A COMPLETENESS PROOF FOR PORTE'S S_a^o AND S_a

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Section I: Background

Motivation for the systems discussed in this note presupposes some familiarity with Jean Porte's [1] and [2]. However, these directions for constructing a completeness proof for systems S_a^o and S_a are accessible prior to a reading of Porte's paper to those familiar with modal logics. In [1] Porte investigated several weak modal logics. Two such systems, S_a^o and S_a are presented below. A basic theorem of [1] was numbered 2.3 and stated: $S_a \vdash L(X)$ iff. $PC \vdash X$.

In this note X,Y,Z are variables ranging over formulas of a modal sentential language whose primitive operators are: $\rightarrow$ (material implication), $\sim$ (negation), L() (it is necessary that). The operators: & (and), v (or), and $\leftrightarrow$ (material equivalence) are defined as usual in classical sentential logic. Throughout this note standard conventions are used in the presentation of formulas and schemas. If S names a system of logic, $S \vdash X$ says that X is a theorem of S. PC names a system for getting exactly the theorems of classical sentential logic. A PC-tautology is any formula in the modal language which is a substitution instance of a tautology in the language of classical sentential logic.

Porte admitted that he failed to prove his Theorem 2.3 of [1]. He wrote that a major goal of his [2] was to present a correct proof of 2.3 of [1] which is numbered Corollary 2.8 in [2]. Unfortunately, Porte's proof of 2.8 of [2] is flawed by a confusion of soundness with completeness. In his argument for Corollary 2.8 he states two theses which he correctly calls completness theorems. They are according to the numbering of [2]:

Lemma 2.4: If X is E-valid, then $S_a^o \vdash X$
Theorem 2.7: If X is E-valid, then $S_a \vdash X$.

In effect, Theorem 2.7 is a corollary of Lemma 2.4. It is the proof of Lemma 2.4. which is seriously flawed. Porte does not prove Lemma 2.4. Instead he proves the converse of Lemma 2.4 which is soundness rather than completeness! Fortunately, for Porte's correction of his original error soundness for S_a^o and S_a suffice for his proof of: If $S_a \vdash L(X)$ then $PC \vdash X$. Nevertheless, his confusion of soundness with completeness is not a mere typographical misstatement of the theorem. He does think that he has proved the completeness of S_a^o and S_a with respect to so-called E-validity. He thinks that he can state in a theorem numbered 4.3 that S_a is complete with respect to a certain kind of Kripke model structure on the basis of his putative completeness proofs with respect to E-validity. Hence, his confusion of soundness with completeness robs his paper of having established completeness for these systems and of having found Kripke semantics for them.

The goals of this paper are to prove Porte's Lemma 2.4 and Theorem 2.7 of his [2] and then to justify his claim that he has found Kripke semantics for S_a. To reach these goals systems S_a^o and S_a are presented in Section II, the notion of E-validity is presented in Section III, directions for a "normal form" completeness proof are given in Section IV, and in Section V Porte's claim to have found Kripke semantics for S_a is vindicated.

Section II: The Systems S_a^o and S_a

The system S_a^o has axiom schemas for classical sentential logic (PC) plus the normalization (necessitation) of these axioms.

P1: $X \rightarrow . Y \rightarrow X$
P2: $X \rightarrow (Y \rightarrow Z) \rightarrow . (X \rightarrow Y) \rightarrow . X \rightarrow Z$
P3: $\sim X \rightarrow \sim Y \rightarrow . Y \rightarrow X$
rP1: L(P1)
rP2: L(P2)
rP3: L(P3)

The rules of proof are: D(modus ponens) $X \rightarrow Y$, $X \vdash Y$, and the so-called normalized modus ponens nD: $L(X \rightarrow Y)$, $L(X) \vdash L(Y)$.

System S_a is S_a^o plus the rule of weakening W: $L(X) \vdash X$. The first three PC axioms together with rule D suffice to give all PC-tautologies as S_a theorems. The normalized PC axioms show us that if X has the form of a PC axiom then $S_a \vdash X$. This observation suffices for the basis step in an induction of the length of a PC proof to establish what I will call Lemma II.1 which is Porte's Theorem 2.1 of his [1].

Lemma II.1: If X is a PC-tautology, then $S_a \vdash X$.

Before leaving this syntactic section let us take note of initially placed L( )s. An initially placed L( ) in a formula X is an L( ) which does not occur in the scope of any other L( ) in X. If we need to emphasize that an L( ) is initially placed we will designate it with L*( ).

Section III: The notion of E-validity

The following is an adaptation of Porte's notion of E-validity. Let e( ) denote the following translation of formulas of S_a's language into a language S_ae. S_ae is the language of S_a extended to include denumerably many new sentential variables q_i which are in one-one correspondence with S_a formulas of the form $L_i()$, where the subscript i indicates some ordering of the L ( ) formulas.

i) $e(X) = X$ if X is a sentential variable.
ii) $e(X \rightarrow Y) = e(X) \rightarrow e(Y)$
iii) $e(X \& Y) = e(X) \& e(Y)$
iv) $e(X \vee Y) = e(X) \vee e(Y)$
v) $e(X \leftrightarrow Y) = e(X) \leftrightarrow e(Y)$
vi) $e(\sim X) = \sim e(X)$
vii) $e(L(X)) = X$ if X is a PC tautology
viii) $e(L_i(X)) = q_i$ if X is not a PC tautology.

Note that the translation of a formula does not "pass beyond" an initially placed L( ). For instance, $e(p \rightarrow L(L(p \rightarrow p) \rightarrow . p \rightarrow p))$ is a formula such as $p \rightarrow q_i$. But $e(p \rightarrow L((p \rightarrow p) \rightarrow . p \rightarrow p))$ is the PC-tautology: $p \rightarrow ((p \rightarrow p) \rightarrow . p \rightarrow p)$. Call the result of applying the e( ) translation to a formula its *e-transformation*. A *formula is E-valid* if its e-transformation is a PC-tautology.

The difference between the notion of E-validity used here and Porte's notion stems from translation rule (vii) above. Porte specifies: If X is a PC tautology, then e(L(X)) = t, where t is the truth value True. Porte then specifies that a formula is E-valid of its e-transformation is t under all classical valuations, viz., is a PC-tautology. We get the same E-valid formulas. But I differ from Porte because I want to show that in certain crucial cases we can prove L(X) with tautologous X by first proving its e-translation; hence we do not want to "lose track" of formulas L(X) with tautologous X by mapping them all to the same tautology or fixed t.

In his Lemma 2.4 and Theorem 2.8 of [2] Porte establishes that S_a^o and S_a are sound. Porte's soundness result provides the main premiss for proving the following lemma.

Lemma III.1: If $S_a \vdash X \leftrightarrow Y$, then X is E-valid iff. Y is E-valid.

Section IV: The completeness of S_a^o *and* S_a *with respect to E-validity*

Porte also uses his soundness result to establish in his Lemma 2.6 of [2] that rule W is admissible in S_a^o. Hence, we can prove the completeness of S_a^o and S_a by focusing on S_a. The strategy of this completeness proof is based on having $S_a \vdash X \leftrightarrow C(X)$, and hence, $S_a \vdash X$ iff. $S_a \vdash C(X)$ where C(X) is a certain conjunctive normal form of X. If $C_i(X)$ represents a conjunct of C(X), we have $S_a \vdash C(X)$ iff. $S_a \vdash C_i(X)$ for each i where i "counts" the conjuncts in C(X). Also C(X) is E-valid iff. $C_i(X)$ is E-valid for each i. Directions will be given for showing that if a conjunct $C_i(X)$ is E-valid then $S_a \vdash C_i(X)$. So, we will get: If C(X) is E-valid, then $S_a \vdash C(X)$, which with standard logical techniques along with Lemma III.1 will give us completeness of S_a with respect to E-validity. So, the crucial tasks for this section are to present a third lemma on normal forms and then to give directions for proving a crucial lemma on developing proofs for certain simple disjunctions which comprise the conjuncts of the conjunctive normal forms.

A *C-conjunctive normal form of X*, C(X), is a conjunction of disjunctions of atoms of X. An *atom in X* is a sentential variable in X, the negation of a sentential variable in X, a subformula of X of the form L*(Y), or the negation of a subformula L*(Y) where L*() is an initially placed L().

By use of PC techniques such as replacing $(X \rightarrow Y)$ with $(\sim X \vee Y)$, DeMorgan's equivalences, and judicious use of distribution we can reduce any formula X of the S_a language to an equivalent C-conjunctive normal form of it. Since PC is included in S_a we can present Lemma IV.1.

Lemma IV.1: If X is a formula in the language of S_a, there is a C-conjunctive normal form C(X) such that $S_a \vdash X$ iff. $S_a \vdash C(X)$.

We now need to show that if a C(X) is E-valid it is provable in S_a. And, as noted earlier, the crucial phase of showing the provability of an E-valid C(X) is showing that a E-valid $C_i(X)$ is S_a provable. Before establishing the crucial lemma about the provability of E-valid $C_i(X)$ it is useful to establish a sublemma about the conditions for the E-validity of a disjunction of atoms because the $C_i(X)$ are such disjunctions.

Sublemma: If only the e-transformation of an E-valid $C_i(X)$ is a PC-tautology, then $C_i(X)$ has an atom of the form L*(Y) where Y is a PC-tautology.

Proof: If E-valid $C_i(X)$ is not a PC-tautology prior to taking its e-transformation, then $C_i(X)$ contains as disjuncts no pair of atoms $(a_j, \sim a_j)$; and, in particular $C_i(X)$ contains no pair of disjuncts $(P_j, \sim P_j)$ where P_j is a sentential variable in X. Now taking the e-transformation of $C_i(X)$ will only disjoin sentential variables and formulas to a disjunction of the sentential variables and their negations from X already occurring in $C_i(X)$. If Y is not a PC-tautology, then e(L*(Y)) gives us only a new q_j to disjoin with the other sentential variable atoms and such a disjunct will not be the negation of any other atom; so it won't give us a PC-tautology. If Y is a PC-tautology but the atom is $\sim$L*(Y), then e($\sim$L*(Y)) gives us a PC-contradiction, $\sim$Y, to disjoin with the other atoms and sentential variables. But the disjunction of a PC-contradiction with a formula which is not already a PC-tautology is not going to give us a PC-tautology. Hence, only if we have an atom of the form L*(Y) where Y is a PC-tautology will we disjoin a formula with the other atoms and formulas which turns the formula into a PC-tautology.

With this sublemma we have arrived at the stage where we can establish our crucial lemma about the provability of E-valid $C_i(X)$.

Lemma IV.2: If $C_i(X)$ is an E-valid disjunction of atoms, then $S_a \vdash C_i(X)$.

Proof: There are two cases. E-valid $C_i(X)$ is a PC-tautology before taking its e-transformation or only the e-transformation of $C_i(X)$ is a PC-tautology. In the first case we have $S_a \vdash C_i(X)$ because PC is included in S_a. In the second case the sublemma tells us there is an atom L*(Y) in $C_i(X)$ where Y is a PC-tautology. My Lemma II.1 tells us that by PC procedures we can get an S_a proof of L*(Y) and then by PC procedures we can disjoin formulas to L*(Y) to get $C_i(X)$ and, thereby, $S_a \vdash C_i(X)$.

On the basis of Lemma IV.2 and the interspersed remarks we are justified in asserting the following completeness theorem.

Theorem 1: If X is E-valid, then $S_a^o \vdash X$ and $S_a \vdash X$.

Let us note some connections between E-validity and Kripke semantics.

Section V: E-validity as validity in certain Kripke model structures

This section presupposes familiarity with the techniques for evaluating formulas in model structures or systems of worlds ordered in some way by an accessibility relation R. Worlds are characterized as normal, semi-normal, and non-normal on the basis of restrictions on assigning the truth-values (t,f) to L(X) formulas. If world w is *normal*, v(L(X),w) = t requires that v(X,w′) = t for all worlds w′ accessible from w, viz., all w′ such that wRw′. Furthermore, if w is normal v(L(X),w) = f requires that there be a w′, wRw′, such that v(~X,w′) = t. If w is *semi-normal*, v(L(X),w) = f requires that there be a w′, wRw′, such that v(~X,w′) = t; but there are no restrictions on having v(L(X),w) = t. For our purposes, the interest of semi-normal worlds lies in the fact that in a semi-normal world G we cannot set v(L*(X),G) = f if X is a PC tautology because in no type of world can we have v(~X, w) = t if X is a PC-tautology. But in a semi-normal world we can treat L(X) with non-tautologous X as if it were a sentential variable. If w is *non-normal*, we can set v(L(X),w) as t or f regardless of the character of X because no worlds are accessible from a non-normal world. In a non-normal world, any L(X) can be treated as if it were a sentential variable.

Consider now the model structures (G,K,R) in which G, the actual world, is normal or semi-normal, K contains G and other worlds normal, semi-normal, or non-normal, and R is an accessibility relation on K. Call these S_a model structures. Let us say that a formula *X is S_a-valid* if there is no G in an S_a model structure such that v(X,G) = f.

Reconsider the notion of E-validity which Porte reminds us "is but a disguise" of S_a-validity. If X is not E-valid then there is a way of assigning (t,f) to positive atoms of X so that v(X) = f as long as we set v(L*(Y)) = t for subformulas L*(Y) where Y is a PC-tautology. Such an assignment showing that X is not E-valid can be re-expressed as an assignment falsifying X in a semi-normal G. On the other hand, if X is not S_a-valid there is a way of assigning (t,f) in a G to positive atoms of X so that v(X,G) = f as long as we set v(L*(Y),G) = t where Y is a PC-tautology. Such a falsification of X in a G can be re-expressed as an assignment showing that X is not E-valid. So we have the following lemma.

Lemma V.1: X is E-valid iff. X is S_a-valid.

Now Lemma V.1 provides the crucial premiss for a proof of Porte's Theorem 4.3 of [2] which gives Kripke semantics for S_a and which I offer, in closing, as a second completeness theorem in this paper for S_a and S_a^o.

Theorem 2: If X is S_a-valid, then $S_a \vdash X$.

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