THE NON-INDEPENDENCE OF AXIOMS IN A PROPOSITIONAL CALCULUS FORMULATED IN TERMS OF AXIOM SCHEMATA

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An axiom is independent if it cannot be derived from the other axioms. Such a definition is perfectly in order for systems that do not employ axiom schemata. For those that do employ such schemata another definition is advisable if it is to be shown that a particular schema is, in some sense, indispensable. The system P, originally due to Łukasiewicz, which I consider below has proved to be a popular one in logic texts. (1) Given in terms of axiom schemata rather than as three axioms plus substitution rule I show that each axiom can be proved from the other axioms. Thus in terms of the first definition given none of the axioms of P is independent. This proof is the main point of this paper.

Careful definitions of the independence of axiom schemata are sometimes given in logic texts but often the reader is expected to construct his own definition from that given in the text for the independence of axioms. (2) At worst, however, is the conflation of the two as exemplified by such a text as Copi's *Symbolic Logic*. (3) His

(1) In its schematic formulation it occurs in A. Church, *Introduction to Mathematical Logic*, Princeton 1956, p. 149; B. Mates, *Elementary Logic*, Oxford 1965, p. 156; J. N. Crossley, *What Is Mathematical Logic?*, Oxford, 1972, p. 19; G. Hunter, *Metalogic*, Berkely, 1971, p. 72. There are of course many other texts that use this system.

(2) None of the books in footnote 1 defines independence of axiom schemata though each employs schematic formulations of P and all but Crossley give definitions for the independence of axioms! A text which does give a definition, coincident with that given in this paper, is E. Mendelson, *Introduction to Mathematical Logic*, New York 1963, p. 38. The text does not use P.

(3) I. Copi, *Symbolic Logic*, New York, 4th ed. 1973. The definition is given on page 182. The proof of independence follows immediately. Copi seems unaware that there is any discrepancy between his definition and his proof caused by his referring to his axiom schemata as axioms.

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system R.S. is an axiom schemata formulation of the propositional calculus which contains the definition of independence given above which is only suitable for those systems employing axioms plus substitution rule. The fault in Copi's case is that the term 'axiom' is used not only to denote both the true axioms of the system R.S. but also the *schemata*. The proof set out below demonstrates the necessity for distinguishing between the independence of axioms and the independence of axiom schemata. For in the case of the system P it is well known that the axiom schemata are independent i.e. at least one instance of an axiom schema is not provable from axioms which are instances of the other two schema. What is new is that no *axiom* of the system is independent.

Further questions can now be considered. Are there any systems formulated in terms of axiom schemata in which some or all of the axioms are independent? Is it possible to formulate a set of schemata in which each axiom is independent? The answers will have to await further investigation.

The system P of propositional calculus contains an infinite list of propositional variables and the connectives '$\sim$' and '$\supset$'. The usual recursive definitions of wff, proof and theorem are assumed. The axioms of P are

$$A \supset (B \supset A)$$
$$(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$$
$$(\sim A \supset \sim B) \supset (B \supset A)$$

where A, B and C are any wffs of P. The only rule is modus ponens (m.p.). There are thus an infinite number of axioms. Each proves to be provable from the others.

The proof utilises in each case another instance of the schema from that instance whose proof is sought, making essential use of the fact that $\sim\sim A \supset A$ and $A \supset \sim\sim A$ are both provable for any wff A. To prove any instance of $A \supset (B \supset A)$ we use $\sim\sim A \supset (B \supset \sim\sim A)$; to prove any instance of $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$ we use $(\sim\sim A \supset (B \supset C)) \supset ((\sim\sim A \supset B) \supset (\sim\sim A \supset C))$ and $(A \supset (B \supset \sim\sim C)) \supset ((A \supset B) \supset (A \supset \sim\sim C))$; to prove any instance of $((\sim A \supset \sim B) \supset (B \supset A))$ we use $(\sim\sim\sim A \supset \sim\sim\sim B) \supset (\sim\sim B \supset \sim\sim A)$. Certain particular instances will need specific proofs to avoid circularity: these will be dealt with as we proceed. The proof begins with the standard proofs of several lemmata which are needed throughout the subsequent proof: transitivity, transposition, identity and the double negation rules. They are included since the reader can check that no circularity has been made in the proofs – a constant danger given that the lemmata are used so often and these lemmata themselves use many instances of the axioms.

Transitivity (trans)

1. $((B \supset C) \supset ((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))) \supset (((B \supset C) \supset (A \supset (B \supset C))) \supset ((B \supset C) \supset ((A \supset B) \supset (A \supset C))))$ Axiom 2
2. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$ Axiom 2
3. $((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))) \supset ((B \supset C) \supset ((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))))$ Axiom 1
4. $(B \supset C) \supset ((A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C)))$ 2,3 m.p.
5. $((B \supset C) \supset (A \supset (B \supset C))) \supset ((B \supset C) \supset ((A \supset B) \supset (A \supset C)))$ 1,4 m.p.
6. $(B \supset C) \supset (A \supset (B \supset C))$ Axiom 1
7. $(B \supset C) \supset ((A \supset B) \supset (A \supset C))$ 5,6 m.p.

Therefore, if $\vdash B \supset C$ and $\vdash A \supset B$ then $\vdash A \supset C$ by m.p. on 7 twice. The only axioms used in this proof are those on lines 1,2, 3 and 6. Whenever transitivity is used in the proof of the non-independence of an axiom, inspection of the A, B and C will reveal that no surreptitious use is made of the particular *axiom* the non-independence of which is being proved.

Transposition (transp)

Suppose 8. $\vdash A \supset (B \supset C)$

9. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$ Axiom 2
10. $(A \supset B) \supset (A \supset C)$ 8,9 m.p.
11. $((A \supset B) \supset (A \supset C)) \supset (B \supset ((A \supset B) \supset (A \supset C)))$ Axiom 1
12. $B \supset ((A \supset B) \supset (A \supset C))$ 10,11 m.p.
13. $(B \supset ((A \supset B) \supset (A \supset C))) \supset ((B \supset (A \supset B)) \supset (B \supset (A \supset C)))$ Axiom 2
14. $(B \supset (A \supset B)) \supset (B \supset (A \supset C))$ 12,13 m.p.

15. $B \supset (A \supset B)$ Axiom 1
16. $B \supset (A \supset C)$ 14,15 m.p.

Therefore, if $\vdash A \supset (B \supset C)$ then $\vdash B \supset (A \supset C)$. The only axioms used in this proof are those on lines 9, 11, 13 and 15. Whenever transposition is used, inspection of the proof will again reveal no circularity.

Identity (id)

17. $(A \supset ((A \supset A) \supset A)) \supset ((A \supset (A \supset A)) \supset (A \supset A))$ Axiom 2
18. $A \supset ((A \supset A) \supset A)$ Axiom 1
19. $(A \supset (A \supset A)) \supset (A \supset A)$ 17, 18 m.p.
20. $A \supset (A \supset A)$ Axiom 1
21. $A \supset A$ 19,20 m.p.

The only axioms used in this proof are those on lines 17, 18 and 20. Whenever identity is used in the proof of the non-independence of an axiom, inspection of the A will reveal that no surreptitious use is made of that particular *axiom* in the identity proff.

Double negation cancellation (dnc)

22. $(\sim\sim\sim\sim A \supset \sim\sim A) \supset (\sim A \supset \sim\sim\sim A)$ Axiom 3
23. $((\sim\sim\sim\sim A \supset \sim\sim A) \supset (\sim A \supset \sim\sim\sim A)) \supset (\sim\sim A \supset$
$((\sim\sim\sim\sim A \supset \sim\sim A) \supset (\sim A \supset \sim\sim\sim A)))$ Axiom 1
24. $\sim\sim A \supset ((\sim\sim\sim\sim A \supset \sim\sim A) \supset (\sim A \supset \sim\sim\sim A))$ 22,23 m.p.
25. $(\sim\sim A \supset ((\sim\sim\sim\sim A \supset \sim\sim A) \supset (\sim A \supset \sim\sim\sim A))) \supset ((\sim\sim A \supset$
$(\sim\sim\sim\sim A \supset \sim\sim A)) \supset (\sim\sim A \supset (\sim A \supset \sim\sim\sim A)))$ Axiom 2
26. $(\sim\sim A \supset (\sim\sim\sim\sim A \supset \sim\sim A)) \supset (\sim\sim A \supset (\sim A \supset \sim\sim\sim A))$
24,25 m.p.
27. $\sim\sim A \supset (\sim\sim\sim\sim A \supset \sim\sim A)$ Axiom 1
28. $\sim\sim A \supset (\sim A \supset \sim\sim\sim A)$ 26,27 m.p.
29. $(\sim A \supset \sim\sim\sim A) \supset (\sim\sim A \supset A)$ Axiom 3
30. $\sim\sim A \supset (\sim\sim A \supset A)$ 28,29 trans
31. $(\sim\sim A \supset (\sim\sim A \supset A)) \supset ((\sim\sim A \supset \sim\sim A) \supset (\sim\sim A \supset A))$
Axiom 2

33. $(\sim\sim A \supset \sim\sim A) \supset (\sim\sim A \supset A)$ 30,31 m.p.
34. $(\sim\sim A \supset \sim\sim A)$ id
35. $(\sim\sim A \supset A)$ 33,34 m.p.

The only axioms used in this proof are those on lines 22, 23, 25, 27, 29, 31 and those, with appropriate A, B and C that occur in the proofs of trans and id. In this proof the instance of axiom schema 3 at line 29 will cause different proofs to be given for the non-independence of axioms of the form $(\sim A \supset \sim B) \supset (B \supset A)$ depending on whether $B = \sim\sim A$.

Double negation introduction (dni)

A similar proof would give instead of 35, *mutatis mutandis*

36. $\sim\sim\sim A \supset \sim A$
37. $(\sim\sim\sim A \supset \sim A) \supset (A \supset \sim\sim A)$ Axiom 3
38. $A \supset \sim\sim A$ 36,37 m.p.

By using these lemmata we are now in a position to show that any axiom of the form of axiom schema 1 may be proved from the other axioms. Each time a lemma is used the reader may check that the *axiom* under consideration does not in fact occur in that lemma.

Proof that any axiom of the form $A \supset (B \supset A)$ *can be proved from the other axioms*

39. $A \supset \sim\sim A$ (dni)
40. $\sim\sim A \supset (B \supset \sim\sim A)$ Axiom 1
41. $A \supset (B \supset \sim\sim A)$ 39,40 trans
42. $\sim\sim A \supset A$ (dnc)
43. $(\sim\sim A \supset A) \supset (B \supset (\sim\sim A \supset A))$ Axiom 1
44. $B \supset (\sim\sim A \supset A)$ 42,43 m.p.
45. $(B \supset (\sim\sim A \supset A)) \supset ((B \supset \sim\sim A) \supset (B \supset A))$ Axiom 2
46. $(B \supset \sim\sim A) \supset (B \supset A)$ 44,45 m.p.
47. $A \supset (B \supset A)$ 41,46 trans

Proof that any axiom of the form $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$ *can be proved from the other axioms*

First assume that B is not the same wff. as $\sim\sim C$.

48. $C \supset \sim\sim C$ (dni)
49. $(C \supset \sim\sim C) \supset (B \supset (C \supset \sim\sim C))$ Axiom 1
50. $B \supset (C \supset \sim\sim C)$ 48, 49 m.p.
51. $(B \supset (C \supset \sim\sim C)) \supset ((B \supset C) \supset (B \supset \sim\sim C))$ Axiom 2
52. $(B \supset C) \supset (B \supset \sim\sim C)$ 50, 51 m.p.
53. $((B \supset C) \supset (B \supset \sim\sim C)) \supset (A \supset ((B \supset C) \supset (B \supset \sim\sim C)))$ Axiom 1
54. $A \supset ((B \supset C) \supset (B \supset \sim\sim C))$ 52, 53 m.p.
55. $(A \supset ((B \supset C) \supset (B \supset \sim\sim C))) \supset ((A \supset (B \supset C)) \supset (A \supset (B \supset \sim\sim C)))$ Axiom 2
56. $(A \supset (B \supset C)) \supset (A \supset (B \supset \sim\sim C))$ 54, 55 m.p.
57. $(A \supset (B \supset \sim\sim C)) \supset ((A \supset B) \supset (A \supset \sim\sim C))$ Axiom 2
58. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset \sim\sim C))$ trans, 56, 57
59. $\sim\sim C \supset C$ (dnc) [See the paragraph following this proof.]
60. $(\sim\sim C \supset C) \supset (A \supset (\sim\sim C \supset C))$ Axiom 1
61. $A \supset (\sim\sim C \supset C)$ 59, 60 m.p.
62. $(A \supset (\sim\sim C \supset C)) \supset ((A \supset \sim\sim C) \supset (A \supset C))$ Axiom 2 [Note $B \neq \sim\sim C$]
63. $(A \supset \sim\sim C) \supset (A \supset C)$ 61, 62 m.p.
64. $((A \supset \sim\sim C) \supset (A \supset C)) \supset ((A \supset B) \supset ((A \supset \sim\sim C) \supset (A \supset C)))$ Axiom 1
65. $(A \supset B) \supset ((A \supset \sim\sim C) \supset (A \supset C))$ 63, 64 m.p.
66. $((A \supset B) \supset ((A \supset \sim\sim C) \supset (A \supset C))) \supset (((A \supset B) \supset (A \supset \sim\sim C)) \supset ((A \supset B) \supset (A \supset C)))$ Axiom 2
67. $((A \supset B) \supset (A \supset \sim\sim C)) \supset ((A \supset B) \supset (A \supset C))$ 65, 66 m.p.
68. $(A \supset (B \supset C)) \supset ((A \supset B) \supset (A \supset C))$ 58, 67 trans

The above proof would be circular if $B = \sim\sim C$ for we use this axiom at line 62. Also, if A and B were both the same wff as $\sim\sim C$ then 59 will also depend on the axiom as can be seen by examining the proof of $\sim\sim C \supset C$ at line 31.

If $B = \sim\sim C$ a different proof must be found. This can be constructed from $(\sim\sim A \supset (B \supset C)) \supset ((\sim\sim A \supset B) \supset (\sim\sim A \supset C))$ instead of the crucial 51 and utilising $\sim\sim A \supset A$ and $A \supset \sim\sim A$. The proof runs along the same lines as the one just given, though it does make use of more transitivity moves and some transposition moves. The reader will find, however, that this proof will involve the wff $(\sim\sim A \supset C) \supset ((A \supset \sim\sim A) \supset (A \supset C))$ which itself is proved by using the wff $(A \supset (\sim\sim A \supset C)) \supset ((A \supset \sim\sim A) \supset (A \supset C))$ [see lines 2 and 7]. Now, if $B = \sim\sim A$ the proof is again circular.

In the case where $B = \sim\sim A$ and $B = \sim\sim C$, a different but much simpler proof exists for the form of the axiom must then be . In this case we can use identity to establish $A \supset A$ then use Axiom 1 twice over to indroduce first $A \supset \sim\sim A$ and then $A \supset (\sim\sim A \supset A)$ as antecedents. All possible cases are then covered.

Proof that any axiom of the form $(\sim A \supset \sim B) \supset (B \supset A)$ *can be proved from the other axioms*

First assume that B is not the same wff as $\sim\sim A$

69. $\sim\sim\sim A \supset \sim A$ (dnc)
70. $(\sim A \supset \sim B) \supset ((\sim\sim\sim A \supset \sim A) \supset (\sim\sim\sim A \supset \sim B)$ trans
71. $(\sim\sim\sim A \supset \sim A) \supset ((\sim A \supset \sim B) \supset (\sim\sim\sim A \supset \sim B))$ 70 transp
72. $(\sim A \supset \sim B) \supset (\sim\sim\sim A \supset \sim B)$ 69,71 m.p.
73. $\sim B \supset \sim\sim\sim B$ (dni)
74. $(\sim B \supset \sim\sim\sim B) \supset ((\sim\sim\sim A \supset \sim B) \supset (\sim\sim\sim A \supset \sim\sim\sim B)$ trans
75. $(\sim\sim\sim A \supset \sim B) \supset (\sim\sim\sim A \supset \sim\sim\sim B)$ 73,74 m.p.
76. $(\sim A \supset \sim B) \supset (\sim\sim\sim A \supset \sim\sim\sim B)$ 72,75 trans
77. $(\sim\sim A \supset A) \supset ((\sim\sim B \supset \sim\sim A) \supset (\sim\sim B \supset A))$ trans
78. $\sim\sim A \supset A$ (dnc) [see following paragraph]
79. $(\sim\sim B \supset \sim\sim A) \supset (\sim\sim B \supset A)$ 78,79 m.p.
80. $(\sim\sim B \supset A) \supset ((B \supset \sim\sim B) \supset (B \supset A))$ trans
81. $(B \supset \sim\sim B) \supset ((\sim\sim B \supset A) \supset (B \supset A))$ 80 transp
82. $B \supset \sim\sim B$ (dni)
83. $(\sim\sim B \supset A) \supset (B \supset A)$ 82,83 m.p.
84. $(\sim\sim B \supset \sim\sim A) \supset (B \supset A)$ 79,83 trans
85. $(\sim\sim\sim A \supset \sim\sim\sim B) \supset (\sim\sim B \supset \sim\sim A)$ Axiom 3

86. $(\sim\sim\sim A \supset \sim\sim\sim B) \supset (B \supset A)$ 84, 85 trans
87. $(\sim A \supset \sim B) \supset (B \supset A)$ 76, 86 trans

If $B = \sim\sim A$ then this proof is circular since the proof of 78 will have used 29. A different proof is then required for axioms of the form $(\sim A \supset \sim\sim\sim A) \supset (\sim\sim A \supset A)$. I prove this below.

88. $(\sim\sim(A \supset A) \supset \sim\sim A) \supset (\sim A \supset \sim(A \supset A))$ Axiom 3
89. $((\sim\sim(A \supset A) \supset \sim\sim A) \supset (\sim A \supset \sim(A \supset A))) \supset (\sim\sim A \supset ((\sim\sim(A \supset A) \supset \sim\sim A) \supset (\sim A \supset \sim(A \supset A))))$ Axiom 1
90. $\sim\sim A \supset (\sim\sim(A \supset A) \supset \sim\sim A) \supset (\sim A \supset \sim(A \supset A)))$ 88, 89 m.p.
91. $(\sim\sim A \supset ((\sim\sim(A \supset A) \supset \sim\sim A) \supset (\sim A \supset \sim(A \supset A)))) \supset ((\sim\sim A \supset (\sim\sim(A \supset A) \supset \sim\sim A) \supset (\sim\sim A \supset (\sim A \supset \sim(A \supset A))))$ Axiom 2
92. $(\sim\sim A \supset (\sim\sim(A \supset A) \supset \sim\sim A)) \supset (\sim\sim A \supset (\sim A \supset \sim(A \supset A)))$ 90, 91 m.p.
93. $\sim\sim A \supset (\sim\sim(A \supset A) \supset \sim\sim A)$ Axiom 1
94. $\sim\sim A \supset (\sim A \supset \sim(A \supset A))$ 92, 93 m.p.
95. $(\sim A \supset \sim(A \supset A)) \supset ((A \supset A) \supset A)$ Axiom 3
96. $\sim\sim A \supset ((A \supset A) \supset A)$ 94, 95 trans
97. $(A \supset A) \supset (\sim\sim A \supset A)$ 96 transp
98. $A \supset A$ id
99. $\sim\sim A \supset A$ 97, 98 m.p.
100. $(\sim\sim A \supset A) \supset ((\sim A \supset \sim\sim\sim A) \supset (\sim\sim A \supset A))$ Axiom 1
101. $(\sim A \supset \sim\sim\sim A) \supset (\sim\sim A \supset A)$ 99, 100 m.p.

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