods which really constitute a natural extension of the common two-valued truth-table decision method of the classical propositional calculus. In fact, we conjecture that any calculus that is decidable at all, is decidable by the corresponding semantics of valuations, i.e., more precisely, by the device which we shall name 'valuation tableaux' (for these tableaux see, for example, [4], [9], [10] and [11]).

2. *The semantics of valuations*

A (logical) calculus $\mathscr{C}$ is an ordered pair $<\Delta, \mathscr{R}>$, in which Δ is a set of formulas of a given language $\mathscr{L}$, and $\mathscr{R}$ is a collection of inference rules; Δ and $\mathscr{R}$ are supposed to be nonempty, and Δ is called the set of axioms of $\mathscr{C}$. In this paper, $\mathscr{L}$ will a propositional language. In general, we assume that $\mathscr{L}$ contains propositional variables (normally a denumerable infinite set of such variables), parentheses and connectives. The connectives are supposed to have finite ranks (although this point is not essential to our discussion). Usually, the language contains connectives for implication ($\rightarrow$), conjunction (&), disjunction ($\vee$) and negation ($\urcorner$); there may also be intensional ones, like the modal and deontic connectives. The concept of formula is introduced as usual. Capitan Roman letters will stand for formulas; sets of formulas will be denoted by capital Greek letters. The notion of inference rule could be made precise, but this is not essential to our objectives here. We recall only that a rule relates a new formula (the conclusion) to a given set of formulas (the premisses). For a given rule, the number of premisses is always finite and fixed.

In a calculus $\mathscr{C}$, it is easy to define when a formula A is a syntactical consequence of a set $\Gamma$of formulas. When this is done, we write $\Gamma \vdash_{\mathrm{C}} \mathrm{A}$. If $\Gamma = \Phi$, A is said to be a thesis or a theorem of $\mathscr{C}$, and this fact is symbolized as follows: $\vdash_{\mathscr{C}} \mathrm{A}$. The symbol '$\vdash_{\mathscr{C}}$' has all the expected properties. We write simply '$\vdash$', instead of '$\vdash_{\mathscr{C}}$', when there is no doubt about the calculus we are considering.

Definition 1. – Let $\mathscr{C} \multimap < \Delta, \mathscr{R} >$ be a calculus and e a function from the set of formulas of the language L of $\mathscr{C}$ into $\{0, 1\}$. We say that e is a (two-valued) evaluation associated with $\mathscr{C}$ if we have:

1) If $\mathrm{A} \in \Delta$, then $e(\mathrm{A}) = 1$;

2) If all premisses of an application of a rule belonging to $\mathscr{R}$ assume the value 1 under e, then the corresponding conclusion also assumes the value 1;

3) There exist at least one formula A such that $e(\mathrm{A}) = 0$.

Let $\mathscr{E}$ be the set of evaluations of a calculus $\mathscr{C}$, $e \in \mathscr{E}$, and Γ a set formulas of the language of $\mathscr{C}$. We say that e satisfies Γ if, for every $\mathrm{A} \in \Gamma$, $e(\mathrm{A}) = 1$. We can easily prove the following properties.

i) If $\Gamma \vdash \mathrm{A}$, then, for every $e \in \mathscr{E}$, if e satisfies Γ, then $e(\mathrm{A}) = 1$;

ii) $\mathscr{C}$ is trivial if, and only if, $\mathscr{E} = \Phi$ ($\mathscr{C}$ is said to be trivial if, for every formula A of its language, we have that $\vdash \mathrm{A}$).

Let $\Sigma \cup \{A\}$ be a set of formulas of a nontrivial calculus $\mathscr{C}$. Σ is called A-saturated if $\Sigma \nvdash A$ and, for every $B \notin \Sigma$, $\Sigma \cup \{B\} \vdash A$. We clearly have (see [8] and [9]):

iii) If Σ is A-saturated, then $\Sigma \vdash B$ if, and only if, $B \in \Sigma$;

iv) If $\Gamma \nvdash A$, then there exists an A-saturated set Σ such that $\Gamma \subset \Sigma$.

v) The characteristic function of an A-saturated set is an evaluation.

An evaluation which is the characteristic function of an A-saturated set is called a *valuation. Let* $\mathscr{V}$ be the set of all valuations associated with $\mathscr{C}$. Using ii, iii, iv, and v, we can easily show that:

vi) $\mathscr{E} = \Phi$ if, and only if, $\mathscr{V} = \Phi$.

The notion of semantical consequence, with respect to a non-trivial calculus $\mathscr{C}$, is introduced without difficulty. We say that A is a semantical consequence of Γ if, for every valuation v which satisfies Γ, $v(A) = 1$ (in which case we write $\Gamma \models_{\mathscr{C}} A$, or simply $\Gamma \models A$). If $\Gamma = \Phi$, we say that A is valid in $\mathscr{C}$ (and we write $\models_{\mathscr{C}} A$ or $\models A$).

Any nontrivial calculus (logical system) has a two-valued semantics, in the sense of the following theorem:

Theorem 1. – $\Gamma \vdash A$ if, and only if, $\Gamma \models A$.

Proof. – Follows from i-v above.

The soundness and completeness of the classical propositional calculus are special cases of the preceeding theorem. The same is true of several other calculi (see, for example, [4] and [10].

A valuation v, such that $v(A) = 1$ for every A belonging to a collection Γ of formulas, is called a model of Γ.

A theory based on a calculus $\mathscr{C}$ is any set $\mathscr{T}$ of formulas of $\mathscr{L}$, the language of $\mathscr{C}$, such that if $\mathscr{T} \vdash A$, then $A \in \mathscr{T}$. When $\mathscr{T} = \{A: \mathscr{K} \vdash A\}$, $\mathscr{K}$ is called a set of axioms for $\mathscr{T}$. A model of $\mathscr{T}$ is any valuation v of $\mathscr{C}$ such that $v(A) = 1$ for every $A \in \mathscr{T}$. The theorems (or theses) of $\mathscr{T}$ are the formulas that belong to $\mathscr{T}$.

$\mathscr{C}$ is paraconsistent if, and only if, there are theories, based on $\mathscr{C}$, having models v for which $v(A) = v(\neg A) = 1$, for some $A \in \mathscr{L}$. Similarly, $\mathscr{C}$ is paracomplete if for some theory $\mathscr{T}$, based on $\mathscr{C}$, $\mathscr{T}$ has a model v such that, for some formula A, $v(A) = v(\neg A) = 0$, and conversely.

Remark. – Sometimes, in order to define the relation $\Gamma \vdash A$ with respect to a given calculus $\mathscr{C} = \langle \Delta, \mathscr{R} \rangle$, it may occur that some rules of $\mathscr{R}$ have to undergo certain restrictions, which take into account the cases where $\Gamma \neq \Phi$ (as occurs in modal logic with the rule of necessitation and in predicate logic with the rule of generalization). However, Theorem 1 remains valid under convenient, clear adaptations.

It seems worthwhile to observe that the semantics of valuations satisfies Tarski's conditions of formal correctness and of material adequacy (see [14] and [15]). In particular, Tarski's criterion $\mathscr{T}$ remains valid. These aspects of the semantics of valuations would become more evident if we were to consider predicate logic instead of propositional logic (cf. [2]).

A large part of our results and comments apply to first-order logic, with or without identity, and even to higher-order logic; of course, profound adaptations are required, since, in particular, we have to analyse the behaviour of the quantifiers.

3. The system π

We introduce now a paraconsistent and paracomplete calculus, π, with a very weak primitive negation, but where a kind of classical negation is definable. The language of π contains the set $\{\rightarrow, \&, \vee, \neg\}$ of primitive connectives, and an axiomatic basis for it is given by the following postulates (where $A^\circ \underset{\text{Def}}{=} \neg(A \& \neg A) \& (A \vee \neg A)$):

1) $A \rightarrow (B \rightarrow A)$
2) $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow C))$
3) $A, A \rightarrow B / B$
4) $(A \& B) \rightarrow A$
5) $(A \& B) \rightarrow B$
6) $A \rightarrow (B \rightarrow (A \& B))$
7) $A \rightarrow (A \vee B)$
8) $B \rightarrow (A \vee B)$

9) $(A \to C) \to ((B \to C) \to ((A \vee B) \to C))$
10) $A^{\circ} \vee (A \,\&\, \neg A) \vee \neg (A \vee \neg A)$
11) $\neg(A \vee \neg A) \to \neg(A \,\&\, \neg A)$
12) $\neg(A \,\&\, \neg A) \to ((A \,\&\, \neg A) \to B)$
13) $\neg(A \vee \neg A) \to ((A \vee \neg A) \to B)$
14) $(A^{\circ} \,\&\, B^{\circ}) \to ((A \,\&\, B)^{\circ} \,\&\, (A \vee B)^{\circ} \,\&\, (A \to B)^{\circ} \,\&\, (\neg A)^{\circ})$

Theorem 2. – The following schemes are not valid in π;

1) $\neg\neg A \to A$
2) $A \to \neg\neg A$
3) $\neg(A \,\&\, \neg A)$
4) $(A \,\&\, \neg A) \to B$
5) $A \vee \neg A$
6) $(A \vee \neg A) \to B$
7) $(A \to \neg A) \to \neg A$
8) $\neg A \vee \neg\neg A$
9) $(A \to B) \to ((A \to \neg B) \to \neg A)$
10) $(\neg B \to \neg A) \to (A \to B)$
11) $(A \to B) \to (\neg B \to \neg A)$
12) $\neg A \to (A \to B)$
13) $\neg(A \to A) \to B$
14) $\neg(A \vee B) \to (\neg A \,\&\, \neg B)$
15) $\neg(A \,\&\, B) \to (\neg A \vee \neg B)$
16) $(\neg A \vee \neg B) \to \neg(A \,\&\, B)$
17) $(\neg A \,\&\, \neg B) \to \neg(A \vee B)$
18) $(A \to B) \to \neg(A \,\&\, \neg B)$
19) $\neg(A \,\&\, \neg B) \to (A \to B)$
20) $\neg\neg(A \vee \neg A)$

Theorem 3. – The following schemes are provable in π:

T1) $\neg(A \vee \neg A) \to (A \to B)$
T2) $(\neg A \,\&\, \neg(A \,\&\, \neg A)) \to (A \to B)$
T3) $A \vee (A \to B)$
T4) $(A \,\&\, \neg A) \vee \neg(A \,\&\, \neg A)$
T5) $((A \,\&\, \neg A) \,\&\, \neg(A \,\&\, \neg A)) \to B$
T6) $\neg((A \,\&\, \neg A) \,\&\, \neg(A \,\&\, \neg A))$
T7) $(A \vee \neg A) \vee \neg(A \vee \neg A)$

T8) $\neg(A \vee \neg A) \rightarrow ((A \,\&\, \neg A) \rightarrow B)$
T9) $A \vee \neg(A \,\&\, \neg A)$
T10) $\neg A \vee \neg(A \,\&\, \neg A)$
T11) $\neg((A \vee \neg A) \,\&\, \neg(A \vee \neg A))$
T12) $(A \,\&\, \neg A)^{\circ}$
T13) $(\neg(A \,\&\, \neg A))^{\circ}$
T14) $(A \vee \neg A)^{\circ}$
T15) $A^{\circ\circ}$
T16) $(A \rightarrow B) \rightarrow ((A \rightarrow (B \rightarrow \neg(B \vee \neg B)) \rightarrow (A \rightarrow \neg(A \vee \neg A)))$
T17) $((A \rightarrow \neg(A \vee \neg A)) \rightarrow \neg((A \rightarrow \neg(A \vee \neg A)) \vee$
$\neg(A \rightarrow \neg(A \vee \neg A)))) \rightarrow A$
T18) $(\neg(A \,\&\, \neg A) \,\&\, (B \vee \neg B)) \rightarrow ((\neg B \rightarrow \neg A) \rightarrow (A \rightarrow B))$
T19) $(\neg(B \,\&\, \neg B) \,\&\, (A \vee \neg A)) \rightarrow ((A \rightarrow B) \rightarrow (\neg B \rightarrow \neg A))$
T20) $A^{\circ} \rightarrow (A \rightarrow \neg\neg A)$
T21) $A^{\circ} \rightarrow (\neg\neg A \rightarrow A)$

Definition 2. – $\sim A \underset{\text{Def}}{=} A \rightarrow \neg(A \vee \neg A)$. ('$\sim$' is the *strong* or *classical negation* of π).

Theorem 4. – '$\sim$' has all properties of the classical negation.
Proof. – In fact, we have in π:

$\vdash (A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$ (by T16),

and

$\vdash \sim\sim A \rightarrow A$ (by T17).

Thus, since postulates 1-9 above, together with the schemes $(A \rightarrow B) \rightarrow (\sim B \rightarrow \sim A)$ and $\sim\sim A \rightarrow A$, constitute an axiomatic basis for the classical propositional calculus (in which '$\rightarrow$', '&', '$\vee$' and '$\sim$' are the primitive connectives), π contains, in an obvious sense, that calculus.

4. *Valuation semantics and decision method for* π

Let V_π be the set of functions from the set of formulas of π into $\{0, 1\}$, such that, for every $\nu \in V_\pi$, we have:

1) $\nu(A \,\&\, B) = 1$ if, and only if, $\nu(A) = 1$ and $\nu(B) = 1$;
2) $\nu(A \vee B) = 1$ if, and only if, $\nu(A) = 1$ or $\nu(B) = 1$;
3) $\nu(A \rightarrow B) = 1$ if, and only if, $\nu(A) = 0$ or $\nu(B) = 1$;
4) If $\nu(\neg\neg A) = \nu(\neg A)$, then $\nu(\neg A) = \nu(A)$;

5) If $\nu(A \S B) = \nu(\neg(A \S B))$, then $\nu(A) = \nu(\neg A)$ and $\nu(B) = \nu(\neg B)$, for $\S \in \{\rightarrow, \wedge, \vee\}$;
6) $\nu(\neg(A \& \neg A)) \neq \nu(A \& \neg A)$;
7) $\nu(\neg(A \vee \neg A)) \neq \nu(A \vee \neg A)$.

Theorem 5. – The set $\mathscr{V}_\pi$ is the set of valuations associated with π.

Proof. – It is easy to demonstrate that $\nu \in \mathscr{V}_\pi$ if, and only if, there exists a set of formulas $\Sigma \cup \{A\}$, such that Σ is A-saturated and ν is its characteristic function.

Remark. – Since every theorem of π is classically valid, it is easy to see that every classical valuation belongs to $\mathscr{V}_\pi$; thus, there are valuations v for π such that, for some A, $v(A) \neq v(\neg A)$. But there are also valuations ν of $\mathscr{V}_\pi$ such that, for some formula A, $v(A) = v(\neg A)$; in this case, we say that A is not 'semantically well-behaved'. Otherwise, A is 'semantically well-behaved'. Our system π has an interesting property: the semantical behaviour of any given formula A can be expressed, in a certain sense, by a formula. For example, $A \& A^\circ$ *means* that A is true and well behaved, in this sense: for every $v \in \mathscr{V}_\pi$, if $v(A \& A^\circ) = 1$, then $v(A) = 1$ and $v(\neg A) = 0$. In the same way: $A \& \neg A$ *means* 'A is true but not well-behaved'; $\neg A \& A^\circ$ *means* that A is false and well-behaved; finally, $\neg A \& \neg(A \vee \neg A)$ *means* that A is false and not well-behaved.

With the help of $\mathscr{V}_\pi$, we can obtain a decision method for the calculus π.

Let $\{A_1, A_2, \ldots, A_n\}$ be a set of formulas of the language π. We say that $A_1, A_2, \ldots, A_n$ constitutes a π-sequence if, for $1 \leq i \leq n$, we have:

a) If B is a subformula of A_i, then, for some $j \leq i$, $B = A_j$;
b) If A_i is $\neg(A \S B)$, where $\S \in \{\rightarrow, \&, \vee\}$, then there are i, $k < i$ such that A_j is $\neg A$ and A_k is $\neg B$;
c) For $1 \leq j < i$, $A_j \neq A_i$.

Definition 3. – Suppose that $A_1, A_2, \ldots, A_n$ is a π-sequence. The tableaux for $A_1, A_2, \ldots, A_n$, denoted by $t_n(A_1, A_2, \ldots, A_n)$ or simply by t_n, when the formulas $A_1, A_2, \ldots, A_n$ are manifest, is a function from $I_n \times J(A_1, A_2, \ldots, A_n)$ into $\{0, 1\}$, where $I_n = \{1, 2, \ldots, n\}$ and

1) $J(A_1) = \{1, 2\}$, $t_1(1, 1) = 1$, and $t_1(1, 2) = 0$;

2) $J(A_1, A_2, \ldots, A_{n-1}) \subseteq J(A_1, A_2, \ldots, A_n)$ and, for $J(A_1, A_2, \ldots, A_{n-1}) = \{1, 2, \ldots, m\}$, one has:

2.1) For $1 \leqslant i \leqslant n$, $1 \leqslant j \leqslant m$, $t_n(i,j) = t_{n-1}(i,j)$;

2.2) a) If A_n is a propositional variable, $J(A_1, A_2, \ldots, A_n) = \{1, \ldots, 2m\}$, and, for $j \leqslant m$, $t_n(n,j) = 1$, and for $j > m$, $t_n(n,j) = 0$;

b) If A_n is a $A_k \& A_l$, $J(A_1, A_2, \ldots, A_n) = \{1, 2, \ldots, m\}$; for $1 \leqslant j \leqslant m$, $t_n(n,j) = 1$ if, and only if, $t_n(k,j) = 1$ or $t_{n(l,j)} = 1$;

c) If A_n is $A_k \vee A_l$, $J(A_1, A_2, \ldots, A_n) = \{1, 2, \ldots, m\}$; for $1 \leqslant j \leqslant m$, $t_n(n,j) = 1$ if, and only if, $t_n(k,j) = 1$ or $t_n(l,j) = 1$;

d) If A_n is $A_k \rightarrow A_l$, $J(A_1, A_2, \ldots, A_n) = \{1, 2, \ldots, m\}$; for $1 \leqslant j \leqslant m$, $t_n(n,j) = 1$ if, and only if, $t_n(k,j) = 0$ or $t_n(l,j) = 1$;

e) If A_n is $\neg A_k$, then:

e_1) If A_k is a propositional variable, $J(A_1, A_2, \ldots, A_n) = \{1, 2, \ldots, 2m\}$ and, for $j \leqslant m$, $t_n(n,j) \neq t_n(k,j)$ for $j > m$, $t_n(n,j) = t_n(k,j)$;

e_2) If a_k is $A_p \& \neg A_p$ or A_k is $A_p \vee \neg A_p$, then $J(A_1, A_2, \ldots, A_n) = \{1, 2, \ldots, m\}$; for $1 \leqslant i \leqslant m$, $t_n(n, j) \neq t_n(k, j)$;

e_3) If A_k is $A_p \& A_q$ or A_k is $A_p \vee A_q$, where A_q is different from $\neg A_p$, then let $\bar{p}$ and $\bar{q}$ be such that $A_{\bar{p}}$ is $\neg A_p$ and $A_{\bar{q}}$ is $\neg A_q$, and let $\overline{J}(A_1, A_2, \ldots, A_n) = \{j \in J(A_1, A_2, \ldots, A_{n-1})\colon t_n(p, j) = t_n(\bar{p}, j)$ or $t_n(q, j) = t_n(\bar{q}, j)\}$; if $J(A_1, A_2, \ldots, A_n) = \{m_1, m_2, \ldots, m_r\}$, then $J(A_1, A_2, \ldots, A_n) = \{1, 2, \ldots, m + r\}$; for $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant r$, $t_n(i, m+s)$; for $j \leqslant m$, $t_n(n, j) \neq t_n(k, j)$; for $j > n$, $t_n(n, j) = t_n(k, j)$;

e_4) If A_k is $A_p \rightarrow A_q$, everything goes as in case e_3) above.

e_5) If A_k is $\neg A_p$, let $\overline{J}(A_1, A_2, \ldots, A_n) = \{j \in J(A_1, A_2, \ldots, A_{n-1}\colon t_n(p,j) = t_n(k,j)\}$; if $\overline{J}(A_1, A_2, \ldots, A_n) = \{m_1, \ldots, m_r\}$. then, $J(A_1, A_2, \ldots, A_n) = \{1, 2, \ldots, m+r\}$; for $1 \leqslant i \leqslant n-1$, $1 \leqslant s \leqslant r$, $t_n(i, m+s) = t_n(i, m_s)$; for $j \leqslant m$, $t_n(n,j) \neq t_n(k, j)$; for $j \leqslant m$, $t_n(n, j) = t_n(k, j)$.

Let $t_n(A_1, A_2, \ldots, A_n)$ be the tableau for a π-sequence $A_1, A_2, \ldots, A_n$. We have:

Lemma 1. – For every $j \in J(A_1, A_2, \ldots, A_n)$ there is a $\nu \in \mathscr{V}_\pi$ such that, for $1 \leqslant i \leqslant n$, $\nu(A_i) = t_n(i, j)$.

Proof. – Construct the set α as follows:

1) If $A \in \{A_1, A_2, \ldots, A_n\}$, then $A \in \alpha$ if, and only if, $t_n(i, j) = 0$:
2) If $A \notin \{A_1, A_2, \ldots, A_n\}$, then $A \in \alpha$ if, and only if:
   a) A is B & C and $B \in \alpha$ or $C \in \alpha$;
   b) A is $B \vee C$ and $A, B \in \alpha$;
   c) A is $B \to C$ and $A \in \alpha$ and $B \notin \alpha$;
   d) A is $\neg(B \,\&\, \neg B)$ and $B \,\&\, \neg B \in \alpha$;
   e) A is $\neg(B \vee \neg B)$ and $B \vee \neg B \in \alpha$;
   f) A is $\neg\neg B$ and $B \in \alpha$ and $\neg B \in \alpha$;
   g) A is $\neg(B \,\S\, C)$, $B \,\S\, C \notin \alpha$ and $B \in \alpha$ if and only if $\neg B \notin \alpha$ and $C \neq \alpha$ if and only if $\neg C \notin \alpha$, for $\S \in \{\to, \&, \vee\}$

Now, it is not difficult to show that the complement of α, with respect to the set of all formulas of π, is A-saturated relative to every formula A of the form $B \,\&\, \neg B \,\&\, \neg(B \,\&\, \neg B)$. Thus, its characteristic function ν belongs to $\mathscr{V}_\pi$ and, by construction, for any $A_i \in \{A_1, A_2, \ldots, A_n\}$, $\nu(A_i) = t_n(i, j)$.

Lemma 2. – For every $\nu \in \mathscr{V}_\pi$, there is a $j \in J(A_1, A_2, \ldots, A_n)$, such that, for $1 \leqslant i \leqslant n$, $t_n(i, j) = \nu(A_i)$.

Proof. – By induction on n.

Proof. – $\vdash A_i$ if, and only if, $\models A_i$ by Theorem 1. Now, by lemmas 1 and 2, $\models A_i$ if, and only if, for every $j \in J(A_1, A_2, \ldots, A_n)$, $t_n(i, j) = 1$.

On the other hand, we can easily stipulate a canonical way of associating, to any formula A, a π-sequence $A_1, A_2, \ldots, A_n$ such that A_n is A.

Therefore, π is decidable by our *tableaux*, based on the valuation semantics.

Example: The graphic below presents the tableau for the schematic π-sequence 'A, $\neg A$, $A \vee \neg A$, $\neg(A \vee \neg A)$, $A \,\&\, \neg A$, $\neg(A \,\&\, \neg A)$, $\neg(A \vee \neg A) \to \neg(A \,\&\, \neg A)$' and shows the validity of the scheme' $\neg(A \vee \neg A) \to \neg(A \,\&\, \neg A)$' as well as the non-validity of '$A \vee \neg A$' and '$\neg(A \,\&\, \neg A)$'.

A	$\neg A$	$A \vee \neg A$	$\neg(A \vee \neg A)$	$A \& \neg A$	$\neg(A \& \neg A)$	$\neg(A \vee \neg A) \rightarrow \neg(A \& \neg A)$
1	0	1	0	0	1	1
0	1	1	0	0	1	1
1	1	1	0	1	0	1
0	0	0	1	0	1	1

Using valuation-tableaux we can prove an interesting result: π is not a finite many-valued logic. The proof runs as follows:

Lemma 3. – Let $A_1, A_2, \ldots, A_n$ be the π-sequence where A_1 is the propositional variable 'p' and, for $1 < i \leqslant n$, $A_i = \neg A_{i-1}$. Then, a) $J(A_1, A_2, \ldots, A_n) = \{1, 2, \ldots, 2_n\}$; b) for $1 \leqslant i \leqslant n$, $t_n(i, 2n-1) = 1$, $t_n(i, 2_n) = 0$; c) for $n \geqslant 2$, $t_n(n, 2n-3) = 0$ and, for $1 \leqslant i \leqslant n-1$, $t_n(i, 2n-3) = 1$; d) for $n \geqslant 2$, $t_n(n, 2n-2) = 1$ and, for $1 \leqslant i \leqslant n-1$, $t_n(i, 2n-2) = 0$.

Proof. – By induction on *n*.

Corollary. – a) $\nvdash \neg_k p \rightarrow \neg_{k+m} p$, for $k \geqslant 0$, $m > 1$;
b) $\nvdash \neg_{k+m} p \rightarrow \neg_k p$, for $k \geqslant 0$, $m > 1$, (where '$\neg_n p$' is the formula obtained by putting *n* occurrences of '$\neg$' before 'p').

Proof. – Take the π-sequence 'p, $\neg p, \ldots, \neg_k p, \ldots, \neg_{k+m} p$'. By Lemma 3, $t_{k+m+1}(k+m+1, 2(k+m+1)-3) = 0$ and $t_{k+m+1}(k+1, 2(k+m+1)-3) = 1$; thus, by Lemma 1, there is some $\nu \in \mathscr{V}_\pi$ such that $\nu(\neg_{k+m} p) = 0$ and $\nu(\neg_k p) = 1$, hence $\nu(\neg_k p \rightarrow \neg_{k+m} p) = 0$ and, by the Theorem 6, $\nvdash \neg_k p \rightarrow \neg_{k+m} p$; on the other hand, by Lemma 3, $t_{k+m+1}(k+m+1, 2(k+m+1)-2) = 1$ and $t_{n+m+1}(k+1, 2(k+m+1)-2) = 0$; therefore, by lemma 1, there is some $\nu \in \mathscr{V}_\pi$ such that $\nu(\neg_{k+m} p) = 1$ and $\nu(\neg_k p) = 0$, thus $\nu(\neg_{k+m} p \rightarrow \neg_k p) = 0$ and, by Theorem 6, $\nvdash \neg_{k+m} p \neg_k p$.

Theorem 7. – π has not a finite characteristic matrix; thus, π is not a finite many-valued logic.

Proof. – In any finite matrix there is some $k \geqslant 0$ and some $m > 1$ such that, for every value x of the matrix, $\neg_k(x) = \neg_{k+m}(x)$. Since $A \rightarrow A$ is π-valid, in any adequate matrix for π, and for every value y, we should have a distinguished value for → (y, y); thus both '$\neg_k p \to \neg_{k+m} p$' and '$\neg_{k+m} p \to \neg_k p$' would be valid in the matrix.

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