PROBABILISTIC SEMANTICS FOR ORTHOLOGIC AND QUANTUM LOGIC

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In [3], Goldblatt gives an axiomatic characterization of consequence relations defining two propositional logics; one he calls "orthologic", and the other he calls "quantum logic". His characterization of the consequence relation may conveniently be thought of as a natural deduction system for the two logics. Goldblatt sketches a proof that orthologic is characterized by the class of ortholattices, and he claims a similar result holds for quantum logic and orthomodular lattices. He then goes on to provide a semantics of the possible worlds sort for each of the logics. In this paper we provide an alternative semantics based on conditional probability theory, and we prove soundness and completeness results for both logics. The probability theories are shown to be non-trivial (i.e., not restricted to a finite number of values). We discuss an application of the probability functions to bets on theorems provable from an unknown set of assumptions.

1. *Syntax*

We assume expressions of our formal language are built up in the usual way from a denumerable number of sentence letters $p_1, p_2, \ldots$, and the sentence connectives "$\wedge$" for conjunction and "$\sim$" for negation. We use left and right parentheses for punctuation; where parentheses are omitted from a conjunctive string, association is assumed to be to the left. We will use A, B, C, and D, with connectives and parentheses, as metalinguistic entities to stand for object language expressions. We will use Γ with standard set theoretic notation to stand for sets of expressions. We adopt the following definitions:

(D.1) $A \vee B =_{df} \sim(\sim A \wedge \sim B)$
(D.2) $A \rightarrow B =_{df} \sim A \vee (A \wedge B)$

(The "$\rightarrow$" notation is not used in [3].)

In the framework of [3], a logic L is defined as a set of ordered pairs of expressions from the language. To say "$\langle A, B\rangle \in L$" is to indicate that B can be inferred from A, i.e., that B is a consequence of A. We will use the notation "$A \vdash B$" instead of "$\langle A, B\rangle \in L$" to indicate that B is a consequence of A. The following axioms and rules governing the consequence relation are taken from [3].

(# 1) $A \vdash A$
(# 2) $A \wedge B \vdash A$
(# 3) $A \wedge B \vdash B$
(# 4) $A \vdash \sim\sim A$
(# 5) $\sim\sim A \vdash A$
(# 6) $A \wedge \sim A \vdash B$
(# 7) If $A \vdash B$ and $B \vdash C$, then $A \vdash C$.
(# 8) If $A \vdash B$ and $A \vdash C$, then $A \vdash B \wedge C$.
(# 9) If $A \vdash B$, then $\sim B \vdash \sim A$.
(#10) $A \wedge (A \rightarrow B) \vdash B$

The system orthologic is defined as the smallest logic satisfying conditions (#1-#9), while quantum logic is defined as the smallest logic satisfying conditions (#1-#10).

An expression A is said to be derivable from a set of expressions Γ, written as $\Gamma \vdash A$, if and only if for some finite number n of members of Γ, say $B_1, \ldots, B_n$, we have $B_1 \wedge \ldots \wedge B_n \vdash A$. An expression A is said to be a theorem, written $\vdash A$, if and only if $\{A \vee \sim A\} \vdash A$. We say that a set Γ of expressions is deductively closed if and only if for every expression A, if $\Gamma \vdash A$, $A \in \Gamma$. Obviously these notions are relative to the particular logic under consideration.

2. *Semantics*

For the purposes of comparison, we will briefly sketch the probabilistic semantics for classical logic before presenting the new material. When doing probabilistic semantics, it has been usual to consider functions defined on the set of ordered pairs of expressions of the language. The usual notation is "Pr(A, B) = r", which is read "the probability of A, given B, is r", where r is a real number such that $0 \leq r \leq 1$. The following constraints can be shown to exactly characterize classical propositional logic (see [4]):

(PR.1) $0 \leq \Pr(A, B) \leq 1$
(PR.2) $\Pr(A, A) = 1$
(PR.3) If $\Pr(B, D) = \Pr(C, D)$ for every expression D, then $\Pr(A, B) = \Pr(A, C)$ for every expression A.
(PR.4) If there is at least one expression C such that $\Pr(C, B) \neq 1$, then for every expression A, $\Pr(\sim A, B) = 1 - \Pr(A, B)$.
(PR.5) $\Pr(A \wedge B, C) = \Pr(A, C) \bullet \Pr(B, A \wedge C)$
(PR.6) $\Pr(A \wedge B, C) = \Pr(B \wedge A, C)$

A given probability function is said to certify an expression A just in case for every expression B, $\Pr(A, B) = 1$. An expression is said to be p-valid if and only if it is certified by every probability function. One can show (as is done in [4]) that the theorems of classical propositional logic are exactly the p-valid expressions, as long as the probability functions satisfy (PR.1-6). We will use the term "classical" to refer to restrictions (PR.1-6) and to the functions satisfying those restrictions.

For our purposes, it will be convenient to consider a slightly more general type of function. Instead of defining our probability functions on ordered pairs of *expressions*, we will define our functions on ordered pairs whose first element is an expression and whose second element is a set of expressions. Let EX be the set of all expressions of our language. Our probability functions, designated by "Ps", will be maps from $EX \times \mathscr{P}(EX)$ into the closed interval [0,1]. We will use "$Ps(A, \Gamma) = r$" to mean "the probability of A, given the assumptions in Γ, is r".

Using the set formulation yields several advantages. For one thing, the set representation of the "given" information follows actual practice more closely. It also allows for the explicit representation of an infinite number of background assumptions. And as we will see below, use of this form allows the formulation of general constraints characteristic of conditional probability, without consideration of the sentential connectives.

In order to have a reasonable *conditional* probability theory, we must impose the following constraints:

(PS.1) $0 \leq \mathrm{Ps}(A, \Gamma) \leq 1$
(PS.2) If $A \in \Gamma$, then $\mathrm{Ps}(A, \Gamma) = 1$.
(PS.3) $\mathrm{Ps}(A, \Gamma) \bullet \mathrm{Ps}(B, \Gamma \cup \{A\}) = \mathrm{Ps}(B, \Gamma) \bullet \mathrm{Ps}(A, \Gamma \cup \{B\})$

Condition (PS.1) sets the usual range, and corresponds directly tp (PR.1). (PS.2) is a slight generalization of (PR.2); if A is one of the assumptions given, then its conditional probability must be 1. (PS.3) is just a version of a classical theorem. In the more familiar form, the theorem follows directly from (PR.5) and (PR.6).

We have not imposed a condition parallel to (PR.3), which says any two expressions which are equally supported by every conceivable piece of evidence must support every conceivable hypothesis equally. The desired condition is just a simple consequence of (PS.2-3).

Theorem 1: If $\mathrm{Ps}(B, \Gamma) = \mathrm{Ps}(C, \Gamma)$ for every set Γ, then $\mathrm{Ps}(A, \Gamma \cup \{B\}) = \mathrm{Ps}(A, \Gamma \cup \{C\})$ for every set Γ.

Proof: Let B and C be arbitrary expressions such that:

(1.1) $\mathrm{Ps}(B, \Gamma) = \mathrm{Ps}(C, \Gamma)$ for every set Γ

Let Γ' be an arbitrary set of expressions. Then by (1.1) and (PS.2) we have:

(1.2) $\mathrm{Ps}(C, \Gamma' \cup \{B\}) = 1$

Let A be an arbitrary expression. Then from (1.2) we know:

(1.3) $\mathrm{Ps}(A, \Gamma' \cup \{B\} \cup \{C\}) = \mathrm{Ps}(C, \Gamma' \cup \{B\}) \bullet \mathrm{PS}(A, \Gamma' \cup \{B\} \cup \{C\})$

Applying (PS.3) to the right side of (1.3) gives:

(1.4) $\mathrm{Ps}(A, \Gamma' \cup \{B\} \cup \{C\}) = \mathrm{Ps}(A, \Gamma' \cup \{B\}) \bullet \mathrm{Ps}(C, \Gamma' \cup \{B\} \cup \{A\})$

But by (1.1) and (PS.2) we know:

(1.5) $\mathrm{Ps}(C, \Gamma' \cup \{B\} \cup \{A\}) = 1$

Putting (1.4) and (1.5) together gives:

(1.6) $Ps(A, \Gamma' \cup \{B\} \cup \{C\}) = Ps(A, \Gamma' \cup \{B\})$

Exchanging B and C in the above steps gives:

(1.7) $Ps(A, \Gamma' \cup \{B\} \cup \{C\}) = Ps(A, \Gamma' \cup \{C\})$

Hence from (1.6) and (1.7) we have the desired result:

(1.8) $Ps(A, \Gamma' \cup \{B\}) = Ps(A, \Gamma' \cup \{C\})$

Thus the proof of Theorem 1 is complete.

Conditions (PS.1-3) set up a framework for conditional probability functions which is independent of the connectives employed in the object language. We must now turn our attention to the restrictions appropriate for each of the connectives of the language.

In both logics, conjunction is treated in a Boolean way. Thus there seems to be no reason to deviate from the classical constraint (PR.5), so we require the following:

(PS.4) $Ps(A \wedge B, \Gamma) = Ps(A, \Gamma) \bullet Ps(B, \Gamma \cup \{A\})$

Note that given our constraint (PS.3), we can derive the semantic commutativity of conjunction from (PS.4). Hence we do not require an additional constraint corresponding to (PR.6).

Negation is slightly more problematic. Both logics under consideration require only that negation act as orthocomplement. Thus we cannot make the very strong restriction parallel to the classical requirement (PR.4). Instead we impose restrictions corresponding to those for orthocomplements (see [1] and [2]).

(PS.5) $Ps(\sim\sim A, \Gamma) = Ps(A, \Gamma)$
(PS.6) $Ps(A \wedge \sim A, \Gamma) \leq PS(B, \Gamma)$
(PS.7) If $Ps(A, \Gamma) \leq PS(B, \Gamma)$ for every set Γ, then $Ps(\sim B, \Gamma) \leq Ps(\sim A, \Gamma)$ for every set Γ.

Note that these conditions can be derived from the classical constraints. We will discuss negation in greater detail below.

We now turn our attention to the conditional of quantum logic. Given (PR.1-6), one can derive the following condition for the material conditional of classical logic (see [6]):

(MI) $Pr(A \supset B, C) = 1 - Pr(A, C) + Pr(A \wedge B, C)$

The conditional of quantum logic is known to be stronger than the material conditional, in the sense that all theorems of the quantum conditional are theorems of the material conditional, while the reverse is not true. Semantically speaking, it should require more evidence to support a quantum conditional than what is required to support a material conditional. Or to put the matter a bit differently, a given body of evidence should support a material conditional to at least as high a degree as that evidence supports a quantum conditional. Thus, for the quantum conditional we require the following:

$$\text{(PS.8)}\quad Ps(A \to B, \Gamma) \leq 1 - Ps(A, \Gamma) + Ps(A, \Gamma) \bullet Ps(B, \Gamma \cup \{A\})$$

We will say that a probability function is an O function just in case it staisfies all of (PS.1-7). We will say that a probability function is a Q function just in case it satisfies all of (PS.1-8). We will say that an inference from a set Γ of expressions to an expression A is O probabilistically valid, O valid, or valid in the sense of orthologic just in case for every O probability function Ps and every set of expressions Γ', $Ps(A, \Gamma \cup \Gamma') = 1$. Intuitively, an inference from Γ to A is probabilistically valid in this sense just in case there is no information which could be added to that in Γ which would make us have any doubt about A. We will say "Γ semantically entails A in the sense of orthologic" to indicate that the inference from Γ to A is O valid. We will use similar terminology for the Q system. If it is clear in the context which logic is under discussion, we may drop explicit reference to O and Q. We will use the notation "$\Gamma \models A$" to indicate that the inference from Γ to A is valid in the appropriate sense.

3. *Soundness*

There are several different ways one could prove soundness. One approach is algebraic. We could show that the equivalence classes of expressions determined by an arbitrary O function (Q function) form an ortholattice (orthomodular lattice). For each O function (Q function) Ps and each set Γ, the valuation function defined on equivalence classes [A] by $V([A]) = Ps(A, \Gamma)$ is then a valuation on an ortholattice. Goldblatt indicates in [3] that if $A \vdash B$ in orthologic (quantum logic), than $\vee([A]) \leq \vee([B])$ for every valuation $\vee$ on the ortholattice (orthomodular lattice) of equivalence classes of expressions. We could than make use of this relationship to prove the soundness result.

There is an alternative approach which does not make direct use of any algebraic arguments. Since is seems much simpler, we use it here. We first state and prove two elementary theorems, and then we prove two versions of the soundness result.

Theorem 2: If $B_1, \ldots, B_n$ are members of the set Γ, then for every O or Q function Ps, $Ps(B_1 \wedge \ldots \wedge B_n, \Gamma \cup \Gamma') = 1$, for every set Γ'.

Proof: The proof is by a simple induction on n. For $n = 1$, the result follows immediately from (PS.2). For the induction step, the result follows from the induction hypothesis, (PS.2), and (PS.4).

Theorem 3: If $B \vdash A$ in orthologic (or quantum logic), then for every O function (or Q function), $Ps(B, \Gamma) \leq Ps(A, \Gamma)$, for every set Γ.

Proof: We only need to show that the proeprty holds for pairs of expressions given by (#1-#6) and (#10), and that the property is preserved by (#7-#9). In all cases the result is straightforward, if not entirely trivial. For example, we will consider (#10). By (PS.4) we know:

$$(3.1)\quad Ps(A \wedge (A \rightarrow B), \Gamma) = Ps(A, \Gamma) \bullet Ps(A \rightarrow B, \Gamma \cup \{A\})$$

From (PS.2) and (PS.8), we have:

$$(3.2)\quad Ps(A \rightarrow B, \Gamma \cup \{A\}) \leq Ps(B, \Gamma \cup \{A\})$$

Then (3.1) and (3.2) together yield:

$$(3.3)\quad Ps(A \wedge (A \rightarrow B), \Gamma) \leq Ps(A, \Gamma) \bullet Ps(B, \Gamma \cup \{A\})$$

Applying (PS.3) to the right side of (3.3) gives:

$$(3.4)\quad Ps(A \wedge (A \rightarrow B), \Gamma) \leq Ps(B, \Gamma) \bullet Ps(A, \Gamma \cup \{B\})$$

Finally, (3.4) with (PS.1) gives the required result:

$$(3.5)\quad Ps(A \wedge (A \rightarrow B), \Gamma) \leq Ps(B, \Gamma)$$

Thus any pair of expressions given by (#10) has the desired property. The other cases are similar.

We are now ready to prove our first soundness theorem. Note that given the present characterization of both orthologic and quantum logic, the consequence relation $\Gamma \vdash A$ is not defined for Γ empty. However, the definition of our probability theory allows for the empty set of assumptions. Our first theorem deals with the usual case in which Γ is not empty. The second theorem deals with the special case in which the assumption set may be empty.

Theorem 4: If $\Gamma \vdash A$ in orthologic (or quantum logic), then $\Gamma \models A$ in the sense of orthologic (or quantum logic).

Proof: Suppose $\Gamma \vdash A$. Then by definition, for some finite number of members of Γ, say $B_1, \ldots, B_n$, A is derivable from the conjunction of the B_i. For notational simplicity, let CB denote the conjunction of the B_i. Then we have $CB \vdash A$. Then by Theorem 3, we know that for every O (or Q) function, $Ps(CB, \Gamma') \leq Ps(A, \Gamma')$, for every set Γ'. So in particular, $Ps(CB, \Gamma \cup \Gamma') \leq Ps(A, \Gamma \cup \Gamma')$, for every set Γ'. Then by Theorem 2 and (PS.1), we have $Ps(A, \Gamma \cup \Gamma') = 1$, for every set Γ', as required. Thus the proof of Theorem 4 is complete.

Theorem 5: Suppose $\vdash A$, in orthologic (or in quantum logic). Then for every O (or Q) function: (a) $Ps(B, \Gamma) \leq Ps(A, \Gamma)$, for every expression B and every set Γ; and (b) $Ps(A, \Gamma) = 1$, if Γ is not empty.

Proof: Suppose $\vdash A$ in orthologic (or in quantum logic). Then by definition:

(5.1) $A \vee \sim A \vdash A$

Then from (5.1) we know by Theorem 3 that for every set Γ:

(5.2) $Ps(A \vee \sim A, \Gamma) \leq Ps(A, \Gamma)$

By (PS.6) we have for every set Γ:

(5.3) $Ps(\sim A \wedge \sim\sim A, \Gamma) \leq Ps(\sim B, \Gamma)$.

Applying (PS.7), (PS.5), and (D.1) to (5.3), we know that for every set Γ:

(5.4) $Ps(B, \Gamma) \leq Ps(A \vee \sim A, \Gamma)$

Thus part (a) is proved. In order to prove part (b), simply note that if Γ is not empty, then the B in (5.4) may be taken to be a member of Γ.

Then the desired result follows from (PS.1-2). That completes the proof of Theorem 5.

Unlike the classical situation, we cannot identify the theorems of either of the two logics with those expressions which take probability 1 on all evidence. Our constraints are compatible with the assignment of 0 to every expression when given only the empty set of information. However, as Theorem 5 shows, the probability of any theorem is maximal, even on empty evidence. Of course it would be a trivial matter to rule out functions which assign 0 to every expression on empty evidence, but there seems to be no need to do so. We still have the result that the theorems of the logics are just those expressions which are maximally probable no matter what the evidence.

4. *Completeness*

We now turn to some completeness results. The first theorem is the standard strong completeness result.

Theorem 6: For every expression A and every set of expressions Γ, if $\Gamma \models A$ in the sense of orthologic (quantum logic), then $\Gamma \vdash A$ in orthologic (quantum logic).

Proof: We will argue for the contrapositive of the theorem. Let A' be some particular expression and let Γ' be some particular set of expressions, and suppose that it is not the case that $\Gamma' \vdash A'$ in orthologic (quantum logic). We must show that it is not the case that $\Gamma' \models A'$ in the sense of othologic (quantum logic). It will be sufficient to define an O function (a Q function) Ps such that ps(A', Γ') $\neq 1$. We can define the required function as follows:

$$\begin{aligned} Ps(A, \Gamma) &= 1 \text{ if and only if } \Gamma \vdash A \\ &= 0, \text{ otherwise} \end{aligned}$$

Clearly Ps(A', Γ') = 0. So to complete the proof, we only need to show that Ps satisfies (PS.1-7) for orthologic and additionally (PS.8) for quantum logic. (PS.1) is trivially satisfied by definition. (PS.2) corresponds to (#1). (PS.5) corresponds to (#4-#5). (PS.6) corresponds to (#6). And (PS.7) corresponds to (#9). The only conditions which call for any comment are (PS.3), (PS.4), and (PS.8). In each case, the proof that hte condition is satisfied is easy, but a bit tedious.

To see that (PS.3) is satisfied, first suppose that the left side is 1, i.e., suppose both of the following hold:

(6.1) $\mathrm{Ps}(A, \Gamma) = 1$
(6.2) $\mathrm{Ps}(B, \Gamma \cup \{A\}) = 1$

By the definition of Ps we know immediately that:

(6.3) $\Gamma \vdash A$
(6.4) $\Gamma \cup \{A\} \vdash B$

The definition of "$\Gamma \vdash A$" assures that if $\Gamma_1 \vdash A$, then $\Gamma_1 \cup \Gamma_2 \vdash A$, for any two sets Γ_1 and Γ_2. Hence (6.3) assures that:

(6.5) $\Gamma \cup \{B\} \vdash A$

So by the definition of Ps we have;

(6.6) $\mathrm{Ps}(A, \Gamma \cup \{B\}) = 1$

The definition of "$\Gamma \vdash A$" and (6.3) assure that for some finite conjunction $C_1$of members of Γ, we have:

(6.7) $C_1 \vdash A$

Similarity from (6.4) we know that for some finite conjunction C_2 of members of $\Gamma \cup \{A\}$:

(6.8) $C_2 \vdash B$

From (#1-#3) and (#7-#8) we may derive the commutativity and associativity of conjunction on the left of "$\vdash$". These facts with (#2), (#7), and (6.8) guarantee that for some finite conjunction C_3 of members of Γ, we have:

(6.9) $C_3 \wedge A \vdash B$

But (6.7) and (6.9) with (#2-#3) and (#7-#8) yield:

(6.10) $C_1 \wedge C_3 \vdash B$

But by commutativity and associativity, $C_1 \wedge C_3$ is equivalent to the conjunction of a finite number of members of Γ. Hence we have:

(6.11) $\Gamma \vdash B$

So by the definition of Ps, we know:

(6.12) $Ps(B, \Gamma) = 1$

From (6.6) and (6.12) we know the right side of (PS.3) must be 1. Consequently, if the left side of (PS.3) is 1, then the right side must be 1. But interchanging A and B in the above argument shows that if the right side of (PS.3) is 1, then the left side must be 1. Since each side must be either 1 or 0 by the definition of Ps, we know (PS.3) must be satisfied.

To see that (PS.4) is satisfied, first suppose the left side is 1, i.e., suppose:

(6.13) $Ps(A \wedge B, \Gamma) = 1$

Then by the definition of Ps, we have:

(6.14) $\Gamma \vdash A \wedge B$

Using (#2) and (#7) with (6.14) yields:

(6.15) $\Gamma \vdash A$

The definition of Ps with (6.15) assures that:

(6.16) $Ps(A, \Gamma) = 1$

Using (#3) and (#7) with (6.14) yields:

(6.17) $\Gamma \vdash B$

But by definition of "$\Gamma \vdash B$", (6.17) assures that:

(6.18) $\Gamma \cup \{A\} \vdash B$

Hence by definition of Ps, (6.18) yields:

(6.19) $Ps(B, \Gamma \cup \{A\}) = 1$

So (6.16) and (6.19) assure that the right side of (PS.4) must be 1. Next, suppose the right side of (PS.4) is 1. Then (6.16) and (6.19) must hold. But (6.16) guarantees (6.15), and (6.19)guarantees (6.18). But by moves parallel to those used to obtain (6.11) from (6.3) and (6.4), we know that (6.15) and (6.18) guarantee (6.17). Then (6.15) and (6.17) with (#8) guarantee (6.14). By definition of Ps, (6.14) guarantees (6.13), so the left side of (PS.4) is 1. Since each side must be either 0 or 1, (PS.4) must be satisfied.

For the case of quantum logic, we must assure that our definition of Ps satisfies (PS.8). If $Ps(A, \Gamma) = 0$, then the inequality is trivially satisfied, since the right side will be 1. Similarly, if $Ps(A \cup B, \Gamma) = 0$, then the inequality is trivially satisfied, since the left side will be 0. So we consider the case when the following hold:

(6.20) $Ps(A, \Gamma) = 1$
(6.21) $Ps(A \rightarrow B, \Gamma) = 1$

Then (6.20) and the definition of Ps give:

(6.22) $\Gamma \vdash A$

And (6.21) and the definition of Ps give:

(6.23) $\Gamma \vdash A \rightarrow B$

But (6.22) and (6.23) with (#8) yield:

(6.24) $\Gamma \vdash A \wedge (A \rightarrow B)$

Then (6.24) and (#10) give:

(6.25) $\Gamma \vdash B$

Given the definition of "$\Gamma \vdash B$", (6.25) assures that:

(6.26) $\Gamma \cup \{A\} \vdash B$

The definition of Ps and (6.26) assure that:

(6.27) $Ps(B, \Gamma \cup \{A\}) = 1$

But given (6.27), the right side of (PS.8) is 1, and the inequality is satisfied.

We can conclude that Ps defined as above is a genuine O function (Q function). Since $Ps(A', \Gamma') = 0$, we know it cannot be the case that $\Gamma' \models A'$. This our strong completeness theorem is proved.

We now turn our attention to weak completeness. Recall that in our discussion of Theorem 5, we indicated that the theorems of the two logics are maximally probable, regardless of the evidence. But recall that being maximally probable does not necessarily mean having a probability of 1. Our weak completeness result simply says that maximally probable expressions are theorems.

Theorem 7: If for every O function (Q function) Ps, every expression B, and every set of expressions Γ, $Ps(B, \Gamma) \leq Ps(A, \Gamma)$, then $\vdash A$ in orthologic (quantum logic).

Proof: Assume the hypothesis of the theorem to be true. Simply take B to be $A \vee \sim A$, and take Γ to be $\{A \vee \sim A\}$. The hypothesis of the theorem with (PS.1-2) then guarantees that $Ps(A, \{A \vee \sim A\}) = 1$. From Theorem 6, we can then conclude that $\{A \vee \sim A\} \vdash A$, which by definition means $\vdash A$. Thus, Theorem 7 is proved.

5. *Non-triviality and examples*

It is easy to show that our O and Q probability functions are not limited to a finite range of values, because in a sense the constraints are weaker than the classical constraints. Consider a classical language built up from only a finite number of sentence letters. Any set Γ of expressions from the language is then deductively equivalent to some expression, which we will designate by $E(\Gamma)$. There will be infinitely many classical probability functions Pr definable on the language. To each such function, there corresponds a function of the sort discussed here; simply set $Ps(A, \Gamma) = Pr(A, E(\Gamma))$. It is easy to see in light of our discussion that any function Ps so defined must satisfy all of (PS.1-8). In a similar fashion, we may obtain examples of our functions with infinitely many values. Let our language have infinitely many sentence letters. Then there will be infinite valued classical functions Pr on the language. We can then define $Ps(A, \Gamma)$ as the limit of $Pr(A, C(\Gamma, n))$, where $C(\Gamma, n)$ is the conjunction of the first n elements (in some alphabetic order) of Γ.

From a perusal of the conditions (PS.1-7), it should be obvious that the deviation from the classical theory centers on the treatment of negation. Given the classical probability theory, one can derive a single functional expression for the probability of $\sim A$. That is, we can obtain a result of the form $Pr(\sim A, B) = f$. (In fact, $f = 1 - Pr(A, B) + Pr(\sim B, B)$; see[7].) It may seem desirable to obtain a similar expression for the negation(s) of orthologic and quantum logic. That is, one may wish to have a result of the form $Ps(\sim A, \Gamma) = g$. However, recall that orthologic and quantum logic are weaker than classical logic, in the sense that they have smaller sets of theorems. Thus it seems reasonable to expect that the classical probability functions will be just a subset of those probability functions appropriate for orthologic and quantum logic. If we insist on finding a function g for the negation of the weaker logics, then we must have $g \neq f$, or else the set of classical functions would coincide with the set of O functions. But if $g \neq f$, then at best the classical functions and the O functions will overlap; neither will be a subset of the other.

There are many functional forms which one could use other than (PR.4) in order to obtain the orthocomplement conditions. We will say that a function, designated by "c", which maps the closed interval [0,1] into itself, is a complement function if it satisfies both of the following:

(c.1) $c(c(n)) = n$
(c.2) If $n \leq m$, then $c(m) \leq c(n)$.

It is not difficult to show that if c is an arbitrary complement function, then the orthocomplement conditions (PS.5-7) would all be satisfied by the following:

(PSC.1) $Ps(\sim A, \Gamma) = c(Ps(A, \Gamma))$, unless $Ps(B, \Gamma) = 1$ for all B

In fact, for each set Γ, we could specify a complement function $c[\Gamma]$, and satisfy the orthocomplement conditions with the following:

(PSC.2) $Ps(\sim A, \Gamma) = c[\Gamma](Ps(A, \Gamma))$, unless $Ps(B, \Gamma) = 1$ for all B

It would only be necessary to assure that logically equivalent sets use the same complement function.

It is of course very easy to cook up functions on the closed unit interval which satisfy (c.1-2). For just one type of example, note that for each positive real number n, the following defines a complement function:

(c.3) $c_n(m) = (1-m^n)^{1/n}$

There is often a reluctance to consider alternate probability theories. Part of the reason for that reluctance has to do with a lack of familiarity with situations to which such theories could be applicable. We will now consider a family of examples in which it can be shown that bets should *not* be made in accordance with the classical theory; however, it can be shown that one should not deviate from (PS.1-8). Thus a probability theory at least as strong as that for quantum logic, but weaker than the classical theory, is appropriate. The probability functions appropriate for the examples will be multi-valued. Hence the O functions and Q functions are nontrivial, even when they deviate from classical constraints.

For the examples, we will restrict ourselves to a language with only finitely many sentence letters, using "$\wedge$" and "$\sim$" as primitive sentence connectives. We assume (D.1-2), above. Suppose we are faced with an ideal computer which is initially given some set Γ' of expressions, the content of Γ' being unknown to us. The computer is then set up to answer questions of the following sort: Is it the case that $\Gamma' \vdash A$ in classical logic? That is, the computer responds to input A with a "yes" or "no", the answer depending on whether or not A is classically derivable from Γ'.

Now there will be only a finite number of logically distinct deductively closed sets, since our logic is classical and we have only finitely many sentence letters. So, our example is equivalent to supplying the device with a deductively closed set and asking about membership in the set. Suppose we know that each logically distinct deductively closed set is equally likely to be placed in the machine. Our problem is the following: Given a knowledge of the set Γ of expressions already known to be derivable, what is the probability that a specified expression A is derivable? (This situation is *not* the same as that described in [5].)

It is obvious that correct probabilities are given by a relative frequency scheme: The probability of A, given Γ, is the number of deductively closed sets containing $\Gamma \cup \{A\}$ divided by the number of deductively closed sets containing Γ. Using "$\#(\Gamma)$" to stand for "the number of deductively closed sets containing Γ", we can represent the appropriate function as follows:

$$\text{(P.1)}\quad P(A, \Gamma) = \#(\Gamma \cup \{A\})/\#(\Gamma)$$

It is a simple matter (left to the reader) to verify that the function P satisfies all of (PS.1-8).

To clearly see that functions defined by (P.1) are not classical, we will consider the simplest case. Suppose our language contains only one sentence letter, say p. Then there will be only four logically distinct expressions: p, $\sim p$, $p \vee \sim p$, and $p \wedge \sim p$. There will also be only four distinct deductively closed sets, which we will represent as follows:

$$\Gamma_1 = \{p \vee \sim p\}$$
$$\Gamma_2 = \{p \vee \sim p,\ p\}$$
$$\Gamma_3 = \{p \vee \sim p,\ \sim p\rangle$$
$$\Gamma_4 = \{p \vee \sim p,\ p,\ \sim p,\ p \wedge \sim p\}$$

Then the function P is defined by the following table:

$P(A, \Gamma)$	Γ_1'	Γ_2	Γ_3	Γ_4
$p \vee \sim p$	4/4	2/2	2/2	1/1
p	2/4	2/2	1/2	1/1
$\sim p$	2/4	1/2	2/2	1/1
$p \wedge \sim p$	1/4	1/2	1/2	1/1

Clearly this function deviates drastically from the calssical constraints. In particular, the negation condition (PR.4) is violated. It is also interesting to note that for no set Γ is it the case that $P(p \wedge \sim p, \Gamma) = 0$. Some authors take the following condition to be required of all reasonable probability functions:

$$P(A, \Gamma) + P(B, \Gamma) = P(A \wedge B, \Gamma) + P(A \vee B, \Gamma)$$

However, it is easy to see that this condition fails for our example, even though the probability is derived from a relative frequency.

Quantum logic is supposed to be particularly appropriate for situations in which quantum mechanical considerations are crucial. Since our quantum probability theory exactly characterizes quantum logic, one would think it must be relevant to the statistics of quantum mechanics. If that is the case, then it would seem that the statistics of conjunctive events should be classical, while the statistics of complementary events should not be. I leave such problems for others to ponder.

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