ON THE GENERALISED CONVERSE IN RELATIONAL LOGIC

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1. *Basic Properties of the g-Converse*

In a paper [1] which recently appeared I introduced the concept of a relation schema of dimension n and order m and showed that the logic of relations, if based upon this concept, displays some interesting analogies with both linear algebra and syllogistics. I also explained, among other things, what is meant by the generalised converse (or g-converse for short) of a relation schema. In the present paper I shall analyse this notion of the g-converse further. I shall, however, confine myself from the start to relation schemata of order 2, that is, to relation schemata whose basic relations are *binary* relations.

Let R and R' be two relations of a relation schema R_n^2 (of order 2 and dimension n), and let them be decomposed into their components. These are defined as the basic relations R_i whose coefficients δ_i, in the representation

$$R = \bigvee_{i=1}^{n} \delta_i R_i, \tag{1}$$

differ from 0. The g-converse can then be defined by the following conditions: (1) for any $R \neq R_v$, $R' = C(R)$ holds if, and only if, R'(y,x) holds whenever R(x,y) holds, and R' is the relation with the smallest number of components for which this is so; (2) $C(R_v) = R_v$. x and y are elements belonging to the set of individuals associated with the relation schema R_n^2. R_v is the void relation.

It is easy to show that C is a 'linear' correspondence in the sense that for any two elements R, R'' of R_n^2 we have

(1) 'On the Logic of Relations', *Dialectica*, vol. 34, No. 3 (1980), pp. 167-182. (Abbreviated reference: LR).

$$C(R \vee R'') = C(R) \vee C(R'') . \tag{2}$$

Applying (1) and (2), we find

$$C(R) = C(\bigvee_{i=1}^{n} \delta_i R_i) = \bigvee_{i=1}^{n} \delta_i C(R_i) = \bigvee_{i=1}^{n} \delta_i \bigvee_{j=1}^{n} c_{ji} R_j = \bigvee_{j=1}^{n} (\bigvee_{i=1}^{n} \delta_j c_{ji}) R_j . \tag{3}$$

The coefficients c_{ji} ($i = 1, \ldots, n$; $j = 1, \ldots, n$) form a quadratic matrix C which may be taken to represent the correspondence C:

$$C = \begin{bmatrix} c_{11} & c_{12} & c_{13} & \ldots & c_{1n} \\ c_{21} & c_{22} & c_{23} & \ldots & c_{2n} \\ \vdots & & & & \\ c_{n1} & c_{n2} & c_{n3} & \ldots & c_{nn} \end{bmatrix} ; \tag{4}$$

its ith column contains the coefficients of the representation of the basic relation R_i.(²) As the coefficients can only assume the values 1 or 0, the elements of C are either 1 or 0. It follows further that C must be symmetric. For, if R_i is a component of $C(R_j)$, then R_j must be a component of $C(R_i)$. Since the g-converse of each basic relation must contain at least one basic relation as a component, each column of C must contain at least one digit 1; and because of the symmetric of C this must also apply to each row of C. Finally, there must be at least one basic relation which applies to couples (x,x) and which is therefore a component of its own g-converse; this implies that the diagonal elements of C cannot all vanish.

(²) C(R) is calculated by using the following 'multiplication' and 'addition' rules: $0.0 = 0, 0.1 = 0, 1.0 = 0, 1.1 = 1$; in order to simplify the notation, the dot is omitted; e.g. instead of $\delta_j . c_{ji}$ we simply write $\delta_j c_{ji}$.
$0 \vee 0 = 0, 0 \vee 1 = 1, 1 \vee 0 = 1, 1 \vee 1 = 1$; e.g. the expression $\bigvee_{j=1}^{n} \delta_j c_{ji}$ in (3) is supposed to be calculated in accordance with this 'addition' rule.

2. *Powers of the g-Converse*

The correspondence C can be applied to itself as follows. Let $R' = C(R_i)$. We now exchange the elements in all the ordered couples to which R' applies. Let $R_{j_1}, R_{j_2}, \ldots$ be the basic relations which apply to this set of ordered couples; then $R_{j_1} \vee R_{j_2} \vee \ldots = CC(R_i) = C^2(R_i)$. Similarly, by carrying out the correspondence C m times in succession, we obtain C^m which shares with C the following basic properties:

a) C^m is 'linear' in the sense C is.
Proof: If C^{m-1} is a 'linear' correspondence, then C^m is 'linear'. For, $C^m(R \vee A') = CC^{m-1}(R \vee R') = C(C^{m-1}(R) \vee C^{m-1}(R')) = CC^{m-1}(R) \vee CC^{m-1}(R') = C^m(R) \vee C^m(R')$. Therefore as C is 'linear' C^m must also be, for any integer $m > 1$.

b) For any positive integer m, C^m is symmetric in the following sense: if R_j is a component of $C^m(R_i)$, then R_i is a component of $C^m(R_j)$; and if R_k is not a component of $C^m(R_i)$, then R_i is not a component of $C^m(R_k)$. Hence C^m is a symmetric matrix.
Proof: Let R_j be a component of $C^m(R_i)$. Then there exist couples $(x_1,x'_1), (x_2,x'_2), (x_3,x'_3), \ldots, (x_n,x'_n)$ such that

$$R_i(x_1,x'_1).\ R_{i_1}(x'_1,x_1).\ R_{i_1}(x_2,x'_2).\ R_{i_2}(x'_2,x_2).\ \ldots\ . \qquad (5)$$
$$R_{i_{m-1}}(x_{m-1},x'_{m-1}).\ R_{i_{m-1}}(x_m,x'_m).\ R_j(x'_m,x_m),$$

with suitably chosen basic relations $R_i, R_{i_1}, R_{i_2}, \ldots, R_j$. R_{i_1} is a component of $C(R_i)$ a component of $C^2(R_i)$ and, by assumption, R_j a component of $C^m(R_i)$. However, according to (5) $R_{i_{m-1}}$ must be a component of $C(R_j)$, $R_{i_{m-2}}$ a component of $C^2(R_j)$, and so on. We thus arrive after m steps at R_i which must be a component of $C^m(R_j)$. On the other hand, if R_j is not a component of $C^m(R_i)$, then R_i cannot be a component of $C^m(R_j)$ either; this follows immediately from the result just established.
But (b) may also be proved by making use of its matrix representation. C^m is symmetric if, and only if, the matrix C^m is symmetric. We merely have to show that $(C^m)^T = C^m$, i.e. that the transpose of C^m equals C^m. As $C^T = C$ we have

$$(C^m)^T = (CC^{m-1})^T = (C^{m-1})^T C = (C^{m-2})^T C^2 = \ldots = C^m.$$

c) The matrix C^m must contain at least one digit 1 in each column and in each row.
Proof: $C(R_i)$ cannot be the void relation, for any basic relation R_i, but must have at least one component.(3) Now, if, for any basic relation R_i, $C^{m-1}(R_i)$ cannot be the void relation R_v, then $C^m(R_i) \neq R_v$, for any basic relation R_i. Indeed, let R_j be a component of $C^{m-1}(R_i)$, then $CC^{m-1}(R_i) = C(R_j \vee \ldots) \neq R_v$.
Hence each column of C^m must contain at least one digit 1; and because of the symmetry of C^m this must also apply to each of its rows.

d) The matrix C^m must contain at least one digit 1 in its diagonal.
Proof: There exists at least one basic relation R_i which applies to couples (x, x), i.e. to couples consisting of two identical elements. Each exchange of these elements produces again the same couple to which R_i applies. Hence R_i is a component of $C^m(R_i)$, for all positive integers m; and the ith digit of the diagonal of C^m equals 1.

According to theorems a) to d) the basic properties of C are preserved under multiplication of C with itself or any of its powers. The following propositions e) to m) show that the powers of C have a number of additional features which are worth mentioning.

e) For any basic relation R_i and any positive integer m, R_i is a component of $C^{2m}(R_i)$.
Proof: The theorem is fulfilled for $m = 1$; for, if $R_i(x,y)$ holds and if x and y are exchanged twice, we are back to (x,y) and hence to $R_i(x,y)$. But if the theorem is fulfilled for $m-1$, then it must also hold for m: if $C^{2(m-1)}(R_i) = R_i \vee \ldots$, then $C^{2m}(R_i) = C^2C^{2(m-1)}(R_i) = C^2(R_i \vee \ldots) = R_i \vee \ldots$

f) For any basic relation R_i and any positive integer m, each of the components of $C^m(R_i)$ is also a component of $C^{m+2}(R_i)$.
Proof: According to theorem e) we have $C^2(R_i) = R_i \vee \ldots$
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Hence $C^{m+2}(R_i) = C^mC^2(R_i) = C^m(R_i \vee \ldots) = C^m(R_i) \vee \ldots$

(3) This is obvious from the definitions of a g-converse and of a relation schema. See LR, pp. 168 and 171.

g) For any non-asymmetric basic relation R_i and any positive integers m,k, each of the components of $C^m(R_i)$ is also a component of $C^{m+k}(R_i)$.
Proof: If R_i is non-asymmetric, then R_i is a component of $C(R_i)$. Therefore $C^{m+1}(R_i) = C^mC(R_i) = C^m(R_i \vee \ldots) = C^m(R_i) \vee \ldots$; that is, the components of $C^m(R_i)$ are also components of $C^{m+1}(R_i)$.

h) If $C^m(R_i) = C^{m+1}(R_i)$, for some basic relation R_i and some positive integer m, then $C^m(R_i) = C^{m+k}(R_i)$, for all positive integers k.
Proof: Ik $C^k(R_i) = C^{k+1}(R_i)$, then $C^{k+1}(R_i) = C^{k+2}(R_i)$, for any positive integer k; for, $C^{k+1}(R_i) = CC^k(R_i) = CC^{k+1}(R_i) = C^{k+2}(R_i)$. Hence, if $C^m(R_i) = C^{m+1}(R_i)$, then $C^m(R_i) = C^{m+k}(R_i)$, for all positive integers k.

i) If $C^m(R_i) = C^{m+2}(R_i)$, for some basic relation R_i and some positive integer m, then $C^m(R_i) = C^{m+2k}(R_i)$, for all positive integers k.
Proof: We merely to show that if $C^{m+2(k-1)}(R_i) = C^{m+2k}(R_i)$, then $C^{m+2k}(R_i) = C^{m+2(k+1)}$. But $C^{m+2(k+1)}(R_i) = C^2C^{m+2k}(R_i) = C^2C^{m+2(k-1)}(R_i) = C^{m+2k}(R_i)$.

j) To any basic relation R_i of a relation schema R_n^2 there exists a positive integer N_i such that either $C^m(R_i) = C^{m+k}(R_i)$ or $C^m(R_i) \neq C^{m+1}(R_i)$ and $C^m(R_i) = C^{m+2k}$, for all positive integers k and all integers $m \geqslant N_i$.
Proof: (1) Let R_i be non-asymmetric, and let us consider the series $C^h(R_i)$, $h = 0,1,2,3,\ldots,n$, where $C^o(R_i)$ is by definition R_i itself. According to theorem g) each of the components of $C^h(R_i)$, $h = 1,2,3,\ldots,n-1$, is also a component of its immediate successor. Now either at least two of these elements $C^h(R_i)$, $h = 0,1,2,\ldots,n-1$, are identical; then all the successors of the first of these two elements must be identical with it, according to theorems g) and h). Or all the $C^h(R_i)$, $h = 0,1,2,3,\ldots,n-1$, differ from each other; then $C^n(R_i)$, where n is the dimension of the schema, must be identical with its immediate predecessor; because there are at most n different components available, for a given R_i, to make up the series ending with $C^n(R_i)$. In any case, then, we can find a number $N_i < n$ (the dimension of the schema) such that $C^{m+k}(R_i) = C^m(R_i)$ for $m \geqslant N_i$ and $k = 1,2,3,\ldots$

(2) Let R_i be an asymmetric relation, and let us consider the series

of even powers: $C^o(R_i)$, ..., $C^{2n-2}(R_i)$, and the series of odd powers: $C(R_i)$, $C^3(R_i)$, ..., $C^{2n-1}(R_i)$. Each series contains n members; and each member, except the first, involves all the components of its predecessors belonging to the same series, as is evident from theorem f). But as there exist, for any given R_i, at most n elements to make up the series, then $C^{2n}(R_i)$ must be identical with some of its predecessors, indeed with its immediate predecessor in the first series; while $C^{2n+1}(R_i)$ must be identical with $C^{2n-1}(R_i)$. Hence there exists a number N_i of the required kind.

k) Let R_i be a basic relation of a relation schema of dimension n, then $N_i \leqslant n-1$, if R_i is non-asymmetric; and $N_i \leqslant 2(n-1)$, if R_i is asymmetric.

Proof: The proof for the first part of the theorem follows at once from paragraph (1) of the proof for theorem j). Let R_i be asymmetric; then $C(R_i)$ cannot include R_i as a component. Now consider the series $C^o(R_i)$, $C^2(R_i)$, ..., $C^{2n}(R_i)$. If two subsequent elements in this series are equal, then all the following elements of the series are equal, according to theorem i). Moreover, if $C^h(R_i) = C^{h+2}(R_i)$, then $C^{h+1}(R_i) = C^{h+3}(R_i)$; for we have $C^{h+1}(R_i) = CC^h(R_i) = CC^{h+2}(R_i) = C^{h+3}(R_i)$. Using theorem e) we find that at most n elements in $C^o(R_i)$, $C^2(R_i)$, ..., $C^{2n}(R_i)$ can be different from each other. Hence $C^{2n}(R_i)$ must be equal to $C^{2n-2}(R_i)$. Consequently, in the series consisting of all odd powers of $C(R_i)$ repetition must occur at the latest after $C^{2n-1}(R_i)$.([4])

l) If a basic relation R_i is a component of some odd power h of $C(R_i)$, then $C^{m+k}(R_i) = C^m(R_i)$, for $m \geqslant N_i$ and all positive integers k.

Proof: $C^{h+2r}(R_i) = C^{2r}C^h(R_i) = C^{2r}(R_i) \vee \ldots$; that is, all the components of any given even power of $C(R_i)$ are also components of some odd powers of $C(R_i)$. If $h \geqslant N_i$, then $C^{h+2r}(R_i)$ must involve all the components of any even power of $C(R_i)$. On the other hand, we have, for q = 1,3,5, ..., $C^{h+q}(R_i) = C^qC^h(R_i) = C^q(R_i) \vee \ldots$; all the components of any given odd power of $C(R_i)$

([4]) I shall use 'power of $C(R_i)$' as a convenient general term for $C(R_i)$, $C^2(R_i)$, $C^3(R_i)$, ..., instead of using the more clumsy term 'power of C applied to R_i'. (Strictly speaking, the nth power of $C(R_i)$ would be the nth power of the relation $C(R_i)$; but as there is no danger of a confusion here, this simplification may be permissible).

must also be components of some even power of $C(R_i)$. If $h \geqslant N_i$, then $C^{h+q}(R_i)$ involves all the components of any odd power of $C(R_i)$. It follows that the set of components of the odd powers $C^p(R_i)$, $p \geqslant N_i$, is identical with the set of components of the even powers $C^q(R_i)$, $p \geqslant N_i$, and hence $C^{m+k}(R_i) = C^m(R_i)$, for $m \geqslant N_i$ and all positive integers k.

m) If there exists a basic relation R_j which is a component of both $C^h(R_i)$ and $C^{h+1}(R_i)$ for a suitably chosen exponent h, then $C^m(R_i) = C^{m+k}(R_i)$, for all $m \geqslant N_i$ and all positive integers k.
Proof: Because of the symmetry of C and its powers, $C^h(R_i) = R_j \vee \ldots$ implies $C^h(R_j) = R_i \vee \ldots$, and therefore $C^{2h+1}(R_i) = C^hC^{h+1}(R_i) = C^h(R_j) \vee \ldots = R_i \vee \ldots$. Thus there exists an odd power of $C(R_i)$ which involves R_i as a component. Using theorem l) we conclude that $C^{m+k}(R_i) = C^m(R_i)$, for $m \geqslant N_i$ and $k = 1,2,3, \ldots$.

3. *The Index of a Relation Schema; Stable, Alternating and Regular Relation Schemata*

With each of the basic relations R_i of a relation schema is associated a positive integer N_i, as we have seen above; $C^n(R_i)$, for $n > N_i$, is either constant or shows an alternating pattern. Let us now define the *index* of a relation schema as the smallest positive integer N which is equal to, or larger than, each of the numbers N_i associated with the basic relations of the schema. Evidently, N is the index of a given relation schema only if we have, for all positive integers k, either $C^N = C^{N+k}$, or $C^N = C^{N+2k}$ and $C^{N+1} = C^{N+2k+1}$, with (in the second case) $C^N \neq C^{N+1}$. Since the correspondence C is represented by the matrix C, it follows that either $C^N = C^{N+k}$, for all positive integers k, or $C^N \neq C^{N+1}$, $C^N = C^{N+2k}$ and $C^{N+1} = C^{N+2k+1}$, for all positive integers k. We may thus classify relation schemata further depending upon whether C^m is constant or alternating for $m \geqslant N$. We call a relation schema, its g-converse C and the corresponding matrix C *stable* if, and only if, $C^N = C^{N+k}$, for all positive integers k; otherwise we call the schema, C and the matrix C unstable or *alternating*.

a) If all the basic relations of a relation schema are non-asymmetric, then the schema is stable.

The proof follows at once from 2j) and the first paragraph of its proof.

b) If a relation schema contains at least one pair of basic relations R_i, R_j ($R_i \neq R_j$) which are asymmetric and for which $C(R_i) = R_j$ and $C(R_j) = R_i$, then the schema is alternating.
Proof: We have $C^2(R_i) = CC(R_i) = C(R_j) = R_i$. If $C^{2m}(R_i) = R_i$, then $C^{2m+2}(R_i) = R_i$; for $C^{2m+2}(R_i) = C^{2m}(R_i)$. On the other hand, $C(R_i) = R_j$; and if $C^{2m+1}(R_i) = R_j$, then $C^{2m+3}(R_j) = R_j$ as is easily verified. Thus there is no positive integer h such that $C^h(R_i) = C^{h+k}(R_i)$, for all positive integers k, which is sufficient to establish that the relation schema cannot be stable and must therefore be alternating.

c) However, a relation schema may include asymmetric basic relations without being alternating, as the following simple example shows. Let X be the set of natural numbers, and let the first basic relation R_1 be the relation 'larger than' and the second basic relation R_2'smaller than or equal to'. We obtain a relation schema of dimension 2 whose g-converse is given by the matrix

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

Although R_1 is an asymmetric relation, the schema is stable and its index is 2. For we have

$$C^2 = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = C^3 = C^4 = \ldots = C^n = \ldots .$$

d) A relation schema is *alternating* if, and only if, there exist at least two proper and non-empty subsets S and S' of the basis such that (i) S and S' are disjoint and (ii) $C(R_i) \varepsilon$ S', for any $R_i \varepsilon$ S, and $C(R_j) \varepsilon$ S', for any $R_j \varepsilon$ S'.
Proof: Let us first show that if (i) and (ii) hold, then the relation schema is alternating. Under the conditions stated $R_i \varepsilon$ S implies $C(R_i) \varepsilon$ S', and $C^2(R_i) \varepsilon$ S. Furthermore, for any positive integer m, if $C^{2m-1}(R_i) \varepsilon$ S', then $C^{2m}(R_i) = CC^{2m-1}(R_i) = C(R_j)$, where $R_j \varepsilon$ S' and $C(R_j) \varepsilon$ S. Thus $C^{2m-1}(R_i) \neq C^{2m}(R_i)$, for all m; and

from this we derive that the schema cannot be stable. On the other hand, if the schema is an alternating one, there exists at least one basic relation R_i for which it is not the case that $C^m(R_i) = C^{m+p}(R_i)$, for $m \geqslant N_i$ and $p = 1,2,3,\ldots$. Then, according to theorem 2m), $C^r(R_i)$ and $C^{r+1}(R_i)$ cannot have any common components, for any positive integer r. Hence there exist two disjoint sets T and T' of basic relations, T containing the basic relations which are components of $C^{2m}(R_i)$, and T' those which are components of $C^{2m+1}(R_i)$. Moreover, if C is applied to any of the basic relations in T, then we obtain a basic relation in T' and vice versa. That is, T and T' satisfy the conditions for the sets S and S', as stated above.

e) The three preceding theorems suggest the following distinction: an asymmetric relation R is called *fully asymmetric* with regard to a g-converse C, if, and only if, none of the components of R are components of any odd power of C(R). Otherwise R is said to be *partially asymmetric* with regard to C.
The same relation R may be fully asymmetric or partially asymmetric depending upon the g-converse under consideration. Thus the relation 'larger than' (R_1), defined on the domain of natural numbers, say, is fully asymmetric with regard to the relation schema of linear order whose g-converse is given by the matrix

$$C = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix} \quad ;$$

for $C(R_1) = R_3 = C^3(R_1) = C^5(R_1) = \ldots = C^{2m-1}(R_1) = \ldots$, and $C^2(R_1) = R_1 = C^4(R_1) = C^6(R_1) = \ldots = C^{2m}(R_1) = \ldots$. However, in the relation schema mentioned under c) above, R_1 is a partially asymmetric relation. The reader will readily confirm, using theorem d), that a relation schema is alternating if, and only if, it contains at least two fully asymmetric basic relations.

f) If C is a one-one-correspondence of the basis onto itself, then, and only then, C^2 equals the identity I, and C equals its inverse C^{-1}.
Proof: If C is a one-one-correspondence of the basis onto itself, then there is to each basic relation R_i one, and only one, basic relation R_j (which may or may not be identical with R_i) such that $C(R_i) = R_j$. As the same applies to R_j and as $C^2(R_i)$ must involve

R_i, we conclude that $C^2(R_i) = C(R_j) = R_i$. Conversely, if $C^2 = I$, then $C^2(R_i) = R_i$. Given there exists a basic relation R_i such that $C(R_i) = R_j \vee R_k \vee \ldots$, (so that $R_j \neq R_k$ and at least one of the components is not identical with R_i, then we have on the one hand $C^2(R_i) = C^2(R_j) \vee C^2(R_k) \vee \ldots = R_j \vee R_k \vee \ldots$, on the other hand $C^2(R_i) = R_i$, which is impossible.

g) Let us call a relation schema *regular* if, and only if, the second power of its g-converse equals the identity correspondence I; in this case we shall call both its g-converse C and the matrix *C* regular. (To avoid confusions let us term a matrix whose determinant is not zero non-singular). Obviously, every regular g-converse is its own inverse, i.e. $C^{-1} = C$.

h) If all basic relations of a schema are symmetric, then $C = I$, and vice-versa.
Proof: R_i is symmetric if, and only if, for all w,y, $R_i(x,y)$ implies $R_i(y,x)$. Hence, R_i must be a component of $C(R_i)$. If $C(R_i)$ had an additional component R_j, then there would exist a pair (x,y) such that the conjunction $R_i(x,y).R_j(y,x)$ holds, which contradicts the assumption that R_i is symmetric. Hence C must be a one-one-correspondence of the basis onto itself, with $C = I$.
Conversely, if $C = I$, then $C(R_i) = R_i$ for all basic relations R_i. If $C = I$, then $C^2 = I$; hence all *symmetric* relation schemata are regular.

i) The index of a regular relation schema equals 1. Furthermore, a regular relation schema is stable if, and only if, it is symmetric.
Proof: A regular schema C must fulfil the condition $C^2 = I = C^4 = C^6 = \ldots = C^{2m} = \ldots$. Now, either $C = I$, or $C \neq I$. In the first case the schema is stable with index 1. In the second case we have $C = C^3 = C^5 = \ldots = C^{2m-1} = \ldots$, i.e. an alternating schema with index 1. Clearly, the schema can only be stable if $C = I$, that is, if the schema is symmetric.

j) If C is regular, then the determinant of *C* is either 1 or −1. For regular stable schemata we have $\det C = 1$; but if $\det C = -1$ and the schema is regular, then it is also alternating.
Proof: The determinant of *C* is defined as a sum of n! terms (where n is the dimension of the schema) each of which is the product of n elements of *C*, one from each row and one from each column. But if C is regular, then *C* has exactly one digit 1 in each row and one

digit 1 in each column. Therefore only one of the n! terms is not zero, and this term equals 1^n. Hence the determinant must be either 1 or -1, depending upon the sign of the permutation.
If C is regular and stable, then $C = I = C^2 = C^3 = C^4 = \ldots = C^n = \ldots$, according to 3g) and 3i). But det $I = 1$. From this we deduce that any regular relation schema with det $C = -1$ cannot be stable and must therefore be an alternating schema.

k) The schema of linear order mentioned under e) is regular and alternating; the determinant of its g-converse equals -1. But there are, of course, regular alternating relation schemata with det $C = 1$. Of this type are, for instance, schemata whose g-cinverses are given by

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix} \text{, or by } C = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

l) Let us introduce another definition which often useful: a relation schema (or its g-converse C, or the matrix C) is sais to be *bounded* if, and only if, for all basic relations of the schema and all positive integers m, $C^m(R_i) \neq R_u$, where R_u is the universal relation. Obviuosly, all regular and all alternating schemata are bounded. Furthermore, if $C^m(R_j) = R_u$, for some basic relation R_j, and some positive integer m, then the schema is not bounded. Indeed, because of the symmetry of C^m, R_j must be a component of $C^m(R_i)$ for every basic relation R_i of the schema; and we have $C^{2m}(R_i) = C^mC^m(R_i) = C^m(R_j) \vee \ldots = R_u$. Thus either$C^k(R_i) \neq R_u$, for all positive integers $k \geqslant N$ and all basic relations R_i, or the schema is bounded.

4. *Similar Matrices* C

One and the same correspondence C can be represented by different matrices depending upon the numeration of the basic relations of the relation schema in question. For instance, in the relation schema of lattice order mentioned under c) the g-inverse is represented by

$$C = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} .$$

However, if we denote the relation 'larger than' with R_2 and 'smaller than or equal to' with R_i we obtain the matrix

$$C' = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} .$$

Two matrices which represent one and the same correspondence C are said to be similar.

When we speak of the basis of a relation schema of dimension n we mean an ordered set of n relations: $R_1, R_2, \ldots, R_n$. Consequently, we can change the basis by changing the order of the basic relations. A relation schema of dimension n thus admits of n! different bases, n! being the number of permutations of n elements. The transition from one such basis to another is a one-one-correspondence U of the set of basic relations onto itself. This transition is defined if each R'_i is represented by means of the basic relations $R_1, R_2, \ldots, R_n$:

$$R'_i = \bigvee_{k=1}^{n} u_{ki} R_k,$$

where each of the numbers u_{ki} is either 0 or 1. The transition from one basis to another may thus be given by a quadratic transformation matrix

$$U = \begin{bmatrix} u_{11} & u_{12} & \ldots & u_{1n} \\ u_{21} & u_{22} & \ldots & u_{2n} \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ u_{n1} & u_{n2} & \ldots & u_{nn} \end{bmatrix} ,$$

the i-th column of which is the representation of the basic relation R'_i; each of its rows and each of its columns contain exactly one digit 1 while all the other digits are zeros. The elements of U and the elements t_{ik} of the transpose of U satisfy the condition

$$u_{ij} \cdot t_{jk} = u_{ij} \cdot u_{kj} = \begin{cases} 1 \text{ if } i = k \\ 0 \text{ if } i \neq k. \end{cases}$$

That is, U is 'othogonal', and its inverse equals its transpose: $U^{-1} = U^{T}$.

Let $C = (c_{ij})$ represent the g-converse with regard to the basis B_n. The matrix C' representing the same g-converse with regard to the basis B'_n then fulfils the equation

$$C' = U^{T}CU$$

Proof: $C(R_j) = \bigvee_{i=1}^{n} c_{ij}R_i$, $C(R'_j) = \bigvee_{i=1}^{n} c'_{ij}R'_i$, and $R'_i = \bigvee_{k=1}^{n} t_{ki}R_k$.

Therefore

$$C(R'_j) = \bigvee_{h=1}^{n} u_{hj}R_h = \bigvee_{h=1}^{n} \bigvee_{i=1}^{n} u_{hj}c_{ih}R_i.$$

On the other hand

$$C(R'_j) = \bigvee_{i=1}^{n} c'_{ij}R'_i = \bigvee_{i=1}^{n} \bigvee_{k=1}^{n} c'_{ij}u_{ki}R_k.$$

Comparing these expressions we obtain

$$\bigvee_{h=1}^{n} \bigvee_{k=1}^{n} u_{hj}c_{kh}R_k = \bigvee_{i=1}^{n} \bigvee_{k=1}^{n} c'_{ij}u_{ki}R_k$$

and $CU = UC'$ or $C' = U^{T}CU$.

Similarity is an equivalence relation as is obvious from its definition. Moreover, if C is similar to C', then C^{m} is similar to C'^{m}, for any positive integer m. For, let us suppose that $U^{T}C^{k-1}U = C'^{k-1}$, then $U^{T}CUU^{T}C^{k-1}U = C'C'^{k-1}$ and $U^{T}C^{k}U = C'^{k}$. Stability, alternating character, the index N and boundedness of a matrix C are all properties which are invariant under transformations U of the type considered above; this should be evident as these properties have been defined in terms of the correspondences C rather than the matrices C representing them.

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