$KG^{k, l, m, n}$ AND THE EFMP

Brian F. CHELLAS

Introduction

For natural numbers k, l, m, and n, $KG^{k, l, m, n}$ is the smallest normal modal logic containing

$$\mathrm{G}^{k, l, m, n}. \qquad \Diamond^k \Box^l p \rightarrow \Box^m \Diamond^n p$$

The formula $\mathrm{G}^{k, l, m, n}$ embraces such familiar formulas as:

D. $\Box p \rightarrow \Diamond p$
T. $\Box p \rightarrow p$
B. $p \rightarrow \Box \Diamond p$
4. $\Box p \rightarrow \Box \Box p$
5. $\Diamond p \rightarrow \Box \Diamond p$

These figure prominently in such well-known logics as the Feys-von Wright "logique t" or system *M*, Kripke's "Brouwersche system", and Lewis's *S4* and *S5*.

Less well known, but important in what follows, is the instance

G. $\Diamond \Box p \rightarrow \Box \Diamond p$

of $\mathrm{G}^{k, l, m, n}$. This is the characteristic theorem of the modal logic *S4.2*, i.e. *KT4G*.

A logic is said to have the finite model property (fmp) just in case each non-theorem of the logic is false in some finite model of the logic. Alternatively, a logic has the fmp exactly when each formula consistent in the logic is true somewhere in a finite model of the logic. We say of a normal modal logic that it has the finite model property essentially (efmp) if and only if each of its normal extensions has the fmp.

The point of this paper is to examine the question: For what values k, l, m, n does $KG^{k, l, m, n}$ have the efmp? It turns out that a single theorem, essentially due to Kit Fine, provides an answer for all but a handful of quadruples of natural numbers. This is the main result of the paper. Except for one (or, perversely, two) the remaining cases can also be resolved. So a secondary purpose here is to state a problem for future research.

Background

A normal modal logic is based on propositional logic and may be characterized by the presence of the theorems

Df$\Diamond$. $\Diamond p \leftrightarrow \neg\Box\neg p$

K. $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$

and rules of modus ponens, substitution, and necessitation

RN. $\dfrac{A}{\Box A}$.

Models and frames for normal modal logics are structures $\mathcal{M} = \langle W, R, P\rangle$ and $\mathcal{F} = \langle W, R\rangle$ in which W is a set (of "possible worlds"), R is a binary relation in W, and P associates truth values with pairs of atomic formulas and worlds. $\mathcal{M}$ is a model on $\mathcal{F}$; $\mathcal{F}$ is the frame of $\mathcal{M}$. Models and frames are finite iff their sets of worlds are. Truth conditions for modalities are given by:

$\Box$A is true at x in $\mathcal{M}$ iff A holds at every y in $\mathcal{M}$ such that xRy.

$\Diamond$A is true at x in $\mathcal{M}$ iff A holds at some y in $\mathcal{M}$ such that xRy.

$\mathcal{M}$ is a model of a logic iff the logic's theorems are true at every world in $\mathcal{M}$. $\mathcal{F}$ is a frame for a logic iff every model on $\mathcal{F}$ is a model for the logic. A logic is determined by a class of models (frames) iff every member of the class is a model of (frame for) the logic and each non-theorem of the logic is rejected by some member of the class.

For each k, l, m, and n, $KG^{k,l,m,n}$ is determined by the class of k, l, m, n-incestual models and frames – i.e. those in which the relation R satisfies the condition that for every x, y, and z in W

if xR^ky and xR^mz then there is a w in W such that yR^lw and zR^nw.

For *KD, KT, KB, K4*, and *K5* this boils down to the familiar properties of seriality, reflexivity, symmetry, transitivity, and eucli-deanness. In the case of *KG* the condition is called simply incestuali-ty. (1)

S4.2 and the efmp

We begin with a lemma.

LEMMA. $KT4G^{k,l,m,n} = KT4G$ (*S4.2*) for all $k, l, m, n \geq 1$.

Proof. Simply note that every extension of *KT4* (*S4*) contains the "reduction laws" $\Box A \leftrightarrow \Box^i A$ and $\Diamond A \leftrightarrow \Diamond^i A$, for every $i \geq 1$.

THEOREM 1. *S4.2* – and therefore $KG^{k,l,m,n}$ for any $k, l, m, n \geq 1$ – has a normal extension that lacks the fmp.

Proof. In his "Logics containing S4 without the finite model property" [3] Kit Fine defines an extension *KT4X* of *S4* and shows that it does not have the fmp. It turns out that an almost exact copy of Fine's proof yields the result that the extension *KT4GX* of *S4.2* lacks the fmp.

Fine's formula X is the conditional $Y \rightarrow Z$, where Z is the formula

$$\Diamond(\Diamond p \wedge \Diamond q \wedge \neg\Diamond r),$$

and Y is a conjunction of these formulas:

(1) s
(2) $\Box(s \rightarrow \Diamond(\neg s \wedge \Diamond s))$
(3) $\Diamond p$
(4) $\Diamond q$
(5) $\Diamond r$
(6) $\Box(p \rightarrow (\neg\Diamond q \wedge \neg\Diamond r))$

(1) For more background material and information on $KG^{k,l,m,n}$ logics see [1], [4], or [7].

(7) $\Box(q \rightarrow (\neg\Diamond p \wedge \neg\Diamond r))$
(8) $\Box(r \rightarrow (\neg\Diamond p \wedge \neg\Diamond q))$

To show that *KT4GX* lacks the fmp it is enough to establish two lemmas.

Consistency lemma. Y is consistent in *KT4GX*.

Infinity lemma. Y holds at a world in a model $\mathcal{M}$ of *KT4GX* only if $\mathcal{M}$ is infinite.

For the infinity lemma it will do simply to remark that Fine proves the corresponding proposition for the logic *KT4X* – i.e. that any model of this logic is infinite if as much as one of its worlds verifies Y. Thus since *KT4GX* extends *KT4X* the result holds here as well.

To prove the consistency lemma it suffices to construct a model of *KT4GX* at one of the worlds of which Y holds. This may be accomplished by minutely modifying the model Fine employs for the analogous purpose in his proof.

Specifically, we consider $\mathcal{M} = \langle W, R, P\rangle$ in which the worlds are arranged as in the diagram in figure 1. In the diagram the arrows indicating *R* are meant to be transitive; the fat arrows mean that worlds 0 and -1 go by *R* to every world below them; and for the sake of readability self-directed arrows indicating reflexivity are everywhere omitted. As the diagram shows, the worlds other than 4, 0, and -1 are divided by levels, three worlds to a level. Thus worlds 1, 2, and 3 are on the lowest level, and so on.

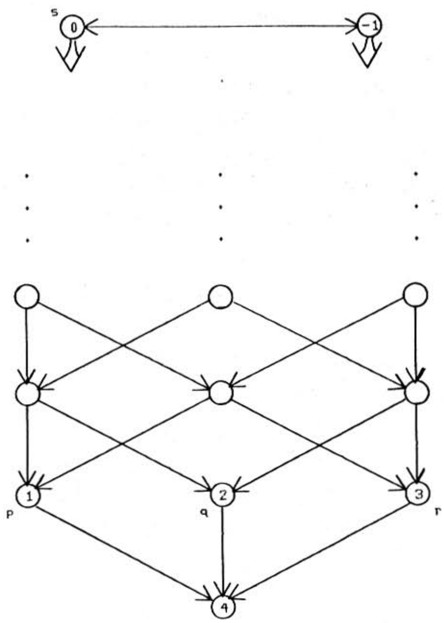
The model $\mathcal{M}$ is exactly like Fine's except for the addition of the world 4.

Note two features of $\mathcal{M}$: First, a world located on a level goes by *R* to all but one of the worlds on every level below it. Second, distinct worlds on the same level between them go by *R* to every world on every level below them. It follows that within the levels three distinct worlds are on the same level if none is an *R*-alternative of any of the others. This is important shortly.

The following two propositions will secure the consistency lemma.

(A) Y is true at 0 in $\mathcal{M}$.

(B) $\mathcal{M}$ is a model of *KT4GX*.



For (A) we verify by inspection that Y's eight conjuncts hold at 0 in $\mathscr{M}$. Thus s holds at 0 by construction, and since s holds only at 0 and $0R-1$ and $-1R0$ the formula $\Box(s \to \Diamond(\neg s \wedge \Diamond s))$ is true at 0. That takes care of (1) and (2). For (3) we note that $0R1$ and p is true at 1; similarly for (4) and (5). Finally, since p holds at 1 alone and none of 1's R-alternatives verifies q or r, $\Box(p \to (\neg\Diamond q \wedge \neg\Diamond r))$ holds at 0. This settles (6), and the reasoning for (7) and (8) is similar.

For (B), note first that the relation R is reflexive, transitive, and incestual. So $\mathscr{M}$ is a model of *KT4G*, and we need to show only that the frame $\mathscr{F} = \langle W, R \rangle$ of $\mathscr{M}$ is a frame for *KT4GX* – i.e. that X is true at every world in every model on $\mathscr{F}$.

Suppose that X's antecedent Y is true at a world w in some model on $\mathscr{F}$. Then Y's first two conjuncts hold at w. This means that the formula $\Diamond(\neg s \wedge \Diamond s)$ holds at w, and hence this world begins an unending R-chain of worlds in $\mathscr{F}$ at which s and $\neg$s hold alternately. Thus w is either 0 or -1, since otherwise both s and $\neg$s hold at 4. Note that w is R-related to every world in $\mathscr{F}$.

Because conjuncts (3)-(5) hold at w this world has R-alternatives x,

y, and z that verify p, q, and r respectively. We argue now that these worlds are all on the same level in $\mathscr{F}$.

Conjunct (6) holds at w. Hence $\Diamond$q and $\Diamond$r are false at x, and so neither y nor z is an R-alternative of x. Using (7) and (8) we conclude in general that none of x, y, and z is an R-alternative of any of the others. R's reflexivity entails moreover that these worlds are distinct. World 4 is an R-alternative to every world in $\mathscr{F}$, and every world is an R-alternative to 0 and -1. So none of x, y, and z is 4, 0, or -1; each is on some level in $\mathscr{F}$. It follows at once that x, y, and z are all on the same level.

Now we can show that X's consequent Z holds at w. Let us designate by xy the world immediately R-above worlds x and y in $\mathscr{F}$. Clearly $\Diamond$p and $\Diamond$q hold at xy. But $\Diamond$r is false at all R-alternatives to xy. So $\Diamond$p, $\Diamond$q, and $\neg\Diamond$r are all true at xy. Therefore, since $wRxy$, $\Diamond(\Diamond p \wedge \Diamond q \wedge \neg\Diamond r)$ is true at w. This ends the proof of (B) and so too the consistency lemma and the theorem.(2)

The remaining cases

Theorem 1 thus resolves, negatively, the question of the efmp for all but at most fifteen of the $KG^{k,l,m,n}$ logics – to wit those in which k, l, m, and n are either 0 or 1 and at least one is 0. Eliminating by duality, and discounting alternative axiomatizations, this number reduces to eight:

$$KG^{0,0,0,0} = K$$
$$KG^{0,0,1,0} = KT_c$$
$$KG^{0,0,1,1} = KB$$
$$KG^{0,1,0,0} = KT$$
$$KG^{0,1,0,1} = KD$$
$$KG^{0,1,1,1}$$
$$KG^{1,0,1,0} = KD_c$$
$$KG^{1,0,1,1} = K5$$

– where T_c and D_c are the converses $p \to \Box p$ and $\Diamond p \to \Box p$ of T and D.

(2) I am very indebted to Alasdair Urquhart, who suggested to me that Fine's proof could be thus adapted to show that *S4.2* lacks the efmp [9].

Four of these logics – K, KT, KD, and $KG^{0,1,1,1}$ – are sublogics of $S4.2$. So by theorem 1 these all lack the efmp.

The case of $K5$ has been resolved affirmatively.

THEOREM 2. $K5$ has the efmp.

This was proved by Michael Nagle in his doctoral dissertation [6] and reported in the *Journal of symbolic logic* [5].

The question of the efmp and the logic KD_c was open until recently.

THEOREM 3. KD_c has the efmp.

Krister Segerberg presented a proof of this in a paper read at the 1983 Conference of the Australasian Association for Logic [8].

The logic KT_c is an extension of both $K5$ and KD_c. So Nagle's and Segerberg's results entail that KT_c has the efmp. But the result is easy to see directly, since generated models for extensions of KT_c have at most one world (0, 0, 1, 0-incestuality is the condition that xRy only if $x = y$). (3)

There remains thus the question of $KG^{0,0,1,1}$: Is there a normal logic containing the "Brouwersche axiom", B, that fails to have the finite model property? I do not know, and so I conclude the paper with this query. (4)

University of Calgary Brian F. CHELLAS

(3) Tne extensions of KT_c are interesting in their own right. Note that they are always normal: for if A is a theorem so is □A by modus ponens and T_c. By examining the generated frames for KT_c, moreover, one can see that its consistent proper extensions are but two in number. One is $K\Box\bot$, also known as *ABS* or *VERUM*. It is determined by the one-element frame in which the relation is empty. The other is KTT_c, also known as *TRIV* (since necessity is equated with truth). This logic is determined by the other one-element frame, in which the relation is total.

(4) The results reported here were included in [2]. The research was supported by a Leave Fellowship from the Social Sciences and Humanities Research Council of Canada and by a Sabbatical Leave Research Grant from the University of Calgary, both of which are gratefully acknowledged. Further subventions from these institutions enabled me twice to travel to New Zealand during my sabbatical leave to work at the University of Auckland. I wish especially to thank my host in Auckland, Professor Krister Segerberg, for our many profitable discussions, and in particular for his resolution of the efmp question for KD_c.

REFERENCES

[1] Chellas, Brian F. *Modal logic: an introduction.* Cambridge and New York: Cambridge University Press, 1980.

[2] Chellas, Brian F. "$G^{k,l,m,n}$ and the finite model property". Paper presented at the 1983 Conference of the Australasian Association for Logic, University of Western Australia, Perth, 18-21 May 1983.

[3] Fine, Kit. "Logics containing S4 without the finite model property". In *Conference in mathematical logic–London '70,* edited by W. Hodges, pp. 98-102. *Lecture notes in mathematics*, no. 255. Berlin: Springer-Verlag, 1972.

[4] Lemmon, E. J. (in collaboration with Dana Scott). *The "Lemmon notes": an introduction to modal logic,* edited by Krister Segerberg. American Philosophical Quarterly Monograph Series, edited by Nicholas Rescher, no. 11. Oxford: Basil Blackwell, 1977.

[5] Nagle, Michael C. "The decidability of normal *K5* logics". *Journal of symbolic logic* 46 (1981): 319-28.

[6] Nagle, Michael C. "Normal *K5* logics". Ph.D. diss., University of Calgary, 1981.

[7] Segerberg, Krister. *An essay in classical modal logic.* 3 vols. Uppsala: University of Uppsala, 1971.

[8] Segerberg, Krister. "A small corner of the big lattice of modal logic". Paper presented at the 1983 Conference of the Australasian Association for Logic, University of Western Australia, Perth, 18-21 May 1983.

[9] Urquhart, Alasdair. Letter to author. 28 September 1981.