ŁUKASIEWICZ'S Ł-MODAL SYSTEM AND CLASSICAL REFUTABILITY

Jean PORTE

1. In [10] it has been proved that the modalities of Łukasiewicz's Ł-system can be defined in a system obtained from the Classical Propositional Calculus (PC) by the addition of a constant, Ω, about which nothing is postulated, and that the modalities of the Ł-system, M (possibility) and L (necessity) can conversely be used to define Ω. The quasi-definitions are:

$$Lx \leftrightarrow (\Omega \wedge x) \quad (1)$$

$$Mx \leftrightarrow (\Omega \rightarrow x) \quad (2)$$

(for every formula x), and

$$\Omega \leftrightarrow (Ma \rightarrow La) \quad (3)$$

(for any constant formula a).

On the other hand, Curry, [1], [2], [4], defines Classical Refutability (CR – called "HE" in [4]) by adding a constant similar to Ω to the negationless part of PC, or Classical Positive Propositional Calculus (PPC – called "HC" in [4]) with

$$\sim x \leftrightarrow (x \rightarrow \Omega) \quad (4)$$

as a quasi-definition of $\sim$, the negation of CR – which will be called here "weak negation", to distinguish it from the negation of PC ($\neg$).

Now, is Ω definable in CR? If so, by the intermediate of the Ω-system of [10], the negationless part of of the Ł-system and CR will be interdefinable, or in the terms of [9] or [11] "equipollent."

It will be proved that it is so: *The negationless part of the Ł-system and the system of Classical Refutability are equipollent.* The results are easy to prove, and utterly counter-intuitive – this last fact meaning only that the modalities of the Ł-system are very far from what everybody calls "possibility" and "necessity" and/or that the weak

negation of CR is very far from what everybody calls "negation."
2. *The negationless part of the Ł-system*, with connectives →, ∧, ∨, L, and M (↔ will be considered a function, defined as usual by means of → and ∧), has a characteristic matrix which is defined as follows:

$x \to y$

x/y	1	2	3	4
1	1	2	3	4
2	1	1	3	3
3	1	2	1	2
4	1	1	1	1

$x \wedge y$

	1	2	3	4
1	1	2	3	4
2	2	2	4	4
3	3	4	3	4
4	4	4	4	4

$x \vee y$

	1	2	3	4
1	1	1	1	1
2	1	2	1	2
3	1	1	3	3
4	1	2	3	4

x:	1	2	3	4
Lx:	2	2	4	4
Mx:	1	1	3	3

D = {1}

This is the natural transformation of the matrix of Łukasiewicz ([6], p. 127) and Smiley [13], completed with the tables for ∧, ∨ and L, and with suppression of the table for classical negation:

x:	1	2	3	4
¬x:	4	3	2	1

The tables for →, ∧ and ∨, with D = {1} as the set of designated elements, constitute the negationless part of a characteristic matrix for PC (a four-elements boolean matrix).

The system of Classical Refutability (CR) has the connectives of PPC((→, ∧, ∨) plus the weak negation (~).

It is well known that PPC has a two-valued characteristic matrix: the ordinary matrix for PC minus the table for ¬.

If weak negation is defined by (4), with a constant about which nothing is said – a constant called "F" in Curry [4] pp. 284-286, and "O" in Wajsberg [16] and [17] (see McCall [7]) – this constant does not differ from the Ω of [10]; CR has, therefore, a four-valued characteristic matrix, just like the Ω-system, where Ω *is assigned the constant value 2.*

Thus, putting together all the connectives ($\rightarrow$, $\wedge$, $\vee$, Ω, $\sim$), CR, the positive part of the Ω-system, and the positive part of the Ł-system *have all the same characteristic matrix.*

L and M being defined by (1) and (2), and Ω being assigned the value 2, we find by immediate computation, using the tables for $\rightarrow$ and $\wedge$:

x	1	2	3	4
$\sim x$	2	1	2	1
$\sim\sim x$	1	2	1	2
$\sim(x \rightarrow x)$	2	2	2	2
$x \wedge \sim x$	2	2	4	4
$\sim x \sim\sim x$	2	2	2	2
$\sim\sim x \rightarrow x$	1	1	3	3
$Mx \rightarrow Lx$	2	2	2	2
$x \rightarrow Lx$	2	1	2	1
$Mx \rightarrow x$	1	2	1	2
$L(x \rightarrow Lx)$	2	2	2	2
$L(x \rightarrow Lx) \rightarrow x$	1	1	3	3

Then, everything can be defined:

– either by Ω:

$Lx \leftrightarrow (\Omega \wedge x)$ (1)

$Mx \leftrightarrow (\Omega \rightarrow x)$ (2)

$\sim x \leftrightarrow (x \rightarrow \Omega)$ (4)

– or by L and M:

$\Omega \leftrightarrow (Mx \rightarrow Lx)$ (3)

$\Omega \leftrightarrow L(x \rightarrow Lx)$ (5)

$\sim x \leftrightarrow (x \rightarrow Lx)$ (6)

but $\sim\sim x \leftrightarrow (Mx \rightarrow x)$ (7)

– or by $\sim$:

$\Omega \leftrightarrow \sim(x \rightarrow x)$ (8)

or: $\Omega \leftrightarrow \sim x \wedge \sim\sim x$ (9)

$Lx \leftrightarrow (x \wedge \sim x)$ (10)

$Mx \leftrightarrow (\sim\sim x \rightarrow x)$ (11)

$Mx \leftrightarrow (\sim(x \rightarrow x) \rightarrow x)$ (12)

What I have said above to be "counter-intuitive" are the results (6), (7), (10), (11) and (12).

Remark: It can be seen in the matrix that M can be defined by L and $\rightarrow$, since

$$Mx \leftrightarrow (L(x \rightarrow Lx) \rightarrow x) \tag{13}$$

It follows that we could suppressed M from the positive part of the Ł-system which, being restricted in such a way would yet remain equipollent to CR. Disjunction could as well be suppressed from both systems, as being definable in PPC after quasi-definition

$$(x \vee y) \leftrightarrow ((x \rightarrow y) \rightarrow y) \tag{14}$$

3. *Axiomatization* – For the "positive Ω-system", or negationless part of the Ω-system as defined in [10], the problem is trivial: the theses are all the substitution instances of PPC-theses; they can be axiomatized, with modus ponens as sole rule by one of the classical sets of axiom schemas for PPC, for instance the one of Kanger [5] (see also T.T. Robinson [12], axioms (1.1) to (1.9)).

For the negationless part of the Ł-system (with $\rightarrow$, $\wedge$, L, M, and without Ω), the general method of [9], chapter 12 (or, better, of [11]) gives the following complete axiomatization, with modus ponens as sole rule, and three axiom schemas:

A1 – t iff t is a substitution instance of a PPC-thesis
A2 – $Lx \leftrightarrow ((My \rightarrow Ly) \wedge x)$
A3 – $Mx \leftrightarrow ((My \rightarrow Ly) \rightarrow x)$
– for all formulas x, y.

Of course, A1 can be replaced by the set of nine axiom schemas derived from Kanger's first axioms, as above. – On the other hand, A2 and A3 can easily be deduced from A4-A8 below; these formulas are valid in the matrix, and A1, A4-A8, with modus ponens, constitute a complete axiomatization (there are more axioms than in A1-A3 but they are simpler):

A4 – $(x \rightarrow y) \rightarrow (Lx \rightarrow Ly)$
A5 – $(x \rightarrow y) \rightarrow (Mx \rightarrow My)$
A6 – $Lx \rightarrow x$
A7 – $x \rightarrow Mx$
A8 – $(Mx \rightarrow Lx) \rightarrow (My \rightarrow Ly)$

I conjecture that in the system A1, A4-A8 every axiom schema is independent.

For the system of Classical Refutability (with →, ∧, ∨, ~, without Ω), the same general method of [9] and [11] gives the following complete axiomatization

B1 – t if t is a substitution instance of a PPC-thesis
B2 – $\sim x \leftrightarrow (x \rightarrow \sim(y \rightarrow y))$
– for all formulas x, y.

But other axiomatizations are known: see Curry [4], ot T.T. Robinson [12].

4. *Non-definability of classical negation* – As could be presumed, ¬ is not definable in any of the three systems considered up to now (→, ∧, ∨, Ω; →, ∧, ∨, L, M; →, ∧, ∨, ~) and just proved to be equipollent.

Proof: In the common characteristic matrix of these systems, the subset of values {1, 2} is closed for all operations, so that no function can take the constant value 3, nor the constant value 4 – while ¬Ω or $\neg(Mx \rightarrow Lx)$ is always assigned value 3, and $\neg(x \rightarrow x)$ is assigned value 4.

The foregoing proof of non-definability is in the line of McKinsey [8]. The objections formulated by Smiley against McKinsey's method do not hold, for the matrix is "normal" in the sense of Smiley [14].

In a similar way it would be seen that, in the tables for →, ∧, ∨, M, the set {1, 3} is closed, so that L is not definable in terms of those connectives – As L is itself definable in terms of ∧ and ~, by quasi-definition (10) above, it follows that ~ is not definable in terms of →, ∧, ∨, M.

5. *Further remarks* – The equivalence ("equipollence") of the Ł-system with the Ω-system, proved in [9], had been discovered independetly by Smiley: In [15] he described in fact the Ω-system under the name of "OPC", the "possibility" of Łukasiewicz being treated as "obligation" in a kind of deontic logic.

Indeed, the idea of defining possibility by (2), using a constant, had been considered previously by Curry in [3]. But he did not pursue the idea.

Université des Sciences Jean PORTE
et de la Technologie Houari Boumedienne, Algiers, Algeria

1, Villa Ornano
75018 Paris

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