PARADOX-TOLERANT LOGIC(*)

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Introduction

In the modern era there have been those who have regarded classical logic as the crock at rainbow's end; and there have those who have preferred a briefer description along the same vascellar lines. It is this dispute as to contents which accounts for the differing degrees of gladness with which these opponents receive the notion that *PL* should be traded for a mess of sensible if less elemental pottage. And inevitably there have been those who have confessed to finding the proferred pottage a trifle lumpy. One of those most recently to retch in print has been D. Lewis (in [8]). Since the philosophical motivation for the logic we study here is intermediate between the one which Lewis cannot digest and the one that he will not swallow, that discussion provides a suitable preparation for our own. The main idea, that of N. Belnap [1], which Lewis considers is one which had gained for relevant logic a welcome measure of even-tempered consideration. This is the notion that distinct sources might assign the same sentence differing truth-values, and when this is so, the requirement that implication must preserve truth may be taken to rule out classical inferential procedures and rule in those licensed by various relevant systems. Lewis's negative point on this score is that in the case Belnap envisages, we will more sensibly try to ke the conflicting information separate, that is, unconjoined, and so he opts for some such approach as that suggested by Jaskowski in [5] or Schotch and Jennings in [11] and elsewhere. The preferred approach is one which, in the words of [6], chooses inferentially pudent conjunction over inferentially prudent implication. As for the relevant systems, they are best reserved for those situations in which the conjunction of contradictions is unavoidable, as when the contradiction arises through an equivocal use of language. This essayed

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rehabilitation of relevant logic is of no intrinsic interest to us here save in one respect. That is that both Belnap's and Lewis's proposed uses of relevant logic arise in circumstances of doxastic irregularity. In Belnap's case it is unavoidable conflict of data; in Lewis's, inadvertent equivocation. This feature of untowardness has marked the motivating examples of Jennings and Schotch, notably, conflicting beliefs and irresolvable conflicts of obligation. In the approach taken by Routley and Priest *et al.* the need for non-classical implication arises from ineluctable inconsistency in some mathematical and physical theories which in the end, we might just decide to accept. This approach, which Lewis labels 'the radical answer', he dismisses without argument.

> The reason why we should reject this proposal is simple. No truth does have, and no truth could have, a true negation. Nothing is, and nothing could be, literally both true and false. This we know for certain, and *a priori,* and without any exception for especially perplexing subject matters. The radical case for relevance should be dismissed just because the hypothesis it requires us to entertain is inconsistent.
>
> That may seem dogmatic. And it is:....

We do not intend specifically to answer this little emesis (to press the metaphor); however, we hope that it will be plain enough that the radical answer deserves better than the superficial consideration with which Lewis dismisses it. The issues raised are, in any case, more directly addressed in Jennings [6] and Routley and Meyer [10]. It is, however, relevant to remark that Lewis's rejection of a view countenances inconsistent worlds seems more the reaction of a possible worlds realist than the reaction of a classicalist. For loyalty to the classicalist banner dictates only the insistence that at inconsistent worlds, if we may be permitted to call these objects 'worlds', every sentence of the chosen language be true. It is this notion that these objects cannot *really* be worlds, but rather must be merely model theoretic devices that seems to give rise to Lewis's illiberality, as well as to B. J. Copeland's claims (in [3]) that the radical treatment is philosophically unilluminating. The two kinds of objection are simply answered. In the first place, the people who have so glutinously adhered to the view that possible worlds have a useful role to play in philosophy have never been able to say what they are, so whether the Routley-Priest objects can count as worlds in their sense must remain a moot point. In the second place, even if they do count as worlds, we ought not to expect their introduction to be any more illuminating in this context than their introduction has been in any other. It surely has to be admitted by now that the explanatory record of worlds in informal philosophy is pretty spotty. At the same time, non-standard worlds have played a useful role in formal studies, even those undertaken by dyed-in-the-wool-and-born-again classicalists. Consider only the Q worlds of frame theory and the heavens of Cresswell [4]. So if in the present case inconsistent and incomplete worlds achieve no more than a technical usefulness, they have achieved as much as any worlds (except possibly the real one) in the modern era. This discriminatory prosecution of the paraconsistentists is entirely misplaced. It would remind one of ordinary ranting fundamentalism save for the feature that in this case the ranters behave as co-authors with the almighty of the scripture in whose defence they rant.

We begin our essay with the remark that the notion of a world, even of the real world, is, for us, a problematic notion. Furthermore, we confess to grave philosophical inadequacy in our understanding of the nature of truth, (What can be meant by the notion of something's being 'literally true', let alone 'literally both true and false'? What is the *literal* meaning of 'truth'?) We have accordingly tried to assume about either notion as little as possible. We assume that the process of saying in general terms how the world is is a process in part of devising a language with which to do so, and thus that the sentences which the world makes true are sentences of a language of human contrivance. We assume further that some sentences of such a language would be true because of the nature of the language rather than, or as well as, because of the way the world is. We also assume that at a particular stage in our attempt to give an account of how the world is, the account that we give may have consequences which we would rather it did not have, such as, for example, that there are physically realisable circumstances in which a particle would have infinite potential energy. We envisage both the likelihood that we should seek to revise our language or the set of sentences of the language to be regarded as true in order to avoid such consequences, and the possibility that we should be unable to do so. Finally, we see no reason to suppose that the best theory we can come up with in the best language we can devise will be free of unwelcome consequences, even contradictions. We do not contend that this would tell us something about *the* world, but do suppose that it would tell us something about *our* world, *i.e.*, the world as viewed by the collective and possibly embarrassed human intellect.

Now if there are some sentences which are true independently of how the world is, and if the objection to true contradiction is just that there is no way that the world could be which would make them true, this still leaves us with the prospect of contraditions which are true independently of how the world is, true that is, because of the nature of the language, true even because of the meanings that we have given to certain words. If it is insisted that such a language would be faulty, we will agree that it is less than ideal, less perhaps even than we will ultimately achieve. But since the contradition has no empirical content, our view of the world need not suffer, provided that we are sensible about what follows from the contradiction.

Implication as a preservation

If you have insisted that implication implies relevance, but have not yet been able to say what relevance is, it might almost seem a divine intervention when you find a plausible story according to which at the very point at which it fails most disasterously on the score of relevance, material implication fails also of the one virtue for which it still finds thin praise, namely the preservation of truth. But if the alternative implication has no more to be said for its preferment than that it preserves truth in that out of-the-ordinary case in which the other does not, the fundamentalist will not be impressed anyway, and the more substantive demands which should be made of implication will be lost sight of. It seems to us wrong to insist exclusively upon the truth-preserving property of implication. Truth is one kind of metalinguistic information about the antecedent, but only one among many, and the kind of implication that one adopts in a given circumstance ought to be designed around the kind of metalinguistic information that one wants to preserve. In various of the polylethic semantics which Lewis lists, implication is required to preserve various combinations of truth and truth only. Jennings and Schotch (in [7] and [11]) have suggested, though in the context of illative rather than implicative systems, that inference should, in certain cases, preserve the level of coherence, that the least partition of a set Σ into inconsistent sub-sets ought to be equinumerous with that of the inferential closure of Σ. A similar requirement can be used to achieve relevant inference. Simply consider the least partition of a set Σ into relevant subsets, subsets in which every sentence is relevant to every other. Call the size of that partition the level of irrelevance of Σ. *r.a.e.* (1) For obvious reasons, none of these is a suitable restriction for implicative systems. But there are other metalinguistic features of antecedents besides truth which even blushing classicalists expect to see preserved by logical implication, for instance, necessity. To say that necessity is monotonic is just to say that logical implication preserves it.

It is in some quarters held to be a virtue of the American style semantics as against the Australian, that it's negation is classical negation. The greater virulence of the classicalist reaction to the Australian semantics is due in part to it's making negation move over (rather a long way, in fact to a neighbouring world) to make room for contradiction. [One might express this perceived virtue of the American plan as that of tolerating contradiction without accommodating it.] The perceived vice of the Australians is that they accomodate contradiction whilst claiming to take no pleasure in it. (They make no such claim about the outrage this generates.)

But this also separates them from the support of a large (and one wants to add 'level headed') contingent of philosophers who will be tempted to regard the dispute (if they should hear about it) as between one troop of possible-world-niks and another. When the most persuasive examples of true contradictions are instances of mathematical paradox, it will seem to have required a particularly virulent attack of the realies to suppose that worlds have somehow to be doctored to allow them their place in the sun. But if this dispute about landscaping requirements seems calenturical, it has to be said that the American idea seems at least equally ill suited to the representatic of *that* kind of contradiction. The Belnap story is one in which conflicting empirical

(1) Fratris aut sororis Roberti filius aut filia es.

information is processed, but in the case of paradox we have instead conflicting necessary truths. Here, the language we have adopted, say the language of naive set theory, is such as to make each of a sentence α and its negation $\neg\alpha$, true in virtue of the meanings of its terms. We may, on that account, regard the language as faulty, but there can be no doubt that at least relatively to that language some sentence is necessarily true, whose negation is also necessarily true. There are many features of this situation which we might want to complain about. We might want to object that this shows that for example the Russell set does not exist. But in what sense does any set exist? Anyone who is not a realist about these matters anyway may nevertheless want to know the properties of such an object, and may well be prepared to accord it, for the purpose, as real an existence as any other mathematical object. But there is a problem about what logic to use, and classical logic will not work here. The problem is not simply that the true contradiction violates a principle of the logic. The true contradiction might well violate a principle of whatever logic we adopted. The problem is that classical logic cannot cope with what happens in this language. In this language, the sentence ϱ is necessarily true: so is $\neg\varrho$. But in classical logic ϱ is equivalent to $\varrho \wedge \neg\varrho$, and accordingly, on the supposition that ϱ is true, so is every other sentence, including those which even in this language concern matters of contingent fact. Moreover, since ϱ is necessarily true, so is every other sentence, including the ones which we had regarded as contingent.

Now this brings us almost to the point of our earlier remark, that the process of devising a theory about the world is in part a process of devising a language. This seems equally to be the case in the process of devising theories other than theories about the world. As we have a notion of something's being true according to a theory, we need a notion of something's being necessarily true according to the language of a theory. What is more we need a logic which preserves this necessity-according-to-a-language. Such a logic must distinguish between sentences which are logical falsehoods or truths because of their logical form, and those which are necessary truths or falsehoods because of their non-logical content. To this end, the logic introduced below will distinguish implicationally between '$\perp$' and '$p \wedge \neg p$' (2)

(2) In fact, the logic keeps distinct all pairs of sentences of the form '$\alpha \wedge \neg\alpha$' in

The point is this: paradoxes and other contraditions can occur beneath the level of sentence structure that the language of the logic enables us to analyse. That is to say, not every contradiction in natural language is of the form '$\alpha \wedge \neg\alpha$'. Of course, classically we might argue that if β is logically false then $\neg\beta$ is logically true and therefore β is equivalent to $\beta \wedge \neg\beta$ (see J. Bennett [2]), but, we maintain, we should distinguish between contradictions of this form in which one conjunct is logically false (and the other logically true) and those in which both conjuncts are contingent. The former sort is taken to represent the sort of paradox of which our logic is, as advertised, tolerant.

Theory and Logic

We distinguish between the language of a particular theory and the language of the logic in which we study the theory. This is not to say that the two may not share vocabulary, 'and', 'or', 'not', *etc.*, but we will imagine ourselves taking sentences in the language of the theory and combining them in various ways which may or may not result in sentences of the theory. Furthermore we will pretend that the sentences that we manipulate in this way are unanalysable in the language of our logic. It would be better to say that if the sentences which are the raw materials of our manipulation *are* analysable (into conjuncts, or disjuncts *autc.*) we simply ignore the fact. An exception is that we will think of ourselves as adding to the language of the theory a kind of implication by which we can give the closure of a given set of sentences of the theory language. These dark doings will become less dark in the sequel.

Language-dependent truth

This paper embodies an oversimplification which must await a later essay for its repair. This is that we take the notion of language-depen-

distinct variables. In distinguishing between compound and non-compound falsities, this work goes beyond [7] and [11] in which the sets $\{p, \neg p\}$ and $\{p \wedge \neg p\}$ are inferentially distinguished.

dence of truth values to be two-valued. So we say that the truth-value of 'some squares are not rectangular' to be determined by the meaning of 'square' and 'rectangle', but the truth-value of 'Mussolini abhorred linguine' to have been determined by material circumstances. This ignores the problem cases such as 'Horses are perissodactyls'. Something a little more Quinian is promised for the future; for the present the dogma must do. In addition we distinguish both of these from logical truths such as 'Either Mussolini abhorred linguine or he didn't', but this last plays no interesting rôle in what follows. We will mark the former distinction semantically by means of a function δ called a dependence function, which maps atomic sentences into 2 and is extended to the set of all formulae by a set of rules representable by matrices. Thus to say that $\delta(\alpha) = 1$ is to be understood informally as saying that the truth value of α is language dependent; if $\delta(\alpha) = 0$, the truth value of α is determined by material fact. Since we have as well a standard valuation function V, we have four possible combinations of truth and dependence values as set out with the corresponding decimally encoded matrix entry in Table 1.

$V(\varrho)$	$\delta(\alpha)$	$\mu(\alpha)$
1	1	3
1	0	2
0	1	1
0	0	0

Table 1.

The matrices

As usual, we take the valid formulae to be those to which V always assigns 1, so the set D of designated matrix values is $\{3,2\}$. The extension of μ to arbitrary formulae *PL* is achieved by rules which are encapsulated to the following matrices:

α	$\neg\alpha$
0	2
1	3
2	0
3	1

[$\mu(\neg)$]

$\vee$	0	1	2	3
0	0	0	2	3
1	0	1	2	3
2	2	2	2	3
3	3	3	3	3

[$\mu(\vee)$]

$\wedge$	0	1	2	3
0	0	1	0	0
1	1	1	1	1
2	0	1	2	2
3	0	1	2	3

[$\mu(\wedge)$]

$\supset$	0	1	2	3
0	2	2	2	3
1	3	3	3	3
2	0	0	2	3
3	0	1	2	3

[$\mu(\supset)$]

$\equiv$	0	1	2	3
0	2	2	0	0
1	2	3	0	1
2	0	0	2	2
3	0	1	2	3

[$\mu(\equiv)$]

Note that these matrices do in fact record truth values in the classical way as well as the dependence of the truth value of truth functional compounds upon the language from which the arguments of the function are drawn. While we may speak informally of the value 3 and 2 as though they distinguished necessary truth from contingent truth this manner of speaking applies, if at all, only to non-compound sentences. If $\mu(p) = 2$ then $\mu(p \vee \neg p) = 2$; thus in the informal idiom $p \vee \neg p$ is a 'contingent' truth, for the matrices ignore tautologies at this level, except of course inasmuch as 2 is a designated value. Two connectives remain to be introduced. Since the paradox-tolerant logic (*PTI*) is to be introduced as an extension of *PL*, axiomatised with $\supset$ and the 0-ary $\perp$ as primitive, we may mention that $\perp$ will take the matrix value 1. The purpose of this is that '$\perp$' is to be thought of as representing a *non-compound* language-dependent falsehood. It will enable us, in the usual way to introduce '$\top$' by the definition:

[df.$\top$] $\quad \top = \text{df.} \perp \supset \perp$

Thus $\top$ will represent (informally) a non-compound language-dependent truth, and receive the matrix value 3.

Finally, we will introduce an implication connective '→' whose interpretation is given by the following matrix:

→	0	1	2	3
0	2	2	2	3
1	1	3	1	3
2	0	0	2	3
3	1	1	1	3

[$\mu(\rightarrow)$]

It is to be noted that → preserves both truth and dependence. The specific entries are explained in the following way: If both truth and dependence are preserved, $\mu(\alpha \rightarrow \beta) \in D$, *i.e.* $V(\alpha \rightarrow \beta) = 1$. Think of $V(\gamma)$ as changeable (by changing circumstances, for example) if $\delta(\gamma) = 0$, else as unchangeable. Think of $\delta(\gamma)$ as always unchangeable. Then if $V(\alpha \rightarrow \beta) = 1$ is changeable because $V(\alpha)$ or $V(\beta)$ is changeable, then $\mu(\alpha \rightarrow \beta) = 2$, else $\mu(\alpha \rightarrow \beta) = 3$. Thus, for example, $\mu_{\rightarrow}(3,2) = 1$, since $\delta(\alpha) = 1$ and $\delta(\beta) = 0$. Now $V(\beta)$ is changeable, but a change in the truth value of β will not affect the truth value of $\alpha \rightarrow \beta$ since $\delta(\alpha)$ would still not be preserved. On the other hand $\mu_{\rightarrow}(2,2) = 2$ since dependence is vacuously preserved as well as truth. But since $\delta(\beta) = 0$, $V(\alpha \rightarrow \beta)$ is changeable to 0 by a change of $V(\beta)$ to 0. Thus the matrix value 2 rather than 3. Similar reasoning underlies the other entries.

This seems to us a natural and intuitive construction for an → matrix in which the language-rootedness of truth-value is to be preserved. The ⊃ matrix is not constructed according to these principles but only by reordering the rows of the V matrix. It is, however, some indication that the V matrix makes the right sort of sense for present purposes that the δ-components of the entries in the ⊃ matrix will bear the same interpretation as those of the → matrix.

An axiomatisation of PTI

Our axiomatisation of *PL* is as in Segerberg [12]:

[PL1] $\vdash p \supset (p \supset p)$

[PL2] $\vdash (p \supset (p \supset r)) \supset ((p \supset q) \supset (p \supset r))$
[PL3] $\vdash ((p \supset \bot) \supset \bot) \supset p$

with uniform substitution ([US]) and *modus ponens* ([MP]). We retain closure under [US] and classical [MP] and add these further axioms (with no claim as to independence or minimality). We call the resulting logic *PTI* (Paradox-tolerant implication).

[PTI 1] $\vdash p \to p$
[PTI 2] $\vdash (p \to q) \supset ((q \to r) \supset (p \to r))$
[PTI 3] $\vdash (p \to q) \supset ((p \supset q))$
[PTI 4] $\vdash (p \to q) \supset ((\bot \to p) \supset (\bot \to q))$
[PTI 5] $\vdash ((\bot \to p) \supset (\bot \to q)) \supset ((p \supset q) \supset (p \to q))$
[PTI 6] $\vdash (\bot \to (p \to q)) \supset (((\bot \to p) \supset (\bot \to q)) \supset (\bot \to (p \supset q))$
[PTI 7] $\vdash (\bot \to (p \supset q)) \supset (\bot \to (p \to q))$
[PTI 8] $\vdash (\bot \to p) \supset (((\bot \to q) \supset \bot) \supset (\bot \to (p \to q)))$
[PTI 9] $\vdash ((\bot \to p) \supset \bot) \supset ((\bot \to (p \supset q)) \supset q)$
[PTI 10] $\vdash ((\bot \to p) \supset \bot) \supset ((\bot \to (p \supset q)) \supset (\bot \to q))$
[PTI 11] $\vdash ((\bot \to p) \supset \bot) \supset (q \supset ((\bot \to q) \supset (\bot \to (p \supset q))))$
[PTI 12] $\vdash (p \supset \bot) \supset ((\bot \to p) \supset (\bot \to (p \supset q)))$
[PTI 13] $\vdash ((\bot \supset \bot) \to p) \supset ((\bot \to (p \supset q)) \supset (\bot \to q))$
[PTI 14] $\vdash ((\bot \supset \bot) \to p) \supset ((\bot \to q) \supset (\bot \to (p \supset q)))$

Soundness

A model $\mathfrak{M}$ for *PTI is a pair* $<V, \delta>$ where both V and δ map *At* (the set of atoms) into 2, and are extended to Φ, (the set of all sentences) by the following set *M* of matrix-defining rules:

[$\bot$0] $V(\bot) = 0$
[$\bot$1] $\delta(\bot) = 1$
[$\supset$0] $V(\alpha \supset \beta) = 1$ if $V(\alpha) \leq V(\beta)$; else $V(\alpha \supset \beta) = 0$
[$\supset$1] if $\delta(\alpha) = 0$, $\delta(\alpha \supset \beta) = \text{Min}[V(\beta), \delta(\beta)]$
[$\supset$2] If $\delta(\alpha) = 1$, $\delta(\alpha \supset \beta) = 1$ if $V(\alpha) = 0$; $= \delta(\beta)$ if $V(\alpha) = 1$; else $\delta(a \supset \beta) = 0$
[$\to$0] $V(\alpha \to \beta) = 1$ if $V(\alpha) \leq V(\beta)$ and $\delta(\alpha) \leq \delta(\beta)$: else $V(\alpha \to \beta) = 0$
[($\to$1] $\delta(\alpha \to \beta) = 1$ if $\delta(\alpha \supset \beta) = 1$ or $\delta(\beta) < \delta(\alpha)$; else $\delta(\alpha \to \beta) = 0$

It is a straightforward matter to show that V maps every substitution instance of every axiom onto 1, and that [MP] preserves validity.

Completeness

To show completeness, it is sufficient to show that every *PTI* maximal consistent set Σ generates a model $\mathfrak{M}_\Sigma = <V_\Sigma, \delta_\Sigma>$ such that $\forall\alpha \in \Phi$, $V_\Sigma(\alpha) = 1$ iff $\alpha \in \Sigma$. Let Σ be *PTI* maximal consistent. We define V_Σ and δ_Σ in the following way:

$$\forall\alpha \in \Phi,\ V_\Sigma(\alpha) = 1 \text{ if } \alpha \in \Sigma$$
$$= 0 \text{ otherwise}$$

$$\forall\alpha \in \Phi,\ \delta_\Sigma(\alpha) = 1 \text{ if } \bot \rightarrow \alpha \in \Sigma$$
$$= 0 \text{ otherwise}$$

The burden of the proof lies in showing that V_Σ and δ_Σ satisfy the matrix defining rules enumerated above, and thus that $\mathfrak{M}_\Sigma$ is a model of the required sort. But the set of axioms and the canonical V and δ functions are designed as though with this in mind.

About the logic PTI

It is not claimed for *PTI* that it is a relevant logic, only that it tolerates paradox. As it happens, it tolerates contradiction in general, since its models treat every contradiction as at least potentially paradoxical. Thus we do not have $\vdash p \wedge \neg p \rightarrow q \wedge \neg q$ since this implication will fail when $\mu(p \wedge \neg p) = 1$ and $\mu(q \wedge \neg q) = 0$. For similar reasons, *ex falso quodlibet* is also rejected. In the other direction, the logic treats every truth as potentially language dependent and, therefore, not only do we lose implications between tautologies, but we lose implications of tautologies by arbitrary formulae. When $\mu(p) = 3$ and $\mu(q \vee \neg q) = 2$, the implication $p \rightarrow (q \vee \neg q)$ will fail. But the toleration of paradox and contradiction, and the disassociation of linguistic truth and falsity from mere tautology and contradiction is achieved at a price. For the implication introduced lacks much that one might not wish to bring into question.

Among the casualities are the formulae listed below, with examples of invalidating matrix assignments:

$p \to (p \vee q)$ <1,0>
$(p \to q) \to (\neg q \to \neg p)$ <0,1>
$(p \wedge q) \to p$ <0,1>
$(p \to q) \to ((q \to r) \to (p \to r))$ <0,1,0>
$(p \to q) \to ((r \to p) \to (r \to q))$ <1,0,0>
$(p \to (q \to r)) \to ((p \wedge q) \to r)$ <0,1,0>
$((p \vee q) \to r) \to ((p \to r) \vee (q \to r))$ <0,3,0>
$(p \to (q \wedge r)) \to (p \to q)$ <1,0,1>
$(p \to q) \to ((p \wedge q) \to q)$ <3,0>
$(p \to q) \wedge (r \to q) \to ((p \wedge r) \to q)$ <0,0,3>

Against this, De Morgan's equivalence, double negation, and various commutations hold, as well as some implications less welcome such as:

$p \to (\neg p \to p)$ which preludes non-trivial relevant necessity, and
$(p \to (q \vee r)) \to ((p \to q) \vee (p \to r))$ which is a defect also of material implication.

Moreover, the logic is as relevant as it is only within the confines of its own limited expressiveness. We have said that '$\bot$' and '$\top$' are to be thought of as language – dependent and unanalysable representatives of the constituents of paradox. So in the language of *PTI* there is only one paradox, namely $\bot \wedge \top$. But if we enrich the formal language by the addition of $\bot_1 \ldots \bot_n$, to represent the makings of orthographically distinct paradoxes, the logic will lack the implicational means of distinguishing them, since they must all bear the matrix value 1. We can at best, by such methods, hope to distinguish more than two classes of contradictions by recognising that the notion of language dependence is multiply valued. As we have remarked earlier, this must await a later study.

Necessity

PTI gives rise to a notion of necessity which can be studied independently of *PTI* itself. The modal operator $\Box$ is introduced by

[Df. $\Box$] $\Box\alpha = \mathrm{df}\, \top \to \alpha$

and $\Diamond$ by the usual

[Df. $\Diamond$] $\Diamond\alpha = \text{df} \neg\Box\neg\alpha$.

On these definitions, the $\Box$ and $\Diamond$ matrices are as follows:

A	$\Box\alpha$	$\Diamond\alpha$
0	1	3
1	1	1
2	1	3
3	3	3

We can suppress the definition and axiomatise the modal logic *LD* directly by adding to the *PL* axioms the principles:

[h] $\Box(p \supset q) \equiv (\Diamond p \supset \Box q)$
[K] $\Box p \wedge \Box q \equiv \Box(p \wedge q)$
[N] $\Box\top$
[T] $\Box p \supset p$
[4] $\Box p \supset \Box\Box p$
[5] $\Diamond\Box p \supset \Box p$

closing under [MP] and [US].

Although this logic has many S5 features, there are some rather interesting differences. For example, it lacks the S5 rules

[RN] $\vdash \alpha \Rightarrow \vdash \Box\alpha$
[RE] $\vdash \alpha \equiv \beta \Rightarrow \vdash \Box\alpha \equiv \Box\beta$.

But it is not a sub-logic of S5: the 'only if' direction of [h] is not a theorem of S5.

Soundness and Completeness

Soundness is straightforwardly proved. For completeness, a Henkin construction is used, with V_Σ and δ_Σ once again defined for an arbitrary maximal consistent set Σ by the following:

$$V_\Sigma(\alpha) = 1 \text{ if } \alpha \in \Sigma$$
$$= 0 \text{ otherwise}$$

$$\delta_\Sigma(\alpha) = 1 \text{ if } \Diamond\alpha \supset \Box\alpha \in \Sigma$$
$$= 0 \text{ otherwise.}$$

The basis of the definition of δ_Σ is easily seen. Since either α or $\neg\alpha$ will have a V-value of 1, and $\delta(\alpha) = \delta(\neg\alpha)$, either $\Box\alpha$ or $\Box\neg\alpha$ will hold whenever $\delta(\alpha) = 1$. But clearly $\Box\neg\alpha \vee \Box\alpha$ is equivalent to $\Diamond\alpha \supset \Box\alpha$.

Again completeness is proved by demonstrating that V_Σ and δ_Σ exhibit those features defined by the matrices given above. For example, we show that $\delta_\Sigma(\bot) = 1$:

$\Box\top \in \Sigma$ by [N] $\quad \therefore \Box\neg\bot \in \Sigma$
$\therefore \Box\neg\bot \vee \Box\bot \in \Sigma \quad \therefore \neg\Box\neg\bot \supset \Box\bot \in \Sigma$
$\therefore \delta_\Sigma(\bot) = 1$

Thus δ_Σ obeys the rule [$\bot$1].

Of course, V_Σ must obey the rule:

$$V(\Box\alpha) = 1 \text{ iff } V(\alpha) = 1 \text{ and } \delta(\alpha) = 1.$$

This can be seen from the $\Box$-matrix. The result is easily proved:

($\Rightarrow$) Assume $\Box\alpha \in \Sigma$ $\quad \therefore \Diamond\alpha \supset \Box\alpha \in \Sigma$
$\therefore \delta_\Sigma(\alpha) = 1$ $\quad$ But $\alpha \in \Sigma$ by [T]
$\therefore V_\Sigma(\alpha) = 1$

($\Leftarrow$) Assume $\alpha \in \Sigma$ and $\Diamond\alpha \supset \Box\alpha \in \Sigma$
But $\alpha \supset \Diamond\alpha \in \Sigma$ by [T]
$\therefore \Box\alpha \in \Sigma \quad \therefore V_\Sigma(\Box\alpha) = 1$

It is a simple matter to show that δ_Σ produces the δ-values required by the $\supset$-matrix. To this end we note the following facts about this matrix:

[$\supset$1'] if $V(\alpha \supset \beta) = 1$, $\delta(\alpha \supset \beta) = 1$ iff
(a) $V(\alpha) = 0$ and $\delta(\alpha) = 1$, or
(b) $V(\beta) = 1$ and $\delta(\beta) = 1$

[$\supset$2'] if $V(\alpha \supset \beta) = 0$, $\delta(\alpha \supset \beta) = 1$ iff $\delta(\alpha) = 1$ and $\delta(\beta) = 1$

We need only show that this holds for δ_Σ, and again this is not difficult. For example, consider:

Lemma : $\delta_\Sigma(\alpha \supset \beta) = 1$ if $V_\Sigma(\alpha) = 0$ and $\delta_\Sigma(\alpha) = 1$

Assume $\neg\alpha \in \Sigma$ and $\Diamond\alpha \supset \Box\alpha \in \Sigma$
$\therefore \neg\Box\alpha \supset \neg\Diamond\alpha \in \Sigma$ i.e. $\Diamond\neg\alpha \supset \Box\neg\alpha \in \Sigma$
But $\neg\alpha \supset \Diamond\neg\alpha \in \Sigma$ by [T]
$\therefore \Box\neg\alpha \in \Sigma$ $\therefore \Box\neg\alpha \vee \Box\beta \in \Sigma$
$\therefore \Diamond\alpha \supset \Box\beta \in \Sigma$
$\therefore \Box(\alpha \supset \beta) \in \Sigma$ by [h]
$\therefore \delta_\Sigma(\alpha \supset \beta) = 1$ by the preceding lemma.

The other lemmas required are proved analogously.

Entailment

The notions of implication introduced above give rise, in the usual way, to an entailment relation ➧ by the definition:

$$\alpha ➧ \beta = \text{df } \Box(\alpha \to \beta)$$

as well as a strict implication $\prec$ by the definition

$$\alpha \prec \beta = \text{df } \Box(\alpha \supset \beta)$$

The ➧ matrix has, as one might expect only δ components of 1

➧	0	1	2	3
0	1	1	1	3
1	1	3	1	3
2	1	1	1	3
3	1	1	1	3

and is moderately relevant.

The $\prec$ matrix has, similarly, only δ components of 1 but is immoderately irrelevant:

$\prec$	0	1	2	3
0	1	1	1	3
1	3	3	3	3
2	1	1	1	3
3	1	1	1	3

The logic *PTE* of entailment is completely axiomatised by PL1-PL3 with E1-E3 and [US] and [MP].

[E1] $(P \blacklozenge Q) = (P \blacklozenge Q)$

[E2] $(Q \supset (P \blacklozenge Q) \equiv (Q \blacklozenge Q))$

[E3] $\neg Q \supset ((P \blacklozenge Q) \equiv (\neg P \wedge (P \blacklozenge P) \wedge (Q \blacklozenge Q))$

Completeness is straightforwardly shown via the canonical definitions:

For Σ a *PTE* maximal consistent set,

$$V_\Sigma(\alpha) = 1 \text{ if } \alpha \in \Sigma\text{; else } V_\Sigma(\alpha) = 0$$
$$\delta_\Sigma(\alpha) = 1 \text{ if } \alpha \blacklozenge \alpha \in \Sigma\text{; else } \delta_\Sigma(\alpha) = 0.$$

The axioms correspond to the three matrix rules.

[$\blacklozenge$1] $\delta(\alpha \blacklozenge \beta) = 1$

[$\blacklozenge$2] if $V(\beta) = 1$, $V(\alpha \blacklozenge \beta) = 1$ if $\delta(\beta) = 1$; else $V(\alpha \blacklozenge \beta) = 0$

[$\blacklozenge$3] if $V(\beta) = 0$, $V(\alpha \blacklozenge \beta) = 1$ if $V(\alpha) = 0$ and $\delta(\alpha) = \delta(\beta) = 1$; else $V(\alpha \blacklozenge \beta) = 0$.

REFERENCES

[1] Belnap, N. "How a computer should think?" in G. Ryle, Contemporary Aspects of Philosophy, Oriel Press, 1977.

[2] Bennett, J. 'Entailment', *The Philosophical Review* LXXVIII (1969) 197-236.

[3] Copeland, B. J. 'On when a semantics is not a semantics: some reasons for disliking the Routley-Meyer semantics for relevance logic', *Journal of Philosophical Logic* 8 (1979) 399-413.

[4] Cresswell, M. *Logics and Languages,* Methuen, London, 1973.

[5] Jaskowski, S. "Propositional Calculus for Contradictory Deductive Systems" *Studia Logica* 24 (1969) 143-57.

[6] Jennings, R. E. 'Paraconsistent Inculpation', Inaugural presentation, Simon Fraser University, 1983.

[7] Jennings R. E. and Schotch, P. K. 'The preservation of coherence', *Studia Logica,* forthcoming.

[8] Lewis, D. 'Logic for Equivocators' *Noûs*, 16 (1982) 431-441.

[9] Martin E. P. and Meyer, R. K. 'Solution to the *P-W* problem', *Journal of Symbolic Logic* 47 (1982) 869-887.

[10] Routley, R. and Meyer, R. K. 'Relevant logics and their semantics remain viable, and undamaged by Lewis's equivocation charge', *Topoi*, forthcoming.

[11] Schotch P. K. and Jennings, R. E. 'Inference and necessity', *Journal of Philosophical Logic* 9 (1980) 327-40.

[12] Segerberg, K. *An Essay in Classical Modal Logic,* Uppsala, 1971.