RELEVANT IMPLICATION AND PROJECTIVE GEOMETRY

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In this paper I give an exposition of the recently discovered connections between relevant implication and projective geometry. One of the consequences of this connection is a simple proof that the propositional logic KR (an extension of the logic R) is undecidable. This proof can be generalized to a proof that any logic between the positive system T_+ of ticket entailment and KR is undecidable [17]. The proof of this result, however, is of necessity long, complex and formal. I present here a separate proof of the undecidability of KR, which is intuitively easy to grasp. An understanding of the paper is excellent preparation for reading [17]. A more important motivation for the paper is to point out the great wealth of ideas, problems and constructions that flow from the connection between geometry and relevant logics, which turns out to be surprisingly intimate.

To avoid cluttering the paper with repetitive material, I assume that the reader is familiar with the basic system R of relevant implication and the fundamentals of its model theory, due to Routley and Meyer [12].

1. *Models for relevant logics*

The present advances in the understanding of R came about (like many advances in logic) by the discovery of a new method for constructing models. Although the basic semantical analysis of R has been around for over a decade, until quite recently disappointingly few *examples* of R model structures were known. If you omit negation, then you can use semilattices to model R_+ [16]. However, semilattice models fail in the worst possible way to extend to the full system R; only the one element semilattice can be used to validate all of R. In the early 1970's only the following models for R were known:

the Sugihara matrix and its finite versions ([1] pp. 335-338) and various small matrices derived by fiddling with many-valued truth-tables, one of which is generalized to an infinite family of models in Belnap [2]. The list of small models was enormously extended by a computer search using some remarkable programs written by Slaney, Meyer, Pritchard, Abraham and Thistlewaite (for an early progress report on this research effort the reader is referred to Slaney's thesis [14]). These programs churned out huge quantities of R matrices and model structures of all shapes and sizes. Clearly, there are lots and lots of R model structures out there! But what are they like? Can we classify them in some intelligent fashion? Are there general constructions that produce interesting examples? The answer to the first two questions is still obscure, though clearer than it was. The answer to the last question is an emphatic "yes!".

I confess here to an old antipathy (now abandoned) to the Routley/Meyer semantics. My dislike of the model theory was based on the unexamined prejudice that it was impossible to "get a picture" of R model structures, in seeming contrast to the case of semilattice models and Kripke-style modal semantics. The main purpose of this paper is to convince you that it is extremely easy to "get a picture" of R model structures. In a literal sense, these models have been staring us in the face for a long time.

2. *The logic KR*

To those who have taken the trouble to read the literature on relevant logic rather than fulminate against it, it has been a familiar fact since the early 70's that there are two conceptually distinct classes of "paradoxes of material implication". The archetype of the first class (paradox of consistency) is $(A \,\&\, \neg A) \rightarrow B$. The archetype of the second (paradox of relevance) is $A \rightarrow (B \rightarrow A)$. It is easy to devise systems of entailment which omit one but not the other. Thinking about the system R, we can see immediately that if we add $A \rightarrow (B \rightarrow A)$ then the result is classical logic with paradoxes of both types. However, the consequences of adding $(A \,\&\, \neg A) \rightarrow B$ to R are not so clear. Here we have a system of relevant logic with regular classical Boolean negation, satisfying all the natural postulates of

R-style negation, including contraposition. The credit for investigating the resulting system KR belongs to Adrian Abraham, R. K. Meyer and R. Routley (see [13] for details of their investigations).

Parenthetically it should be noted that KR is *not* the same as the classical relevant logic CR investigated by Meyer and Routley [9]. This system adds a classical negation operator to R, which is distinct from the negation proper to R. In KR, classical negation and relevant negation are *identified*. One's initial reaction to KR is that it is probably a trivial system, if it doesn't simply collapse into classical logic. As we shall see, this reaction could hardly be wider of the mark. The first indication that KR is indeed non-trivial came from the computer, which churned out reams of interesting KR matrices. In retrospect, this is hardly surprising, because we now know that KR models can be manufactured *ad lib* from projective geometries.

First, though, some definitions. A KR model structure (krms) is a 3-place relation Rabc on a set containing a distinguished element 0, satisfying the postulates:

1. $R0ab \Leftrightarrow a = b$
2. $Raaa$
3. $Rabc \Rightarrow (Rbac \,\&\, Racb)$ (total symmetry)
4. $(Rabc \,\&\, Rcde) \Rightarrow \exists f\,(Radf \,\&\, Rfbe)$ (Pasch's postulate).

Note that a krms is just an rms in the sense of Routley and Meyer, except that we have imposed total symmetry by setting $a^* = a$ for all a. We define truth and falsity with respect to a krms exactly as for an rms, except for negation. Writing "A is true at a" as "$a \models A$", the crucial clauses are:

$$a \models \neg A \Leftrightarrow a \not\models A$$
$$a \models A \rightarrow B \Leftrightarrow \forall bc\,((b \models A \,\&\, Rabc) \Rightarrow c \models B).$$

A slight modification of the usual completeness proof for R shows that KR is complete with respect to the class of all KR model structures.

The total symmetry condition seems especially odd on first acquaintance. To explain how we can construct such strange models in profusion we turn to the theory of projective geometry.

3. *Projective spaces*

In this section, I give a summary of standard material on projective spaces. There are numerous good textbooks on projective geometry. I found the books of Garner [4], Hartshorne [6] and Mihalek [10] helpful; also, the classic by Veblen and Young [18] is very inspiring reading. For the lattice-theoretic approach to projective geometry Birkhoff [3] and Grätzer [5] should be consulted.

DEF. 3.1. A projective space consists of a set of *points* P and a collection of subsets of P called *lines*, satisfying the two conditions:

P1. Two distinct points a,b lie on (i.e. belong to) exactly one line a + b.

P2. If a,b,d,e are distinct points such that some point c lies on both a + b and d + e, then there is a point f lying on both a + d and b + e (see Fig. 1).

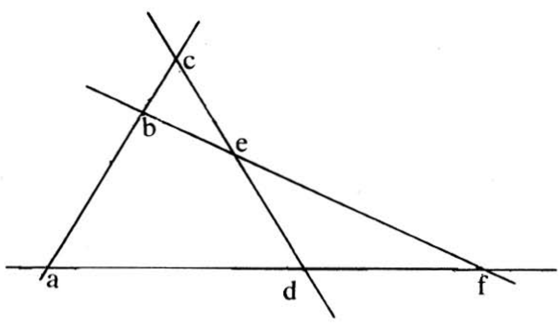

Figure 1

A projective space is said to be irreducible if it satisfies:

P3. No line contains exactly two points.

We shall also make use of the additional postulate:

P4. No line contains exactly three points.

The most familiar example of a projective space is ordinary Euclidean 3-space, enriched by the addition of a point at infinity for each parallelism class in ordinary 3-space, together with the plane and lines at infinity. This is real projective 3-space.

We now define the notion of collinearity. Various ways of doing this are possible. The method adopted by most texts is to define points a,b,c to be collinear if they all lie on a single line. This definition, however, is not suitable for our purposes, because it is "too fat"; it counts as collinear any triple containing repeated points. Instead we use:

DEF. 3.2. If P is a projective space, the collinearity relation Cabc in P is defined by:
Cabc iff (a) $a = b = c$ or (b) a,b,c are distinct and lie on a common line.

Note that if we define $a + a = \{a\}$, Cabc can be given the symmetric definition: Cabc $\Leftrightarrow (a + b = b + c = a + c)$.

LEMMA 3.3 Let P be a projective space satisfying P4. Then the collinearity relation on P satisfies:

1. Cabc $\Rightarrow$ (Cbac & Cacb)
2. Cabb $\Leftrightarrow a = b$
3. (Cabc & Ccde & $a \neq d$) $\Rightarrow \exists$f (Cadf & Cfbe)
4. (Cabc & Cbcd & $a \neq d$) $\Rightarrow$ Cabd.

Proof: A straightforward calculation. The postulate P4 is needed to validate the instances of 3. where $a = b = c$. □

A set of points X in a projective space P is a (linear) subspace of P if:

$$a,b \in X \ \&\ Cabc \Rightarrow c \in X.$$

The family of all linear subspaces of P forms a complete lattice, ordered by containment, in which the lattice join of two subspaces X and Y is

$$X + Y = \bigcup (a + b : a \in X \ \&\ b \in Y).$$

Projective spaces can be characterized by means of their linear subspaces.

DEF. 3.4 A modular geometric lattice is a complete lattice $\langle L, \wedge, \vee \rangle$ satisfying:

1. $a \geqslant c \Rightarrow a \wedge (b \vee c) = (a \wedge b) \vee c$;
2. Every element of L is a join of atoms in L;
3. Every atom in L is compact; that is, $a \leqslant \bigvee X$ implies $a \leqslant \bigvee Y$ for some finite $Y \subseteq X$.

FACT 3.5 The subspace lattices of projective spaces are exactly the modular geometric lattices (up to isomorphism).

For these and other classical results on projective spaces, see Gratzer Chapter IV and Birkhoff Chapter IV.

Given a modular geometric lattice, a projective space can be defined by taking the points as the atoms and the lines all sets of atoms of the form $\{x : x \in a + b\}$, where $a \neq b$. The concepts of projective space and modular geometric lattice are thus completely equivalent.

An enormous variety of projective spaces can be constructed from vector spaces. Let V be a vector space over a division ring (= skew field). Define a point to be one-dimensional subspace and a line to be a 2-dimensional subspace of V. The result is a projective space. This construction applied to the space of 4-vectors over the real numbers produces real projective 3-space.

The language of pure lattice theory is the equational language containing variables $x_1, x_2, \ldots$, constants 0 and 1 and symbols for lattice meet $x \wedge y$ and lattice join $x \vee y$. If K is a class of lattices, the word problem for K is the problem of determining whether or not a given equation can be deduced from a given finite set of equations in the equational theory determined by K.

The undecidability result for KR is based on the following important result proved independently by Hutchinson [7] and Lipshitz [8].

FACT 3.5 Let K be a class of modular lattices which contains the subspace lattice of an infinite-dimensional projective space. Then the word problem for K is unsolvable.

4. *Model structures constructed from projective spaces*

With the notion of collinearity in a projective space to hand, it is not hard to see how we can construct examples of KR model structures. The following construction is general enough for our needs.

LEMMA 4.1 Let C be the collinearity relation in an irreducible projective space satisfying P4. Add a new element 0 and define R to be the smallest totally symmetric relation on $P \cup \{0\}$ containing ƒ. Then $\langle P \cup \{0\}, R \rangle$ is a krms.

Proof: An easy verification, using Lemma 3.3. □

We now have a copious supply of highly non-trivial KR model structures. But how general is the construction? The following lemma answers this question to some extent by showing that the connection between KR and projective geometry is very intimate. It is from the following simple but powerful lemma that all the undecidability results flow.

The definition of linear subspace used in the previous section carries over directly to KR model structures, substituting Rabc for Cabc in the definition.

LEMMA 4.2 Let M be a KR model structure. The non-empty linear subspaces of M form a modular lattice. If M satisfies the condition: (*) Rabb $\Rightarrow$ a = 0 or a = b then the subspace lattice is geometric.

Proof: Before proceeding to the proof proper, we pause to note that the lattice join and meet can be expressed in the language of KR. Lattice meet corresponds to conjunction $A \wedge B$, join corresponds to $A \circ B = \neg(A \rightarrow \neg B)$.

This last formula is the definition of the fusion connective, which as Meyer observed some time ago is the key connective in relevant logics, rather than $\rightarrow$. Here it turns up as a lattice join operation; a somewhat surprising role for the connective to play, since the definition above is the classical definition of *conjunction*.

Now for modularity. Let A, B and C be subsets of M, with $C \subseteq A$ and A a linear subspace of M. If $x \in A \wedge (B \circ C)$, then $x \in A$ and $x \in B \circ C$. Thus there exist y,z such that Rxyz, $y \in B$, $z \in C$, hence $z \in A$. Since $x \in A$, $z \in A$ and Rxyz, $y \in A \circ A = A$ (remember, A is a linear subspace). Thus $y \in A \wedge B$, so that $x \in (A \wedge B) \circ C$.

The second part of the lemma follows easily from the fact that if M satisfies (*) then any set of the form {0,a} is a linear subspace. □

Let's take stock! At this point we have shown that every KR model structure has associated with it in a natural way a modular lattice, which in an important special case is the lattice of subspaces of a projective space. This is enough to give us undecidability. To state the result precisely we need to specify the translation of lattice equations into the language of KR. The translation $(\varphi = \psi)^t$ of a lattice equation $(\varphi = \psi)$ is defined as follows:
$(x_i)^t = p_i$, $(\varphi \wedge \psi)^t = \varphi^t \,\&\, \psi^t$, $(\varphi \vee \psi)^t = \varphi^t \circ \psi^t$, $0^t = t$, $1^t = T$. Now let $I = ((\varphi_1 = \psi_1) \,\&\, \ldots \,\&\, (\varphi_n = \psi_n)) \Rightarrow \delta = \in$ be an implicational formula of pure lattice theory. The translation of I is:

$I^t = ((L(p_1) \,\&\, \ldots \,\&\, L(p_m) \,\&\, (\varphi_1 = \psi_1)^t \,\&\, \ldots \,\&\, (\varphi_n = \psi_n)^t \Rightarrow (\delta = \in)^t$
where $L(A)$ abbreviates $((A \circ A) \leftrightarrow A) \,\&\, t \rightarrow A)$ and $p_1, \ldots, p_m$ contain all the variables in the translation of the lattice equations.

5. *Undecidability*

Now all we have to do is put together Fact 3.5 and Lemma 4.1 and we have proved the undecidability of KR. Actually, we've proved a good deal more. Let P be an irreducible projective space satisfying P4 and L(P) the logic determined by the model structure constructed from P using the procedure of Lemma 4.1.

THEOREM 5.1 Let L be a logic intermediate between KR and L(P), where P is an infinite-dimensional irreducible projective space satisfying P4. Then L is undecidable.

Proof Let M be a krms constructed from a projective space P. It is easy to see that the lattice of linear subspaces of M is isomorphic to the lattice of linear subspaces of P. Furthermore, an implication I holds in this lattice if and only if its translation I^t is valid in the model structure. The undecidability of L now follows immediately from Fact 3.5. □

This theorem is a powerful result. It suggests that we can easily get an undecidability proof for R by some little trick, like embedding KR into R. Originally I thought I had such an embedding, but Bob Meyer soon disabused me of *that* idea. The basic trouble is that the conditions $A \& \neg A \leftrightarrow F \leftrightarrow B \& \neg B$ do not extend inductively to $(A \circ B)$, so that there is apparently no simple way to embed KR in R.

To get an undecidability result for R, we have to pull aside the rug which conceals the trap door leading to the hidden treasures on a lower level. In Theorem 5.1 we have simply used the Hutchinson/Lipshitz result without examining its proof. To deal with R, we need to dig a bit deeper and look at their actual construction, which turns out to be very interesting. They used the von Neumann coordinatization theorem for modular lattices, a powerful technique whose history we now briefly sketch.

Let P be any projective space satisfying the Desargues theorem. Then the lattice of subspaces of P is isomorphic to the subspace lattice of a vector space over a division ring D. This classical result was proved by von Staudt by the "algebra of throws". To coordinatize the space P we single out a fixed line in the space and choose three distinct points on the line as the zero, unit and point at infinity. We can then define multiplication and addition for points on the line using purely geometrical constructions (see Veblen and Young Vol. 1 Chapter 6 and Grätzer pp. 208-210 for details). The resulting algebra is the division ring D. Thus we have constructed an algebra from purely geometrical material; to those familiar with the history of geometry it should come as no surprise that the ancestry of the von Staudt constructions can be traced to the Eudoxan theory of proportion. Figure 2 illustrates the definition of multiplication.

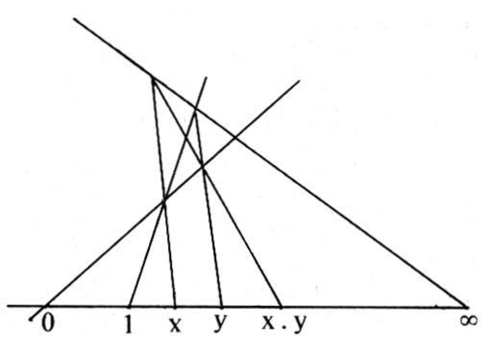

Figure 2

A far-reaching generalization of the von Staudt construction was introduced by von Neumann in the 1930's. He observed that the use of the Desargues theorem could be avoided by postulating the existence of an appropriate coordinate frame. With this modification we can construct a ring with which we can coordinatize any complemented modular lattice containing a 3-dimensional coordinate frame satisfying certain added conditions; for a detailed account of this result the reader is referred to the classic [19].

Von Neumann's proof that the ring multiplication is well-defined and associative uses only the modularity of the given lattice. This observation leads directly to the proof of Fact 3.5, because the existence of a coordinate frame can be expressed in terms of pure lattice equations, and any countable semigroup can be imbedded in the multiplicative semigroup of the von Neumann ring constructed from an infinite-dimensional projective space.

To prove undecidability for R, we have to generalize the construction still farther. First, let's go back and examine the proof of modularity in Lemma 4.2. We can extend the proof to R in a simple way to derive the following result: in an R model structure if A is a linear subspace satisfying $(A \mathbin{\&} \neg A) = \emptyset$, and $C \subseteq A$ then for any B, $A \wedge (B \circ C) = (A \wedge B) \circ C$, that is, A is *modular*. Now if we turn to the multiplication operation in the von Neumann construction, we find that the proof that the operation is well-defined and associative uses only a finite number of instances of modularity; to be precise, von Neumann needs modularity only for elements of the coordinate frame. The combination of these two observations gives us undecidability for R.

The extension to E, T and the positive systems is a messy business which employs the Glivenko double negation construction. The unpleasant details are all to be found in [17]. This particular mix of ingredients is enough to give us the general result:

THEOREM 5.2 Let L be a positive logic (expressed in terms of $\wedge$, $\vee$, $\rightarrow$) intermediate between T_+ and the positive part of L(P), where P is an infinite-dimensional projective space satisfying P3 and P4. Then L is undecidable.

This theorem seems to be just about the best that can be done using these geometrical techniques. Any farther advance on the decision problems for relevant logics (such as the still open decision problem for the semilattice system of [16]) would seem to require some new ideas.

6. *More geometrical ruminations*

As I said at the beginning, one of my main aims in writing this paper was to make logicians aware of the rich possibilities offered by the techniques of classical synthetic geometry in the field of relevant logics. As far as decidability goes, the techniques are probably close to exhaustion; but elsewhere the surface has hardly been scratched. For example, how can we axiomatize the logic determined by the class of all models constructed from projective spaces?

In conclusion, I give one more small result which shows the kind of theorems which can be proved by simply adapting known techniques from geometry and lattice theory.

THEOREM 6.1 There are continuum many logics intermediate between KR and classical logic.

Proof: For p a prime number let L(p) be the subspace lattice of the projective plane coordinatized by the p-element field. By a result of Baker (see Gratzer [5] pp. 239-240), distinct subsets of $\{L(p) : p$ a prime, $p > 2\}$ generate distinct equational classes. It follows that the logics determined by the classes of model structures constructed from such distinct subsets are distinct extensions of KR. □

The connection between projective geometry and relevant logics is both simple and natural and it makes sense to ask why it was not investigated earlier. The connection with geometry was clear to Dunn in the early 1970's; it was Dunn who christened the crucial condition "Pasch's postulate". It is puzzling that the geometrical insight was not exploited until over a decade had expired. Unfortunately the penetrating remarks of Toohey [15] were not followed up, perhaps because of his obscure expository style. I can, however, give a reason in my own case which has to do with the vagaries of geometrical terminology. The postulate we have been calling "Pasch's postulate" is in fact due to Peano [11]. The original Pasch axiom on which Peano improved says that if a line passes through one side of a triangle it either passes through another side or through a vertex. A large number of geometry books attribute the Peano axiom to Pasch. When I went to look up Pasch's postulate in a textbook, however, I found the original Pasch axiom which looks so little like the Peano version that I immediately abandoned geometrical interpretations as hopeless. With only a little more effort I would have discovered that the bloodthirsty troll barring my way could be overcome with the greatest of ease.

There are lots of other possibilities for constructing interesting model structures out of geometries. For example, by using two copies of a geometry, we can construct rms's which are not krms's. In another direction, we can construct rms's from geometrical spaces satisfying the classical axioms of betweenness.

However, I hope that by this time the reader is inspired to explore these possibilities independently, and hence discover some of the wide unexplored territory lying between relevant logics and classical geometry.

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REFERENCES

[1] Anderson A. R. and Belnap N. D. Jr., *Entailment* Vol. 1, Princeton University Press, Princeton and London 1975.

[2] Belnap N. D. Jr., Intensional models for first degree formulas, *The Journal of Symbolic Logic*, Vol. 32, pp. 1-22.

[3] Birkhoff G., *Lattice Theory*. Third edition. American Mathematical Society, Providence, Rhode Island 1967.

[4] Garner L. E., *An outline of projective geometry*, North Holland, New York and Oxford 1981.

[5] Gratzer G., *General lattice theory*, Academic Press, New York and San Francisco 1978.

[6] Hartshorne R., *Foundations of projective geometry*, W. A. Benjamin, 1967.

[7] Hutchinson G., Recursively unsolvable word problems of modular lattices and diagram-chasing, *Journal of Algebra*, Vol. 26, (1973) pp. 385-399.

[8] Lipshitz L., The undecidability of the word problems for projective geometries and

modular lattices, *Transactions of the American Mathematical Society*, Vol. 193 (1974), pp. 171-180.

[9] Meyer R. K. and Routley R., Classical relevant logics (I), *Studia Logica*, Vol. 32 (1973), pp. 51-66.

[10] Mihalek R. J., *Projective geometry and algebraic structures*, Academic Press, New York and London 1972.

[11] Peano G., *I principii di geometria logicamente esposti*, Bocca, Turin, 1889.

[12] Routley R. and Meyer R. K., The semantics of entailment (I), *Truth, syntax, modality*, ed. H. Leblanc, North Holland, Amsterdam 1973 pp. 199-243.

[13] Routley R., Meyer R. K., Plumwood V. and Brady R., *Relevant logics and their rivals*, Ridgeview Publishing Company, California 1983.

[14] Slaney J. K., *Computers and relevant logic: A project in computing model structures for propositional logics*, Australian National University, 1980.

[15] Toohey B., Lines of reasoning, unpublished lecture notes, University of Wollongong 1974.

[16] Urquhart A., Semantics for relevant logics, *The Journal of Symbolic Logic*, Vol. 37 (1972), pp. 159-169.

[17] Urquhart A., The undecidability of entailment and relevant implication, forthcoming *The Journal of Symbolic Logic*.

[18] Veblen O. and Young J. W., *Projective Geometry*, Vol. 1, Ginn, New York 1910.

[19] Von Neumann J., *Continuous geometry* (I. Halperin ed.), Princeton University Press, Princeton, New Jersey, 1960.