ON AN APPLICATION OF TRUTH-FUNCTIONS TO THE LOGIC OF PREDICATES

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1. Truth-tables for a predicate with respect to an operator

The reader is familiar, I suppose, with the way truth-tables are used in the elementary propositional calculus. With basic propositions and logical constants complex propositions are formed; by means of truth-tables it is then possible to determine the truth values of a complex proposition for the various combinations of the truth values of its constituent propositions. It is useful, in view of the following analysis, to generalise this notion of truth-function a little. Let us assume that complex propositions are formed out of certain types of elementary propositions or propositional functions each of which can be assigned either of two 'values'; any function which yields the truth values of the complex proposition for the various combinations of 'values' of the constituent propositions or propositional function I shall call a truth-function. Let us denote the two possible 'values' of the constituents by + and − ; we can then establish truth-tables in a similar way as is done for ordinary truth-functions in the propositional calculus.

One type of complex proposition to which this general notion of a truth-function can be applied is characterised by the form

$$\delta A(x),$$

where x is a variable ranging over a domain X of individuals, A a one-plece predicate for the individuals of X, and δ an operator which can be applied to the propositional function A(x) and which, when so applied, produces a proposition in which the variable x is bound. A we suppose to belong to a class of predicates which contains with every predicate A also its negation ~A. δA(x) then gives rise to truth tables in the following way. A(x) is first given its positive 'value', that is, it is just left as it is, and δ is applied to A(x) and the truth-value of δA(x) determined; A(x) is then given its negative 'value', it is negated, δ applied to ~A(x) and the truth value of δ(~A(x)) determined. It will from now on be assumed that the predicates under consideration are such that $A(x_i)$ is either true or false for every individual x_i of X and that $\sim A(x_i)$ is a false proposition if, and only if, $A(x_i)$ is a true one. Under these conditions every predicate A can be characterised with respect to δ by one of the four truth-functions ①′, ②′, ③′ or ④′ in the table below. (1 stands for 'true', 0 for 'false').

δA	①′	②′	③′	④′
+	1	1	0	0
−	1	0	1	0

For instance, let X be the domain of natural numbers, δ the operator 'For some numbers larger than ten −', and let the predicates include

A : 'even number' (or: '− is an even number'),
A′ : 'successor of some natural number',
A″ : 'even prime'.

Then ①′, ②′, ③′ are the truth-functions characterising A, A′ and A″ respectively.

It is useful to draw the distinction between ordinary and non-ordinary predicates. I shall call a predicate A ordinary with regard to the domain X if, and only if, there are some individuals of X to which A applies and some to which A does not apply. A predicate is said to be non-ordinary if, and only if, it applies either to some individuals of X whereas its negation applies to none, or it applies to none while its negation applies to some of them. In the light of what was said above, it follows that a non-ordinary predicate applies either to all or to none of the individuals of the domain. If a predicate is an ordinary one with respect to the domain X, then ①′ is its characteristic truth-function with regard to the operator 'For some individuals of X −'. Non-ordinary predicates are characterised either by the truth-function ②′ or by ③′ with respect to this operator.

2. *Truth-functions for binary relations between predicates*

Let us now consider ordered pairs (A,A′) of predicates on X and the corresponding conjunctions A(x) . B(x) of propositional functions to which the operator δ is applied. A and B are assigned their positive or negative 'values' as before and for each ordered pair of 'values' we determine the truth-value of the proposition δ(A(x) . B(x)). There are now 16 possibilities in all for the resulting truth-functions, as is shown in the table below.

δ(AB)	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯
+ +	1	1	1	1	0	1	0	1	1	0	0	1	0	0	0	0
+ −	1	1	1	0	1	0	1	1	0	1	0	0	1	0	0	0
− +	1	1	0	1	1	0	1	0	1	0	1	0	0	1	0	0
− −	1	0	1	1	1	1	0	0	0	1	1	0	0	0	1	0

We are particularly interested in the operator 'For some individuals of X −' and in the truth values of the proposition

For some x of X: A(x) . B(x).

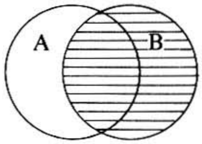
Provided the predicates A and B satisfy the conditions set out above each ordered pair of predicates on X will have as its characteristic truth-function one of the functions ① -⑮ . If X is a sufficiently rich domain (it would have to comprise at least four individuals, if for simplicity we assume that to each sub-class of individuals corresponds at least one predicate), then each function will be the caracteristic truth-function of at least one pair of predicates. No pair of predicates can be assigned two different truth-functions of the table; and ⑯ is assigned to none.

There exist 15 fundamental configurations for ordered pairs of predicates over a domain X. ([5]) Each of these fundamental configurations is characterised by exactly one of the truth-functions ① -⑮ , as can easily be verified. That is, a one-one-correspondence can be established between the fundamental configurations for ordered pairs of predicates and the set of the first fifteen truth-functions of the above

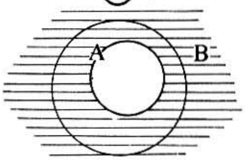
([5]) Given the domain is sufficiently rich, as we shall assume.

table. Let us denote this set by F, and the total set, including (16), by F^+. The fundamental configurations are listed below and represented by means of the well-known Euler diagrams. (3)

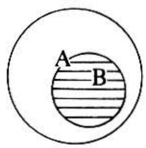
A (1) B



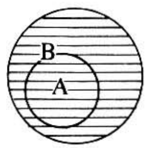
A (2) B



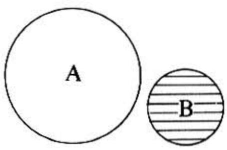
A (3) B



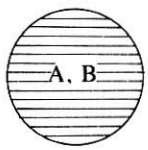
A (4) B



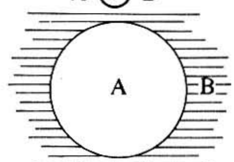
A (5) B



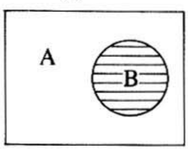
A (6) B



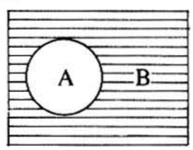
A (7) B



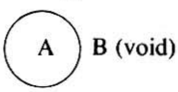
A (8) B



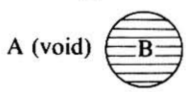
A (9) B



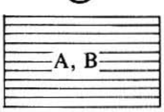
A (10) B



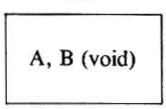
A (11) B



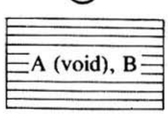
A (12) B



A (13) B



A (14) B



A (15) B

A (void, B (void)

(3) See e.g. O. Bird, *Syllogistic and its Extensions*, Englewood Cliffs N.J., 1964, " 37, where these configurations are discussed.

These configurations fall into three major categories. The first of them comprises the configurations I - VII, which include ordinary predicates only. Their truth-functions involve either four values 1 (in the case of the first function), or three values 1 (functions ② - ⑤), or two values 1 (functions ⑥ and ⑦).

Configurations VI and VII differ from the others of this group in that each consists of a pair of predicates which are not genuinely different – in VI they are co-extensive, in VII complements of each other. To the second category belong the configurations VIII - XI each of which consists of one ordinary and one non-ordinary predicate (a void or a universal predicate). Their truth-functions involve two values 1.

Finally, there is the third category consisting of the configurations XII - XV each of which links two non-ordinary predicates (a universal with a universal, a universal with a void, a void with a universal, or a void with a void predicate). Their truth-functions contain a single value 1. It is in order to emphasise this natural grouping that I have deviated from the usual lexicographical order of the 16 functions.

3. *Linear correspondences on the set of truth-functions* (1)

The four truth-functions of the third category, i.e. ⑫, ⑬, ⑭ and ⑮, from a basis by means of which the truth-functions of F can be represented. For instance, ① can be written as ⑫ + ⑬ + ⑭ + ⑮, ② as ⑫ + ⑬ + ⑭, ③ as ⑫ + ⑬ + ⑮, and so on. In short, any function f of F can be represented as a linear combination

$$f = \sum_{i=1}^{4} \varepsilon_i f_i, \qquad (1)$$

where f_1 = ⑫, f_2 = ⑬, f_3 = ⑭ and f_4 = ⑮ (a notation which will from now on be strictly adhered to), and where the coefficients ε_i take on the values 1 and 0. I shall refer to the right hand side of the equation

(1) The reader is referred to my paper 'On the Logic of Relations', *Dialectica*, 34 (1980), pp. 176-182, in which some of the techniques applied here are explained in greater detail.

as the representation of f with regard to the basis f_1, f_2, f_3, f_4. The basic functions f_i whose coefficients equal 1 are called the components of f. If for two predicates A,A′ f(A,A′) holds,

then $(\sum_{i=1}^{4} \varepsilon_i f_i)(A,A')$ also holds, and vice versa.

But it does not follow that $\sum_{i=1}^{4} \varepsilon_i f_i(A,A')$ is valid; for, as has been pointed out above, the f_i need not apply to the same pairs of predicates to which f applies.

A linear correspondence H on the set F^+ of truth-functions is a correspondence of F^+ into, or onto, itself such that the following condition is fulfilled for any $f \varepsilon F^+$:

$$H(f) = H(\sum_{i=1}^{4} \varepsilon_i f_i) = \sum_{i=1}^{4} \varepsilon_i H(f_i). \tag{2}$$

Trivially, the image of the truth-function ⑯ is ⑯ itself. We shall therefore omit ⑯ and work with the set F rather than F^+. It is obvious from (2) that the image H(f) is fully determined for any given $f \varepsilon F$, provided the images of the basic functions are specified. We therefore introduce the corresponding matrix H whose with column contains the coefficients of the representation of $H(f_i)$:

$$H = \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix}$$

With the help of the 'column vector'

$$f = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \end{pmatrix}$$

and the image H(f) is represented by

$$H(f) = Hf.$$

In order to calculate products of this kind we add and multiply the coefficients in accordance with the rules: $0+0 = 0$, $0+1 = 1+0 = 1+1 = 1$; $0.0 = 0.1 = 1.0 = 0$, $1.1 = 1$.

4. Conversion and counterposition as linear correspondences

Let (A,A′) be an ordered pair of predicates on X; if f is the truth-function which characterises their configuration, then we say that A is related to A′ by f, or that f applies to (A,A′), or that f(A,A′) holds. By exchanging A and A′ we obtain the ordered couple (A′,A), the 'mirror image' of (A,A′). Further, let G be the set of all ordered couples of predicates to which f applies; and let f′ be the function which applies to all the 'mirror images' of the ordered pairs of G; then f′ is said to be the converse of f. The correspondence which associates with each truth function of F its converse is the conversion on F; we denote it by C.

C is a linear correspondence, as can easily be proved. A look at the table of truth-fuctions shows that in accordance with the definition of C C(f) and f agree in their first component; but they also agree in their last component and can only differ with respect to their second and third components. It is also obvious that the second component of f equals the third of f′, while the third of f equals the second of f′. Hence

$$C(\varepsilon_1 f_1 + \varepsilon_2 f_2 + \varepsilon_3 f_3 + \varepsilon_4 f_4) = \varepsilon_1 f_1 + \varepsilon_3 f_2 + \varepsilon_2 f_3 + \varepsilon_4 f_4;$$

but

$$f_1 = C(f_1),\ f_2 = C(f_3),\ f_3 = C(f_2),\ f_4 = C(f_4).$$

The conversion matrix C is given by

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

C is a one-one-correspondence of F^+ into itself. The product of C with itself is the unit correspondence E; or, in matrix notation, $CC = E$. Applying C to all the functions of F^+ we obtain the images given below.

f	①	②	③	④	⑤	⑥	⑦	⑧	⑨	⑩	⑪	⑫	⑬	⑭	⑮	⑯
C(f)	①	②	④	③	⑤	⑥	⑦	⑨	⑧	⑪	⑩	⑫	⑭	⑬	⑮	⑯
	first category							second category				third category				

The functions ①, ②, ⑤, ⑥, ⑦, ⑫, ⑮ and ⑯ are invariant under C. Because C(f) merely results in a permutation of the truth values, for any f the number of its digits 1 remains invariant under C. This, together with the fact that ⑤ and ⑥ are invariant, shows that for any f ε F, f and C(f) are members of the same category.

Counterpositions too constitue linear correspondences of F into itself. (6) Let A and A′ be two predicates such that f(A,A′), and (~A,A′) is the ordered pair which results from (A,A′) by replacing A by its negation ~A. The function f′ which applies to all those ordered pairs which are obtained by this operation from an ordered pair to which f applies, will be called the first counterpositive of f. The first counterposition K_1 can then be defined as the correspondence which assigns to every truth-function of F its first counterpositive function. If f′ is the counterpositive of f, then f must be the counterpositive of f′. That the counterpositive of any f ε F must be uniquely determined can be seen at once by going back to the table of truth-functions: As A has to be replaced by its negation, this simply means that in the table the first row has to be exchanged with the third, and the second with the fourth. Thus, $K_1K_1 = E$; K_1 (just as the conversion C) is an involution. K_1 is linear, as can readily be shown by utilising the fact that $K_1(f)$ is obtained by exchanging the first value of f with its third, and the second with its fourth. Hence we obtain $K_1(f_1) = f_3$, $K_1(f_2) = f_4$, $K_1(f_3) = f_1$, and $K_1(f_4) = f_2$.

(6) For lack of a more suitable term I have introduced the term 'counterpositive'. Provided we are prepared to discard the rather baroque traditional terminology ('permutation', 'obversion', 'inversion' etc.) we may then merely talk of the first, second, and third counterpositions (K_1, K_2, K_3) and refer to L_1, L_2, and L_3 as the first, second, and third contrapositions.

The second and third counterpositions – we shall denote them by K_2 and K_3 – can be defined in a similar fashion. K_2 assigns to each truth-function f of F another such function, f′, such that whenever f(A,A′), then f′(A,~A′); and K_3 assigns to every f a function f′ such that whenever f(A,A′), then f′(~A, ~A′). Again we have $K_2K_2 = E$ and $K_3K_3 = E$, as is evident from the definitions. To any $f \varepsilon F$, $K_2(f)$ and $K_3(f)$ are uniquely determined. In order to obtain $K_2(f)$ the first value of f has to be exchanged with the second, and the third with the fourth; to obtain $K_3(f)$ the first value of f is exchanged with the fourth, the second with the third. The linearity of K_2 and K_3 is thus easily established. f and $K_i(f)$ comprise the same number of digits 1. In addition the counterpositions leave the functions ⑥ and ⑦ invariant or change them into each other. Hence f and $K_i(f)$ belong to the same of the three categories distinguished above.

The three counterpositions are given by the matrices

$$K_1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \quad K_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad K_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

With the help of the matrices C, K_1, K_2 and K_3, or simply by making use of the definitions of the four correspondences, the following equations may be derived

$$\begin{aligned} K_1K_2 &= K_2K_1 = K_3\,, \\ K_1K_3 &= K_3K_1 = K_2\,, \\ K_2K_3 &= K_3K_2 = K_1\,, \\ CK_1 &= K_2C\,, \\ CK_2 &= K_1C\,, \\ CK_3 &= K_3C\,, \end{aligned}$$

and, as has already been pointed out,

$$CC = K_1K_1 = K_2K_2 = K_3K_3 = E.$$

To any of the 15 fundamental configurations, its converse, its counterpositives as well as the result of any sequence of applications of some of these correspondences can thus be determined at once. To this effect we merely have to find its characteristic truth-function and multiply the latter from the left with the matrices concerned, using where necessary the above equations in order to simplify the matrix products.

However, both from a mathematical and from a logical point of view it would be rather unsatisfactory to confine ourselves to the traditional four operations represented by C, K_1, K_2, and K_3. The temptation is irresistable to add (besides E) three additional correspondences, which I denote by L_1, L_2, and L_3, such that

$$\begin{aligned} L_1 &= K_1C = CK_2\,, \\ L_2 &= K_2C = CK_1\,, \\ L_3 &= K_3C = CK_3\,. \end{aligned}$$

Their matrices are

$$L_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \; L_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}, \; L_3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

The eight correspondences E, C, K_1, K_2, K_3, L_1, L_2 and L_3 form a structure which is closed under the multiplications considered. In fact, they form a group whose multiplication table is given below.

	E	C	K_1	K_2	K_3	L_1	L_2	L_3
E	E	C	K_1	K_2	K_3	L_1	L_2	L_3
C	C	E	L_2	L_1	L_3	K_2	K_1	K_3
K_1	K_1	L_1	E	K_3	K_2	C	L_3	L_2
K_2	K_2	L_2	K_3	E	K_1	L_3	C	L_1
K_3	K_3	L_3	K_2	K_1	E	L_2	L_1	C
L_1	L_1	K_1	L_3	C	L_2	K_3	E	K_2
L_2	L_2	K_2	C	L_3	L_1	E	K_3	K_1
L_3	L_3	K_3	L_1	L_2	C	K_1	K_2	E

One interesting feature of this group is that it consists of exactly those correspondences (i.e. of exactly those eight among the 24 possible permutations of the four basic functions) which either leave the functions ⑥ and ⑦ invariant or map them upon each other.

5. *Bilinear correspondences and syllogisms*

For each ordered pair (f_i,f_j) of basic functions we may consider the proposition: for all predicates A,A′,A″ on F, if $f_i(A,A').f_j(A',A'')$, then $f(A,A'')$. If there exist predicates A,A′,A″ such that the conjunction holds, and if there is a truth-function $f \varepsilon F$ which fulfils the condition, then we shall write $S_1(f_i,f_j) = f$. However, if there are no predicates A,A′,A″ for which the conjunction $f_i(A,A').f_j(A',A'')$ is true, then we put $S_1(f_i,f_j) = e$, where e denotes the function ⑯. In this way a function S_1 can be defined on the set of all ordered pairs of basic functions; and going back to the fundamental configurations listed in section 2 it can easily be seen that S_1 is given by the matrix

$$S_1 = \begin{pmatrix} f_1 & f_2 & e & e \\ e & e & f_1 & f_2 \\ f_3 & f_4 & e & e \\ e & e & f_3 & f_4 \end{pmatrix}$$

That is, we obtain $S_1(f_1,f_1) = f_1$, $S_1(f_1,f_2) = f_2$, $S_1(f_1,f_3) = e$, etc.

By means of this matrix S_1 we define a linear correspondence on F, stipulating that for any ordered pair (f,f′) of functions of F the value of $S_1(f,f')$ be calculated on the basis of the equation

$$S_1(f,f') = S_1(\sum_{i=1}^{4} \varepsilon_i f_i, \sum_{j=1}^{4} \varkappa_j f_j) = \sum_{i=1}^{4}\sum_{j=1}^{4} \varepsilon_i \varkappa_j S(f_i,f_j).$$

This is equivalent to stipulating that for any two functions f,f′ of F, $S_1(f,f')$ is obtained by multiplying the matrix S_1 from the left with the transpose of the 'column vector' representing f, and from the right with the 'column vector' representing f′:

$$S_1(f,f') = f^T S_1 f'.$$

Applying this rule [4] we find e.g. that S_1 (④, ④) = ④ and S_1(④, ⑤) = ⑤ :

$$(1,0,1,1)\begin{pmatrix} f_1 & f_2 & e & e \\ e & e & f_1 & f_2 \\ f_3 & f_4 & e & e \\ e & e & f_3 & f_4 \end{pmatrix}\begin{pmatrix}1\\0\\1\\1\end{pmatrix} = (f_1+f_3, f_2+f_4, f_3, f_4)\begin{pmatrix}1\\0\\1\\1\end{pmatrix} = f_1+f_3+f_4 = \textcircled{4}$$

$$(1,0,1,1)\begin{pmatrix} f_1 & f_2 & e & e \\ e & e & f_1 & f_2 \\ f_3 & f_4 & e & e \\ e & e & f_3 & f_4 \end{pmatrix}\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = (f_1+f_3, f_2+f_4, f_3, f_4)\begin{pmatrix}0\\1\\1\\1\end{pmatrix} = f_2+f_3+f_4 = \textcircled{5}$$

In an analogous way we may define bilinear correspondences S_2, S_3 and S_4, using the implications

for all A,A′,A″ on F, if f_i(A,A′).f_j(A″,A′), then f(A,A″),
for all A,A′,A″ on F, if f_i(A′,A).f_j(A′,A″), then f(A,A″),
for all A,A′,A″ on F, if f_i(A′,A).f_j(A″,A′), then f(A,A″).

We then obtain the matrices

$$S_2 = \begin{pmatrix} f_1 & e & f_2 & e \\ e & f_1 & e & f_2 \\ f_3 & e & f_4 & e \\ e & f_3 & e & f_4 \end{pmatrix}, \quad S_3 = \begin{pmatrix} f_1 & f_2 & e & e \\ f_3 & f_4 & e & e \\ e & e & f_1 & f_2 \\ e & e & f_3 & f_4 \end{pmatrix}$$

$$S_4 = \begin{pmatrix} f_1 & e & f_2 & e \\ f_3 & e & f_4 & e \\ e & f_1 & e & f_2 \\ e & f_3 & e & f_4 \end{pmatrix},$$

and the equations

$$S_2 = S_1C,$$
$$S_3 = CS_1,$$
$$S_4 = CS_1C.$$

[4] Observing that $f_i + e = e + f_i = f_i$, and o.e = e.o = o, l.e = e.l = e.

In order to determine the value of $S_i(f', K_j(f''))$ we simply multiply the matrix S_iK_j with f'^T from the left and f'' from the right. And $S_i(K_j(f'), f'')$ is obtained by multiplying K_jS_i with f'^T from the left and f'' from the right. For in the latter case we have

$$S_i(K_j(f'), f'') = (K_jf')^T S_i f'' = f'^T K_j^T S_i f'' = f'^T K_j S_i f''.$$

What is the logical significance of these correspondences S_i? Their definition and the two examples mentioned above – that S_1(④, ④) = ④ and S_1(④, ⑤) = ⑤ – suggest a close link to the syllogisms of the first, second, third and fourth figures. The connection can be characterised as follows: for any predicates A,A′,A″ over X and for any functions f and f′ belonging to the first seven of the table (see section 2), if f(A,A′) and f′(A′,A″), then A is related to A′ by a function all of whose components are also components of $S_1(f,f')$; and analogous theorems hold for S_2 and the syllogisms of the second, S_3 and the syllogisms of the fourth, and S_4 and those of the fourth figure.(2) In order to sketch out the proof for the case of S_1 and the first figure (the proofs for the other three cases follow the same pattern), let us observe that $S_1(f,f')$ with $f = \sum_{i=1}^{4} \varepsilon_i f_i$ and $f' = \sum_{i=1}^{4} \varkappa_j f_j$, reduces to an expression of four terms:

$$\begin{aligned} S_1(f,f') &= (\varepsilon_1\varkappa_1+\varepsilon_2\varkappa_2)f_1 + (\varepsilon_1\varkappa_2+\varepsilon_2\varkappa_4)f_2 + (\varepsilon_3\varkappa_1+\varepsilon_4\varkappa_3)f_3 + \\ &\quad (\varepsilon_3\varkappa_2+\varepsilon_3\varkappa_4)f_4 \\ &= \lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 + \lambda_4 f_4 . \end{aligned}$$

(2) The only syllogisms considered here are basic ones, that is, syllogisms the premises of which are of the form f(A,A′), where f is one of the functions ①–⑦, A and A′ predicates over X. There exists of course a more comprehensive system of syllogisms, including those with premises of the form f(A,A′) ∨ f′(A,A′) ∨ ..., where f,f′, ... are basic functions. This and other comprehensive systems of syllogisms are most elegantly presented and analysed in terms of relation schemata.

It is easy to show that if $\lambda_i = 0$, and provided f(A,A′) and f′(A′,A″) hold, then A and A″ cannot be related by a function $f'' \varepsilon F$ which contains the component f_i. If e.g. $\lambda_1 = 0$, then either $\varepsilon_2 = \varkappa_3 = 0$ or $\varepsilon_2 = \varkappa_1 = 0$; as we are here concerned with functions ① to ⑦, ε_1 and ε_2 cannot vanish together, nor can $\varkappa_1$ and $\varkappa_2$. We first consider the case where $\varepsilon_1 = \varkappa_3 = 0$, $\varepsilon_2 = \varkappa_1 = 1$. It follows that there is no $x \varepsilon X$ to which both A and A′ apply, and no such x to which both A and ~A′ apply; on the other hand, there is at least one x to which both A and ~A′, and at least one to which both A′ and A″ apply. Hence all the individuals x for which A(x) will also fulfil ~A′(x), and there is no individual to which both A and A″ apply. Next, let $\varepsilon_2 = \varkappa_1 = 0$ and $\varepsilon_1 = \varkappa_3 = 1$. There are no individuals to which both A and ~A′ and no individuals to which both A′ and A″ apply; but there is at least one x to which A and A′, and at least one to which A′ and ~A″ apply. Therefore, all individuals x for which A holds are individuals for which A′ holds, and there exist no individuals x to which both A and A″ apply. In either case, then, A is related to A″ by a function which does not contain f_1 as a component. The remaining coefficients λ_2, λ_3, and λ_4 can be dealt with in a similar way. Thus, since A must be related to A″ by one of the functions ① to ⑦, but cannot be related to A″ by any function $f \varepsilon F$ which involves a component which is not also a component of S_1(f,f′), then A must be related to A″ by a function all of whose components are also components of S_1(f,f′).

It should be obvious, then, that the bilinear function S_1 gives rise to a peculiar type of syllogism which we may represent by the schema

$$\frac{\begin{array}{l} f(A,A') \\ f'(A',A'') \end{array}}{A\{S_1(f,f')\}A''},$$

to half-way syllogisms as it were, stating that if A is linked to A′ by f, and A′ to A″ by f′, then A is related to A″ by a function whose representation involves only components of S_1(f,f′). Analogous considerations apply to S_2, S_3, and S_4. The half-way syllogisms of the first figure are given in the table below. Making use of the matrices S_2, S_3, and S_4 all the other half-way syllogisms can easily be derived.

	①	②	③	④	⑤	⑥	⑦
①	①	①	①	①	①	①	①
②	①	①	①	②	③	②	③
③	①	②	③	①	①	③	②
④	①	①	①	④	⑤	④	⑤
⑤	①	④	⑤	①	①	⑤	④
⑥	①	②	③	④	⑤	⑥	⑦
⑦	①	④	⑤	②	③	⑦	⑧

6. *Other linear correspondences involving complex predicates*

In section 4, when introducing the counterpositions, we had to correlate predicates A and their negations ~A. Here we shall in addition consider the union A ∨ A′ and the intersection A.A′ of two predicates on X and by means of them define the following linear correspondences Q_1, Q_2, R_1, and R_2 of F into itself:

Q_1(f) = f′ if, and only if, whenever f(A,A′), then f′(A ∨ A′,A′),
Q_2(f) = f′ if, and only if, whenever f(A,A′), then f′(A, A ∨ A′),
R_1(f) = f′ if, and only if, whenever f(A,A′), then f′(A.A′,A′),
R_2(f) = f′ if, and only if, whenever f(A,A′), then f′(A,A.A′).

The reader will easily verify that these correspondences are given by the matrices

$$Q_1 = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad Q_2 = \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$R_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \end{pmatrix}.$$

They have some obvious common features. Firstly, they are not one-one correspondences of F onto itself, but in each case two of the basic functions f_i have the same image. Secondly, all four of them are projections, as the following identities hold:

$$Q_1Q_1 = Q_1, Q_2Q_2 = Q_2, R_1R_1 = R_1, R_2R_2 = R_2,$$

an obvious consequence of the above definitions of Q_1Q_2, R_1, and R_2. Thirdly, ⑥ is invariant under each of the four correspondences; for the ordered pair (A,A) is identical to (A ∨ A,A), to (A, A ∨ A), to (A.A,A), and to (A,A.A).

To indicate briefly the significance of correspondences of this kind and their interplay with the conversion C and the counterpositions K_1, K_2, and K_3, let us start from the identity ~(A.A′) = ~A ∨ ~A′, from which we derive that e.g. the correspondences given by the products K_1R_1 and $K_2Q_1K_3$ must be identical, or

$$K_1R_1 = K_2Q_1K_3,$$

and hence

$$R_1 = K_2K_1Q_1K_3,$$

but

$$K_2K_1 = K_3$$

and thus

$$R_1 = K_3Q_1K_3.$$

As K_3 is identical with its transpose K_3^T, the above equation shows that R_1 and Q_3 are similar correspondences. K_3 here plays the role of a regular transformation, the change of basis consisting simply in a permutation of the basic functions. It can easily be shown that R_2 and Q_2 are also similar in this sense:

$$R_2 = K_3Q_2K_3,$$

and so are Q_1 and Q_2, and R_1 and R_2:

$$Q_1 = CQ_2C, R_1 = CR_2C,$$

with C as the regular transformation.

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