DEDUCTION SYSTEMS AND VALUATION SPACES

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§ 1. *Introduction*

This paper is a brief summary of two technical reports by the author (Katz [1, 2]), where detailed proofs of the various lemmas and theorems given here, as well as some additionel concepts and results, can be found. Priestley's [3] representation theory for distributive lattices is extended to generalized deduction systems satisfying the structural pre-connective rules (as given in, e.g., Scott [4]) which are common to all conventional deductive logics. A complete duality theory is developed and some of the machinery built in the process is then applied to the study of distributive lattices and generic structures following the ideas of Simmons [5].

§ 2. *Deduction Systems*

Definition 2.1: The pair $(\Psi, \vdash)$ is a deduction system if $\vdash$ is a binary relation on the collection $P_\omega(\Psi)$ of finite subsets of the non-empty set Ψ, satisfying, for all $\psi \in \Psi$ and all $\Gamma, \Delta, \Gamma', \Delta' \in P_\omega(\Psi)$, the following reflexive, monotone and transitive rules:

$$[R] \qquad \psi \vdash \psi$$

$$[M] \quad \frac{\Gamma \vdash \Delta}{\Gamma, \Gamma' \vdash \Delta, \Delta'}$$

$$[T] \quad \frac{\Gamma, \psi \vdash \Delta \quad \Gamma \vdash \psi, \Delta}{\Gamma \vdash \Delta}$$

Here $\Gamma \vdash \Delta$ stands for $(\Gamma, \Delta) \in \vdash$, Γ, Γ' for $\Gamma \cup \Gamma'$ and ψ for $\{\psi\}$. The rule [R] is said to be linear, or unconditional, or an axiom. The other two read: if the deduction(s) above the line hold(s) so does the one below the line.

Definition 2.2: A deduction algebra is a deduction system $(\Psi, \vdash)$ in which $\psi = \Theta$ for all $\psi, \Theta \in \Psi$ s.t. $\psi \vdash \Theta$ and $\Theta \vdash \psi$.

From every deduction system $(\Psi, \vdash)$ a deduction algebra can be obtained by factoring over the relation $\sim$ defined for every $\psi, \Theta \in \Psi$ by

$$\psi \sim \Theta \text{ iff } \psi \vdash \Theta \ \& \ \Theta \vdash \psi \ ,$$

since it can be shown that $\sim$ is an equivalence relation on Ψ and a congruence w.r.t. $\vdash$ in the obvious sense.

The following are some conventional examples of deduction systems and algebras. Note that in each of these examples, as indeed in general, the converse of a deduction is also a deduction.

Ex. 2.3: Let $(\Psi, \leq)$ be a quasi-order. The relation $\vdash$ defined on $P_\omega(\Psi)$ by

$$\Gamma \vdash \Delta \text{ iff } \exists \psi \in \Gamma \ \exists \Theta \in \Delta. \ \psi \leq \Theta$$

forms a deduction system $(\Psi, \vdash)$. If $(\Psi, \leq)$ is a partial order then $(\Psi, \vdash)$ is a deduction algebra.

Ex. 2.4: Let $(\Psi^*, \wedge, \vee)$ be a distributive lattice (with corresponding ordering $\leqslant$), and let $\Psi \subseteq \Psi^*$. The relation $\vdash$ defined on $P_\omega(\Psi)$ by

$$\Gamma \vdash \Delta \text{ iff } \wedge \Gamma \leq \vee \Delta$$

forms a deduction algebra $(\Psi, \vdash)$. Such deductions, called lattice deductions, show how our deduction systems generalize the notion of a distributive lattice.

In particular if Ψ is a collection of sets then the relation $\vdash$ defined on $P_\omega(\Psi)$ by

$$\Gamma \vdash \Delta \text{ iff } \cap \Gamma \subseteq \cup \Delta$$

forms a deduction which we shall call set-deduction.

Ex. 2.5: Let $X \subseteq \{0,1\}^{\Psi}$ for some set Ψ. The relation $\vdash$ defined on $P_{\omega}(\Psi)$ by

$$\Gamma \vdash \Delta \text{ iff } \forall x \in X(\forall \psi \in \Gamma\ (x(\psi) = 1) \to \exists \Theta \in \Delta(x(\Theta) = 1))$$

forms a deduction system $(\Psi, \vdash)$.

This deduction is equivalently defined as follows. For each $\psi \in \Psi$ let

$$i(\psi) = \{x \in X : x(\psi) = 1\}.$$

Then for all $\Gamma, \Delta \in P_{\omega}(\Psi)$

$$\Gamma \vdash \Delta \text{ iff } \bigcap_{\psi \in \Gamma} i(\psi) \subseteq \bigcup_{\Theta \in \Delta} i(\Theta) .$$

On the right side we have here a set-deduction on

$$S^{i} = \{i(\psi) : \psi \in \Psi\} .$$

Whereas S^{i} with its set-deduction is obviously a deduction algebra, $(\Psi, \vdash)$ will in general become a deduction algebra only after factoring over the relation

$$\psi \sim \Theta \text{ iff } \forall x \in X. x(\psi) = x(\Theta) .$$

Ex. 2.6: Let T be a theory based on a certain propositional or predicate logic (classical, intuitionistic, modal or almost any other logic we may wish to consider). Let Ψ be a set of formulae in the language of T. The relation $\vdash$ defined on $P_{\omega}(\Psi)$ by

$$\Gamma \vdash \Delta \text{ iff } \Gamma \vdash_{T} \Delta ,$$

where $\Gamma \vdash_{T} \Delta$ has its usual meaning, forms a deduction system $(\Psi, \vdash)$. Obviously we shall be concerned with this kind of deduction, explicitly or implicitly, throughout this paper. A deduction algebra is obtained from a deduction system of this type through the usual procedure of constructing the appropriate Lindenbaum algebra.

Definition 2.7: Let $(\Psi, \vdash)$ and $(\Omega, |\vdash)$ be two deduction algebras and let h be a map of Ψ into Ω.

(i) h is said to be a homomorphism (of Ψ into Ω) if for every Γ, $\Delta \in P_{\omega}(\Psi)$

$$\Gamma \vdash \Delta \rightarrow h(\Gamma) \mid\vdash h(\Delta)$$

where

$$h(\Gamma) = \{h(\psi) : \psi \in \Gamma\}$$
$$h(\Delta) = \{h(\Theta) : \Theta \in \Delta\}$$

(ii) h is said to be a$\nvdash$-homomorphism (of Ψ into Ω) if for every Γ, $\Delta \in P_\omega(\Psi)$

$$\Gamma \nvdash \Delta \rightarrow h(\Gamma) \mid\nvdash h(\Delta)$$

with $h(\Gamma)$ and $h(\Delta)$ as in (i) .

(iii) h is said to be an isomorphism (of Ψ onto Ω) if it is a homomorphism, a$\nvdash$-homomorphism and onto.

Note that if h is a$\nvdash$-homomorphism then in particular it is 1-1. So if in addition h is onto (and in particular if it is an isomorphism) then the inverse $h^{-1} : \Omega \rightarrow \Psi$ is well-defined.

Clearly if h is an isomorphism of Ψ onto Ω then h^{-1} is an isomorphism of Ω onto Ψ. So if such an isomorphism h exists we say that Ψ is isomorphic to Ω (and Ω is isomorphic to Ψ), or simply that Ψ and Ω are isomorphic. It is obvious that being isomorphic is an equivalence relation on the class of deduction algebras.

Note that all three parts of the above definition are so formulated that they (and hence the representation theorem and the dual constructions involved in it) can apply also to deduction systems which are not algebras. However, in this case a $\nvdash$ -homomorphism is not necessarily 1-1 and the isomorphism relation is not necessarily symmetric.

§ 3. *Valuation Spaces*

Let X be a non-empty set and let S^i be a subset of P(X). Define a binary relation $\leq$ on X, and three additional subsets S^d, S, T of P(X) as follows:

$x \leq y$ iff $\forall Y \in S^i . x \in Y \rightarrow y \in Y$
$S^d = \{Y \subseteq X : X \setminus Y \in S^i\}$
$S = S^i \cup S^d$
T is the topology on X for which S is an open subbase.

Note that $\leq$ is reflexive and transitive (i.e., a quasi-ordering on X), and that it could equivalently be defined by

$$x \leq y \text{ iff } \forall Y \in S^d, y \in Y \rightarrow x \in Y.$$

Definition 3.1: Let S^i be a subset of P(X) for some non empty set X, and define $\leq$, S^d, S, T as above. The sequence (X, T, $\leq$, S) is said to be a valuation space if

(i) (X, T) is compact,
(ii) $\leq$ is anti-symmetric.

The following theorem whose proof is straightforward provides some interesting properties of valuation spaces.

Theorem 3.2: If (X, T, $\leq$, S) is a valuation space then:

(1) (X, T) is a Stone space,
(2) (X, $\leq$) is a partial order,
(3) the elements of S are clopen in T,
(4) (a) the elements os S^i are increasing, i.e., for all $Y \in S^i$ and all $x, y \in X$

$$x \in Y \wedge x \leq y \rightarrow y \in Y$$

(b) the elements of S^d are decreasing, i.e., for all $Y \in S^d$ and all $x, y \in X$

$$x \in Y \wedge y \leq x \rightarrow y \in Y$$

(5) S is a $\leq$-separation set for X, i.e., whenever $x, y \in X$ and $x \not\leq y$ there is a subset Y of X s.t.

$$x \in Y \in S^i \text{ and } y \in X \setminus Y \in S^d \ .$$

Definition 3.3: Let (X_1, T_1, $\leq_1$, S_1) and (X_2, T_2, $\leq_2$, S_2) be two valuation spaces and let f be a map of X_1 into X_2.

(i) f is said to be s-continuous if for every $Y \subseteq X_2$
$Y \in S_2^i \rightarrow f^{-1}(Y) \in S_1^i$
$Y \in S_2^d \rightarrow f^{-1}(Y) \in S_1^d$

(ii) f is said to be s-open if for every $Y \subseteq X_1$

$Y \in S_1^i \to f(Y) \in S_2^i$
$Y \in S_1^d \to f(Y) \in S_2^d$

(iii) f is said to be an s-homeomorphism if it is 1-1, onto, s-continuous and s-open.

It is clear again that if f is an s-homeomorphism (of X_1 onto X_2) then f^{-1} is also an s-homeomoerphism (of X_2 onto X_1). So if such an f exists we say that X_1 is s-homeomorphic to X_2 (and X_2 is a s-homeomorphic to X_1), or simply that X_1 and X_2 are s-homeomorphic. Obviously being s-homeomorphic is an equivalence relation on the class of valuation spaces.

Lemma 3.4: Let $(X_1, T_1, \leqslant_1, S_1)$ and $(X_2, T_2, \leqslant_2, S_2)$ be two valuation spaces and let f be a map of X_1 into X_2.

(1) If f is s-continuous then f is continuous.
(2) If f is 1-1 and s-open then f is open.
(3) If f is an s-homeomorphism then f is a homeomorphism.

Note that in fact s-continuity (etc.) is much stronger than continuity (etc.), since ordinary continuity does not at all require that the inverse image of a subbasic set is also subbasic.

Lemma 3.5: Let $(X_1, T_1, \leqslant_1, S_1)$ and $(X_2, T_2, \leqslant_2, S_2)$ be two valuation spaces and let f be a map of X_1 into X_2.

(1) If f is s-continuous then f is $\leqslant$-preserving, i.e., for all $x, y \in X_1$

$$x \leqslant_1 y \to f(x) \leqslant_2 f(y)$$

(2) If f is 1-1 and s-open then f is $\nleq$-preserving (and $f^{-1} : f(X_1) \to X_1$ is $\leq$-preserving), i.e., for all $x, y \in X_1$

$$x \nleq_1 y \to f(x) \nleq_2 f(y).$$

Equivalenty, for all $x, y \in X_1$

$$f(x) \leq_2 f(y) \to x \leq_1 y.$$

(3) If f is an s-homeomorphism then f is a $\leq$-isomorphism, i.e., for all $x, y \in X_1$

$$x \leq_1 y \leftrightarrow f(x) \leq_2 f(y).$$

Proof: For (1) suppose $x \leq_1 y$ and $f(x) \in Y$ for some $Y \in S_2^i$. Then $x \in f^{-1}(Y) \in S_1^i$ since f is s-continuous. But elements of S_1^i are increasing, so $y \in f^{-1}(Y)$, i.e., $f(y) \in Y$. Thus by the definition of $\leq$, $f(x) \leq_2 f(y)$.

For (2) suppose $f(x) \leq_2 f(y)$ and $x \in Y$ for some $Y \in S_1^i$. Then $f(x) \in f(Y) \in S_2^i$ since f is s-open. But elements of S_2^i are increasing, so $f(y) \in f(Y)$, and since f is 1-1 $y \in Y$. Thus by the definition of $\leq$, $x \leq_1 y$.

Clearly (3) follows from (1) + (2) and so the proof is complete.

§ 4. *Dual constructions*

Definition 4.1: The dual algebra of the valuation space $(X, T, \leq, S)$ is the set-deduction algebra whose domain is S^i.

Now, denote by $(2, \vdash)$ (or just by 2) the deduction algebra whose domain is $2 = \{0,1\}$ and whose deduction is defined by

$$\Gamma \vdash \Delta \text{ iff } \min_{\psi \in \Gamma} \psi \leq \max_{\Theta \in \Delta} \Theta$$

where $\leq$ is the natural ordering on 2 (and min and max are taken w.r.t. this $\leq$), and where for the empty set ϕ we let

$$\min \phi = 1 \quad \text{and} \quad \max \phi = 0.$$

Definition 4.2: A valuation of a deduction algebra $(\Psi, \vdash)$ is a homomorphism of Ψ into 2.

Definition 4.3: Let X be the collection of valuations of the deduction algebra $(\Psi, \vdash)$, and let $\psi \in \Psi$. The truth set for ψ is

$$i(\psi) = \{x \in X : x(\psi) = 1\}$$

and the collection of truth-sets for Ψ is

$$S^i = \{i(\psi) : \psi \in \Psi\}.$$

Definition 4.4: The dual space of the deduction algebra $(\Psi, \vdash)$ is the sequence $(X, T, \leq, S)$, where X is the collection of valuations of Ψ, and $T, \leq, S$ are defined as in the previous section by means of the collection S^i of truth sets for Ψ.

Lemme 4.5: Let T^* be the product topology on 2^Ψ, S^* be the natural subbase of T^*, and for each $x, y \in 2^\Psi$ set (where $\leq$ is the natural ordering on 2):

$$x \leq^* y \text{ iff } \forall \psi \in \Psi.\ x(\psi) \leq y(\psi).$$

Then $(2^\Psi, T^*, \leqslant^*, S^*)$ is a valuation space.

Proof: The lemma follows from the fact that

(1) $(2^\Psi, T^*)$ is compact,
(2) $(2^\Psi, \leq^*)$ is a partial order,
(3) $S^* = S^{*i} \cup S^{*d}$
where

$$S^{*i} = \{i^*(\psi) : \psi \in \Psi\}$$
$$S^{*d} = \{d^*(\psi) : \psi \in \Psi\} = \{Y \subseteq 2^\Psi : 2^\Psi \setminus Y \in S^{*i}\}$$

and for every $\psi \in \Psi$

$$i^*(\psi) = \{x \in 2^\Psi : x(\psi) = 1\}$$
$$d^*(\psi) = \{x \in 2^\Psi : x(\psi) = 0\},$$

(4) $x \leqslant^* y$ iff $\forall Y \in S^{*i}.\ x \in Y \rightarrow y \in Y$.

for all $x, y \in 2^\Psi$.

Lemma 4.6: Let $(X, T, \leq, S)$ be a valuation space, and let X_1 be a subset of X closed in (X, T). $(X_1, T_1, \leq, S_1)$ is a valuation space, where T_1, $\leq_1$, S_1 are the restrictions to X_1 of T, $\leq$, S.

Proof: The lemma follows from the fact that

(1) (X_1, T_1) is compact,
(2) $(X_1, \leq_1)$ is a partial order,
(3) S_1 is a subbase for T_1 and
$S_1 = S_1^i \cup S_1^d$

where

$$S_1^i = \{Y \cap X_1 : Y \in S^i\}$$
$$S_1^d = \{Y \cap X_1 : Y \in S^d\} = \{Y \subseteq X_1 : X_1 \setminus Y \in S_1^i\},$$

(4) $x \leq_1 y$ iff $\forall Y \in S^i.\ x \in Y \rightarrow y \in Y$

for all $x, y \in X_1$.

Theorem 4.7: The dual space of a deduction algebra is a valuation space.

Proof: Check that the topology T in Definition 4.4 is in fact the restriction to X of the product topology T^* on 2^{Ψ} and that (X, T) is closed in (2^{Ψ}, T^*). Thus the theorem follows from the last two lemmas.

§ 5. *Duality Theory*

A complete duality theory for deduction algebras and valuation spaces is developed in this section starting with the representation theorem for deduction algebras. Theorems 5.1 and 5.5 are parts of the representation theorem (5.6), but we bring them separately because of their logical significance.

Theorem 5.1 (Correctness): For every deduction algebra $(\Psi, \vdash)$ the map $i : \Psi \to S^i$ of Definition 4.3 is a homomorphism.

Proof: It is obvious that for every $\Gamma, \Delta \in P_\omega(\Psi)$

$$\Gamma \vdash \Delta \to \bigcap_{\psi \in \Gamma} i(\psi) \subseteq \bigcup_{\Theta \in \Delta} i(\Theta) .$$

Definition 5.2: A prime-filter in the deduction algebra $(\Psi, \vdash)$ is a subset Φ of Ψ s.t. for all $\Gamma, \Delta \in P_\omega(\Psi)$ if $\Gamma \vdash \Delta$ then

either $\Gamma \not\subset \Phi$
or $\Delta \not\subset \Psi \setminus \Phi$

The set $\Psi \setminus \Phi$ is then said to be a prime-ideal of $(\Psi, \vdash)$.

Lemma 5.3: If $(\Psi, \vdash)$ is a deduction algebra and $\Gamma, \Delta \in P_\omega(\Psi)$ are s.t. $\Gamma \nvdash \Delta$, then there is a prime-filter Φ of Ψ s.t.

$$\Gamma \subseteq \Phi$$
$$\Delta \subseteq \Psi \setminus \Phi .$$

Proof: (We assume the axiom of choice and hence impose no restrictions on the size of Ψ).
Note first that by [R] and [M] $\Gamma \nvdash \Delta$ implies

(a) $\Gamma \cap \Delta = \phi$
(b) $\Gamma' \subseteq \Gamma \wedge \Delta' \subseteq \Delta \rightarrow \Gamma' \not\vdash \Delta'$.

Now let $\{\psi_\alpha\}_{\alpha<\beta}$ be an enumeration of Ψ (where α, β denote ordinals) and define transfinite sequences Γ_α and Δ_α (for $\alpha \leqslant \beta$) as follows:

(i) $\Gamma_0 = \Gamma$ and $\Delta_0 = \Delta$
(ii) $\Gamma_{\gamma+1} = \Gamma_\gamma \cup \{\psi_\gamma\}$ and $\Delta_{\gamma+1} = \Delta_\gamma$
if (*) for every $\Gamma', \Delta' \in P_\omega(\Psi)$
$\Gamma' \subseteq \Gamma_\gamma \cup \{\psi_\gamma\} \wedge \Delta' \subseteq \Delta_\gamma \rightarrow \Gamma' \not\vdash \Delta'$
(iii) $\Gamma_{\gamma+1} = \Gamma_\gamma$ and $\Delta_{\gamma+1} = \Delta_\gamma \cup \{\psi_\gamma\}$
iff (*) of (ii) is not the case
(iv) $\Gamma_\lambda = \bigcup_{\gamma<\lambda} \Gamma_\gamma$ and $\Delta_\lambda = \bigcup_{\gamma<\lambda} \Delta_\gamma$
if λ is a limit ordinal.

That for every $\alpha \leqslant \beta$ and every $\Gamma', \Delta' \in P_\omega(\Psi)$ we now have
$(b)_\alpha$ $\Gamma' \subseteq \Gamma_\alpha \wedge \Delta' \subseteq \Delta_\alpha \rightarrow \Gamma' \not\vdash \Delta'$

is provable by induction, using (b) above for the 0-stage, [M] + [T] for the successor stages and the definitions of Γ_λ and Δ_λ for the limit stages. Now let $\Phi = \Gamma_\beta$.
Then by (a) and (i) – (iv) $\Psi \setminus \Phi = \Delta_\beta$, by $(b)_\alpha$ Φ is a prime filter of Ψ, and by (i) – (iv) again, Φ satisfies the assertion of the lemma.

Lemma 5.4: If $(\Psi, \vdash)$ is a deduction algebra then for each prime-filter Φ of Ψ there is a valuation x of Ψ, and for each valuation x of Ψ there is a prime-filter Φ of Ψ, s.t. for all $\psi \in \Psi$

* $\quad x(\psi) = 1$ iff $\psi \in \Phi$.

Proof: If Φ is a prime-filter of Ψ define $x : \Psi \rightarrow 2$ by * and check that it is a valuation of Ψ. If x is a valuation of Ψ define a subset Φ of Ψ by* and check that it is a prime-filter of Ψ.

Theorem 5.5 (Completeness): For every deduction algebra $(\Psi, \vdash)$ the map $i : \Psi \rightarrow S^i$ of Definition 4.3 is a$\not\vdash$homomorphism.

Proof: Let $\Gamma, \Delta \in P_\omega(\Psi)$ be s.t. $\Gamma \not\vdash \Delta$ and let Φ be the prime-filter of Ψ satisfying lemma 5.3 for these Γ and Δ. Then let x be the valuation on Ψ defined from this prime-filter Φ by * of Lemma 5.4. It is obvious that

$$\forall \psi \in \Gamma . x(\Psi) = 1$$
$$\forall \Theta \in \Delta . x(\Theta) = 0$$

and thus we have

$$\bigcap_{\psi \in \Gamma} i(\psi) \not\subset \bigcup_{\Theta \in \Delta} i(\Theta)$$

Theorem 5.6 (Representation): Every deduction algebra is isomorphic to a set-deduction algebra, namely to the dual algebra of its dual space.

Proof: If $(\Psi, \vdash)$ is a deduction algebra then the map i: $\Psi \rightarrow S^i$ of Definition 4.3 is the required isomorphism.

This follows immediately from Theorems 5.1 and 5.5, together with the obvious fact that i is onto by deninitionm

We now turn to the duality theorem for valuation spaces.

Theorem 5.7: Every valuation space is s-homeomorphic (hence, in particular, homeomorphic and order-isomorphic) to the dual space of its dual algebra.

Proof: Let $(X_2, T_2, \leq_2, S_2)$ be the dual space of the dual algebra of the valuation space $(X_1, T_1, \leq_1, S_1)$, and define a map f: $X_1 \rightarrow X_2$ by

$$f(x)(Y) = 1 \text{ iff } x \in Y$$

for each $x \in X_1$ and each $Y \in S_1^i$. It is easy to check that for any $x \in X_1$ f(x) is a valuation of S_1^i, so that it is indeed an element of X_2. The proof that f is s-continuous, s-open, onto and 1-1 is in Katz [1].

In the same paper the reader can find detailed proofs of the remaining results of this section, dealing with duality for morphisms. In the sequel –

X_1 is the dual space of the deduction algebra Ψ,
X_2 is the dual space of the deduction algebra Ω,
h is a map of Ψ into Ω,
f is a map of X_2 into X_1.

(We note that for the following results to hold Ω must be a deduction algebra, while Ψ may be a deduction system).

Definition 5.8: If h and f are s.t. for all $x \in X_2$ and all $\psi \in \Psi$

$$f(x)(\psi) = x(h(\psi))$$

then h said to be dual to f, f is said to be dual to h, and (h, f) is said to be a dual pair.

It is easy to prove that if a dual to f (or to h) exists then it is unique. Thus we may speak of *the* dual of f (or h), and deduce that the dual of the dual of f (of h) is f itself (h itself). Conditions for the existence of duals are given in the following duality theorem for morphisms.

Theorem 5.9: (i) If h is a homomorphism then there is a dual f to h. (ii) If f is s-continuous then there is a dual h to f. (iii) If (h, f) is a dual pair then h is a homomorphism and f is s-continuous.

In the next theorem we provide some connections between various properties of the components of a dual pair.

Theorem 5.10: If (h, f) is a dual pair then (i) f is onto iff h is a $\succ\!\!\!\!\vdash$-homomorphism. (ii) f is $\precsim$-preserving if h is onto. (iii) h is onto if f is 1-1 and s-open. (iv) f is s-open if h and f are onto.

Note that (iii) might be considered a partial converse of (ii) since if f is 1-1 and s-open then in particular it is $\precsim$-preserving. We conclude with an important corollary which follows immediately from the last two theorems.

Corollary 5.11: Let (h, f) be a dual pair. Then h is an isomorphism iff f is an s-homeomorphism.

§ 6 *The Case of a Distributive Lattice*

Let $(\Psi, \wedge, \vee)$ be a distributive lattice with a unit 1 and a zero element 0 s.t. $0 \neq 1$, and let $\vdash$ be the lattice deduction of Example 2.4. If we exclude the trivial valuations (which map all elements of Ψ on 0 or all elements of Ψ on 1) it is easy to see that the valuations of $(\Psi, \vdash)$ are exactly the 0,1-preserving lattice homomorphisms of $(\Psi, \wedge, \vee)$ onto 2, where the natural lattice operations are defined on 2. It is also easy to see that if on the set X of valuations of Ψ we define S^i (and S^d) as in Definition 4.3 then, since Ψ is closed under $\wedge$ and $\vee$, each of S^i and S^d is closed under finite unions and finite intersections.

Now for a given valuation space $(X, T, \leq, S)$ denote by C the set of clopen elements of T, by I the set of increasing subsets of X (w.r.t. $\leq$), and by D the set of decreasing subsets of X (w.r.t. $\leq$). That is

$$I = \{Y \subseteq X: \forall x, y \in X . y \in Y \wedge y \leq x \rightarrow x \in Y\}$$
$$D = \{Y \subseteq X: \forall x, y \in X . y \in Y \wedge x \leq y \rightarrow x \in Y\}.$$

Theorem 6.1: Let $(\Psi, \wedge, \vee), \vdash, X, S^i, S^d$ be as in the first paragraph of this section and let $(X, T, \leq, S)$ be the valuation space obtained from X and S^i in the usual way (i.e., the dual space of $(\Psi, \vdash)$). Define C, I, D as above for this space. Then:

$$S^i = C \cap I$$
$$S^d = C \cap D.$$

The proof of this theorem (Katz [1]) follows, more or less, the relevant steps in Priestley's [3] proof of the representation theorem for distributive lattices.

It follows that for spaces corresponding to distributive lattices instead of starting with S^i and constructing T and $\leq$ from it, we can start with T and $\leq$ and define S^i to be the collection of clopen increasing subsets. Thus Priestley's duality theory for distributive lattices and what she calls 'compact totally-order-disconnected spaces' $(X, T, \leq)$ is a special case of our duality theory for deduction algebras and valuation spaces. (And in this special case we can also prove a full converse of Theorem 5.10 (ii) using the closure of S^i (and S^d) under finite unions and intersections). Furthermore the duality theory for Boolean algebras and Stone spaces is the special case of Priestley's theory where $\leq$ reduces to equality.

We now turn to another type of deduction which can be defined on a distributive lattice.

Definition 6.2: Let $(\Psi, \wedge, \vee)$ be a distributive lattice with (0 and) 1. The complementation deduction on Ψ is the binary relation $|\vdash$ defined on $P_\omega(\Psi)$ by

$$\Gamma \mid\vdash \Delta \text{ iff } (\wedge\Gamma)^+ \subseteq (\vee\Delta)^+$$

where for $\psi \in \Psi$

$$\psi^+ = \{\Theta \in \Psi: \psi \vee \Theta = 1\}$$

Theorem 6.3: Let $(\Psi, \wedge, \vee)$ and $|\vdash$ be as above, let $\vdash$ be the lattice deduction an Ψ, and let $(X, T, \leq, S)$ be the dual space of $(\Psi, \vdash)$. Denote by μX the collection of elements of X which are minimal w.r.t. $\leq$. Then we have:

(i) $\forall x \in X \; \exists y \in \mu X \,.\, y \leq x$
(ii) $\forall x \in X \, (x \in \mu X \leftrightarrow \forall \psi \in \Psi \,.\, x(\psi^{+}) = \{1\} \rightarrow x(\psi) = 0)$
(iii) $\forall \psi, \Theta \in \Psi (\psi \in \Theta^{+} \leftrightarrow \forall x \in \mu X \,.\, x(\psi) = 0 \rightarrow x(\Theta) = 1)$
(iv) $\forall \psi, \Theta \in \Psi (\psi \,|\!\vdash \Theta \leftrightarrow \forall x \in \mu X \,.\, x(\psi) \leq x(\Theta))$.

This powerful theorem is the amalgamation of several results proved in Simmons [5].

Corollary 6.4: Let Ψ and $|\vdash$ be as in Definition 6.2. Then $(\Psi, |\vdash)$ is a deduction system.

Proof: From Theorem 6.3 (iv) we deduce that for all $\Gamma, \Delta \in P_{\omega}(\Psi)$

$$\Gamma \,|\!\vdash \Delta \;\text{ iff }\; \forall x \in \mu X \,.\, x(\wedge \Gamma) \leq x(\vee \Delta) \;.$$

Hence, since deduction valuations preserve lattice operations,

$$(*) \qquad \Gamma \,|\!\vdash \Delta \;\text{ iff }\; \forall x \in \mu X \,.\, \min_{\psi \in \Gamma} \; x(\psi) \leq \max_{\Theta \in \Delta} \; x(\Theta)$$

and thus the complementation deduction $|\vdash$ is a special case of Example 2.5 above.

Now that we know that $(\Psi, |\vdash)$ is a deduction system the natural question to ask is what is the dual space of $(\Psi, |\vdash)$. This question is sound, in view of the remarks in the last paragraph of §2. Only later in this section, when we apply our duality theorems for morphisms, we shall have to make sure that one of the systems involved is an algebra.

Theorem 6.5: Let $(\Psi, \wedge, \vee), |\vdash, \vdash$ and $(X, T, \leq, S)$ be as in Theorem 6.3. The dual space of $(\Psi, |\vdash)$ is the sequence $(C\lambda(\mu X), T_1, \leq_1, S_1)$, where $C\lambda(\mu X)$ is the closure in (X, T) of the set μX of minimal elements of X w.r.t. $\leq$, and $T_1, \leq_1, S_1$ are the restrictions of $T, \leq, S$ to $C\lambda(\mu X)$.

Proof: By Lemmas 4.5 and 4.6 above it is enough to show that $C\lambda(\mu X)$ coincides with the collection Y of valuations of $(\Psi, |\vdash)$.

From (*) in the proof of Corollary 6.4 we deduce that $\mu X \subseteq Y$, and hence, since Y is closed,

(**) $C\lambda(\mu X) \subseteq Y$.

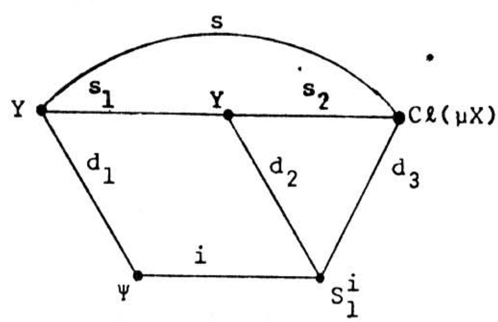
This, together with (*) again, implies

(***) $\Gamma \mid\vdash \Delta$ iff $\forall x \in C\lambda(\mu X).\ \min_{\psi \in \Gamma} x(\psi) \leq \max_{\Theta \in \Delta} x(\Theta)$

for any $\Gamma, \Delta \in P_\omega(\Psi)$. Thus $(\Psi, \mid\vdash)$ is isomorphic to the dual system of $C\lambda(\mu X)$. The dual system of $C\lambda(\mu X)$, being a set-deduction system (whose domain is S_1^i), is a deduction algebra. This allows us to use Corollary 5.11, to deduce that the dual space Y^* of S_1^i is s-homeomorphic to the dual space Y of $(\Psi, \mid\vdash)$. On the other hand, by the duality theorem for valuation spaces, Y^* is s-homeomorphic to $C\lambda(\mu X)$. Since being s-homeomorphic is an equivalence relation (i.e., in particular it is transitive) we conclude that Y is s-homeomorphic to $C\lambda(\mu X)$. Together with (**) this implies

$$C\lambda(\mu X) = Y$$

as required.
The diagram below is intended to clarify the main points of this proof.



The lines connecting objects in this diagram read as follows:

d_1 : Y is the dual space of $(\Psi, \mid\vdash)$
d_2 : Y^* is the dual space of S_1^i
d_3 : S_1^i is the dual algebra of $C\lambda(\mu X)$
i : $(\Psi, \mid\vdash)$ is isomorphic to S_1^i (by (***))
s_1 : Y is s-homeomorphic to Y^* (by d_1, d_2, i)
s_2 : Y^* is s-homeomorphic to $C\lambda(\mu X)$ (by d_2, d_3)
s : Y is s-homeomorphic to $C\lambda(\mu X)$ (by s_1, s_2)

Let us now use the relation $\sim$ (see the remarks following Definition 2.2) to obtain a deduction algebra $(\Psi^0, |\vdash)$ from the system $(\Psi, |\vdash)$. Using Theorem 6.3 we can show that $\sim$ is a congruence w.r.t. the lattice operations on Ψ and thus lattice operations and a lattice deduction $\vdash$ can be defined on the domain Ψ^0 of the factor algebra $(\Psi^0, |\vdash)$ in the obvious way. The following theorem shows that on this factor algebra the complementation deduction coincides with the lattice deduction.

Theorem 6.6: Let $(\Psi^0, |\vdash)$ be the factor algebra of $(\Psi, |\vdash)$ and let $\vdash$ be the lattice deduction on Ψ^0. Then

(i) for all $\psi, \Theta \in \Psi^0$

$$\psi |\vdash \Theta \text{ iff } \psi \vdash \Theta \ .$$

(ii) μX^0 is dense in (X^0, T^0), where $(X^0, T^0, \leqslant^0, S^0)$ is the dual space of $(\Psi^0, \vdash)$.

Proof: (i) That $\psi \vdash \Theta$ implies $\psi |\vdash \Theta$ if obvious. To prve the converse let $\psi |\vdash \Theta$, which means $\psi^+ \subseteq \Theta^+$. Therefor $\psi^+ \cap \Theta^+ = \psi^+$. But this implies for any $\varphi \in \Psi^0$:

$$\varphi \vee \psi = 1$$
iff
$$\varphi \vee \psi = 1 \text{ and } \varphi \vee \Theta = 1$$
iff
$$(\varphi \vee \psi) \wedge (\varphi \vee \Theta) = 1$$
iff
$$\varphi \vee (\psi \wedge \Theta) = 1 \ .$$

So we have $(\psi \wedge \Theta)^+ = \psi^+$, Which implies $\psi \wedge \Theta = \psi$ since we are working here within Ψ^0. (Note that all the operations and relations in this proof, i.e., $\vdash, |\vdash, {}^+, \vee, \wedge, 1, \leq$, are those relating to Ψ^0). Now we conclude that $\psi \leq \Theta$ which is the same as $\psi \vdash \Theta$.

(ii) It follows from (i) that the dual spaces of $(\Psi^0, \vdash)$ and $(\Psi^0, |\vdash)$ coincide. But by Theorem 6.5 the dual space of $(\Psi^0, \vdash)$ is $(C\lambda(\mu X^0), T_1^0, \leq_1^0, S_1^0)$, where $T_1^0, \leq_1^0, S_1^0$ are the restrictions of $T^0, \leq^0, S^0$ to $C\lambda(\mu X^0)$. So $C\lambda(\mu X^0) = X^0$ (and $T_1^0 = T^0$, $\leq_1^0 = \leq^0$, $S_1^0 = S^0$).

In the proof of (ii) we used Theorem 6.5 whose own proof rests on

Theorem 6.3 (iv), via Corollary 6.4. We can provide another, slightly less immediate, proof of (ii), applying directly Theorem 6.3 (iv) instead of Theorem 6.5.

For μX^0 to be dense in (X^0, T^0) it must intersect every non-empty basic subset β of (X^0, T^0). Such a subset would be of the form

$$\beta = \{x \in X^0 : x(\psi_1) = \ldots = x(\psi_m) = 1 \text{ and }$$
$$x(\Theta_1) = \ldots = x(\Theta_n) = 0\} ,$$

where $m, n \in \omega$ and $\psi_1, \ldots, \psi_m, \Theta_1, \ldots, \Theta_n \in \Psi^0$. But since Ψ^0 is closed under its lattice operations $\wedge$, $\vee$, we can write

$$\beta = \{x \in X^0 : x(\psi) = 1 \text{ and } x(\Theta) = 0\},$$

where

$$\psi = \psi_1 \wedge \ldots \wedge \psi_m$$
$$\Theta = \Theta_1 \vee \ldots \vee \Theta_n .$$

(We may identify sub-basic sets with those β's where either ψ is the unit of Ψ^0 or Θ is the zero of Ψ^0).

Now, if β is to be non-empty it must be the case that $\psi \not\vdash \Theta$. But then by (i) $\psi \mid\not\vdash \Theta$ and by Theorem 6.3 (iv)

$$\exists y \in \mu X^0 \, . \; y(\psi) = 1 \text{ and } y(\Theta) = 0.$$

Corollary 6.7: The space $(C\lambda(\mu X), T_1, \leq_1, S_1)$ of Theorem 6.5 is s-homeomorphic to the space $(X^0, T^0, \leq^0, S^0)$ of Theorem 6.6.

Proof: This corollary follows from Corollary 5.11 using the fact that $(\Psi, \mid\vdash)$ is isomrophic to $(\Psi^0, \mid\vdash)$ which by the last theorem coincides with $(\Psi^0, \vdash)$. Note that $(\Psi^0, \mid\vdash)$, which plays here the role of $(\Omega, \mid\vdash)$ of Corollary 5.11 is a deduction algebra and not just a deduction system.

Corollary 6.8: The spaces $(\mu X, T_2)$ and $(\mu X^0, T_2^0)$ obtained from those in the previous corollary by restricting T_1 and T^0 to μX and μX^0 respectively, are homeomorphic.

Proof: The s-homeomorphism f of the previous corollary is the required homeomorphism for this corollary. To see that this is so check that

$$x \in \mu X^0 \to f(x) \in \mu X$$
$$x \in \mu X \to f^{-1}(x) \in \mu X^0.$$

We conclude by briefly noting how Simmons [5] applied the machinery described in this section to the study of generic structures. Detailed proofs of the claims made here can also be found in Katz [2].

Let K be a theory in a first-order language L and let $\forall$ be the set of sentences of L whose prenex normal forms contain only universal quantifiers. Denote by $|\forall|$ the lattice $\forall$ mod K, and by $|\psi|$ the element of $|\forall|$ corresponding to the sentence ψ of $\forall$. The dual space of $|\forall|$ with the lattice deduction is as usual (X, T, $\leqslant$, S).

Denote by G_K and F_K the classes of g-generic and f-generic structures for K and by S_K the class of submodels of K (i.e., the models of $K \cap \forall$). G_K and F_K are both included in S_K. Now define a map $k : S_K \to X$ by

$$k(A) = x \ ,$$

where A is a submodel of K and x is the element of X such that for all $\psi \in \forall$

$$x(|\psi|) = 1 \text{ iff } A \models \psi.$$

Simmons shows that k maps G_K onto μX (the set of minimal elements of X w.r.t. $\leq$) and F_K onto a dense subset of μX (dense w.r.t. to the restriction of T to μX). In addition, for both G_K and F_K the map k is 1-1 on equivalence classes modulo the relation of elementary equivalence. Thus the number of such classes for F_K is smaller than or equal to the number of these classes for G_K.

Simmons also shows that μX is a $G_\delta$subset of X. From this it follows that if the language L is countable but there are uncountabley many equivalence classes of G_K modulo elementary equivalence then there are exactly 2^ωsuch classes. The reason is that if L is countable than so is $\forall$ and hence so is also the subbase S of (X, T) since S^i is isomorphic to $\forall$. Thus (X, T) is second countable and we already know that it is compact and Hausdorff. This implies that for every $G_\delta$subset Y of X, if $\overline{Y}$ denotes the cardinality of Y then

$$\overline{Y} > \omega \to \overline{Y} = 2^\omega.$$

Finally we note that a complementation deduction $|\vdash$ can be defined on $|\forall|$ as in Definition 6.2 above and then, using the relation $\sim$ corresponding to $|\vdash$, a factor lattice $|\forall|^0$ can be obtained from $|\forall|$. We can now apply Corollary 6.8 to replace μX by μX^0 (where X^0 is the dual space of $|\forall|^0$) in all the results mentioned in the preceding two paragraphs.

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