SUBSTITUTIONAL QUANTIFICATION AND THE PROBLEM OF EXPRESSION TYPES

Peter LUDLOW

It has been argued that ontological economy may be attained in mathematics by quantifying substitutionally over numbers. Recently, this idea has been developed by Gottlieb, with his construction of a semantics for substitutional mathematics. (1) The idea behind the substitutional quantifier is that sentences such as

(1) $(\exists x)\phi$

be semantically interpreted as

(2) For some closed term 't', replacing free occurances of 'x' with 't' in ϕ results in a true sentence.

as opposed to objectual quantification, where sentence (1) is semantically interpreted as

(3) For some object in a given domain, ϕ is satisfied.

Ontological economy is supposedly attained in mathematics because one is no longer objectually quantifyling over numbers. Thus, for example,

(4) $(\exists x)$(x is a prime number between 3 and 7)

may be semantically interpreted as

(5) For some closed term 't', 't is a prime number between 3 and 7' is true.

as opposed to the objectual interpretation, which interprets (4) as,

(1) Dale GOTTLIEB, *Ontological Economy: Substitutional Quantification and Mathematics*. (Oxford: Oxford University Press, 1980).

(6) There is an object of which 'is a prime number between 3 and 7' is true.

(6) obviously commits to the existence of numbers (however we may chose to construe them) while (5) does not commit us to the existence of numbers.

There are a number of objections to substitutional quantification which shall not be considered in this paper. For example, the problem of unnamed objects will not be examined here. Rather, attention shall be focused upon the question of whether substitutional quantification provides what it has promised to provide – ontological economy.

It will not be argued that substitutional quantification fails to free us from commitment to numbers. Rather, this investigation will try to determine whether substitutional quantification commits us to the existence of other sorts of entities – expression types – and to determine whether these other entities are not as problematic as numbers.

The argument that substitutional quantification commits us to expression types is as follows. In sentence (2) (which was the semantic interpretation of a substitutionally quantified sentence) there the existence of expression types. That is, we are still committed to how we are to understand the quantifier 'some' in this phrase. If the quantifier is interpreted objectually, it seems that we are committed to the existence of expression types. That is, we are still committed to the existence of abstract entities.

There are three responses to this problem which we shall consider. They are:

(i) Quantification in the meta-language may be interpreted substitutionally, hence there is no committment to expression types,

(ii) Quantification in the meta-language is objectual, but we need not regard expression types as abstract entities,

(iii) We are committed to expression types as abstract entities, but expression types are 'less abstract' than numbers, and therefore are better.

(i)

The idea that the meta-language be interpreted substitutionally has been suggested on several occaisions, but never fully developed. (2) In response to this suggestion, Parsons has observed that if we interpret quantification substitutionally in the meta-language, the same question will arise in the semantics of the meta-language. In short, 'we have embarked upon a regress that we shall have to end at some point.' (3) Presumably, it will have to end in a language which is not in need of interpretation – natural language. The question then arises as to whether quantification over expression types in natural language is substitutional or objectual. Parsons considers a substitutional interpretation of natural language quantification 'implausible,' but let us see how far we can push the argument.

There are two ways in which this matter may be approached. One may either show that quantification over types is substitutional in natural language, or one may show that quantification in general is substitutional in natural language. We shall take the latter approach.

Determining the given sense of quantification in natural language forces us to look at the semantics of natural language. We shall assume that a theory of the semantics of natural language is correct if (among other things) it assigns a value of true to those sentences which we ordinarily take to be true, and assigns a value of false to those sentences which we ordinarily take to be false. Otherwise, a theory of the semantics of natural language will be regarded as incorrect. A case can be made that it is incorrect to regard quantification in the semantics of natural language as being objectual.

Georgette Ioup has suggested that a substitutional interpretation of natural language quantification may resolve the following problem. (4)

(7) Alberta believes that a dragon ate her petunias.

The indefinite article in (7) has a specific and a non-specific

(2) J. Michael Dunn and Nuel Belnap, 'The Substitution Interpretation of Quantifiers,' *Nous* 2 (1968), p. 184.

(3) Charles Parsons, 'A Plea for Substitutional Quantification,' *Journal of Philosophy*, 68 (1971), pp. 231-7.

(4) Georgette Ioup, 'Specificity and the Interpretation of Quantifiers,' *Linguistics and Philosophy* 1 (1977), p. 236.

interpretation. On the non-specific interpretation, Alberta believes that it was any dragon. She had no particular dragon in mind. On the specific interpretation, Alberta has a particular dragon in mind (perhaps Puff). The logical form of (7) on the specific interpretation is taken to be

(8) ($\exists$x: x is a dragon)(Alberta believes x ate her petunias).

But notice that even if Alberta believes Puff ate her petunias, an objectual interpretation of the quantifier in (8) will compel us to regard (8) as false (for clearly, Puff does not exist). Thus, it might be argued, to interpret quantification objectually in natural language is to have a theory of the semantics of natural language which regards as false, sentences which we ordinarily take to be true. On the substitutional interpretation, (7) would be understood as

(9) For some dragon-name 'n', 'Alberta believes n ate her petunias' is true.

And this is unproblematic.

Of course the argument just presented is far from conclusive. First, this apparent anomaly only occurs when we quantify into referentially opaque contexts. If it were determined that the substitutional interpretation were the correct interpretation of quantification in such contexts, it would not immediately follow that the substitutional interpretation was correct in general, or (importantly) a correct interpretation of quantification over expression types in natural language. Second, another way to avoid having our semantics interpret (8) as false is to regard quantification as being objectual but non-existential. Thus, we might extend the range of the quantifier to include possible but unactual objects. The difficulties with such an approach are well-documented,([5]) but in the eyes of some, not insurmountable.([6]) Third, it may be argued that a structure can be

([5]) For example: W.V.O. QUINE 'On What There Is,' in *From a logical Point of View*, (Cambridge: Harvard University Press, 1961), pp. 1-19 and also, Alvin PLANTINGA, *The Nature of Necessity*, (Oxford: Oxford University Press, 1974), ch. 7 & 8.

([6]) Terence PARSONS, 'The Methodology of Nonexistence,' *Journal of Philosophy*, 76 (1979), pp. 649-652.

built up in the object position of propositional attitude verbs. (7) Thus, for example, 'seeks' might be unpacked as 'try to find', and 'want' as 'try to get'. 'John seeks a dragon' would then be represented as

(10) John trys ($\exists$x: x is a dragon)(to get x)

Again, such an approach is not free from difficulty. (8) For instance, it is not clear that all propositional attitude verbs can be unpacked in this manner. Consider the verb 'believes', for a case in point.

The fourth difficulty with the argument that quantification is substitutional in ordinary language hinges upon the substitutional interpretation of identity. Consider

(11) Two people wrote Waverly.

Taking (11) in the sense of independently writing the same book (by sheer coincidence) we would represent it as

(12) ($\exists$x)($\exists$y)(x wrote Waverly & y wrote Waverly & x $\neq$ y & (z)(if z wrote Waverly, z = x or z = y))

Two distinct terms, when standing in the place of 't', will make the sentence 't wrote Waverly' true ('Scott' and 'Sir Walter'). Either the substitutionalist must regard (12) as true (which it clearly is not) or must show that Scott and Sir Walter are identical. But how is the substitutionalist to do this? Appeal to names having the same referent cannot be allowed, for substitutional quantification does not fix a reference. The obvious way out for the substitutionalist is to interpret identity as substitutivity *salva veritate* of the terms 'Scott' and 'Sir Walter'. But 'Scott' and 'Sir Walter' are not substitutable *salva veritate*, for they are not substitutable in opaque contexts. Further, to equate identity with substitutivity of names raises some very curious problems with different persons having the same names. Are two persons with the same names to be considered identical because their names are substitutable in all contexts? Clearly it is folly to contend

(7) For example: Richard MONTAGUE, 'On the Nature of Certain Philosophical Entities,' in *Formal Philosophy*, (New Haven: Yale University Press, 1974), pp. 148-187.

(8) The problems with this approach from the perspective of transformational grammar are discussed in Robert May, 'The Grammar of Quantification,' Diss. MIT 1977, p. 231 ff.

that quantification in natural language is substitutional.

Of course it may be a mistake to entertain the idea of a substitutional meta-language at all. Our intuition is that to interpret a language is to talk about a language and the expressions therein. Thus Kripke remarks,

> What justifies us in calling the language M a *metalanguage* for the object language, L, at all? If nothing in M purports in any way to refer to, or quantify over, expressions of L, how can a formal theory phrased in M possibly say anything about the semantics of L? If the ontology of M is really supposed to be the null ontology, the formula T(x) can no longer be interpreted as a *predicate* satisfied by exactly the true sentences of L, but it is rather a *form* of M with no interpretation whatsoever. How then can the theory prased in M be said to be the theory of *truth* for the language L? (9)

(ii)

The second response to the problem of expression types is to allow that quantification in the meta-language is objectual, but to maintain that we are not committed to expression types as abstract entities. This would supposedly be accomplished by regarding quantification as being over tokens.

The distinction between tokens and types goes back to Peirce. The idea is that (for example) the ink marking on the upper corner of this page is a token of the type '418'. Tokens and types may be thought standing in the same relation as particulars and universals (or members and classes). Goodman has argued that we may do without types altogether and regard the tokens of a so-called type as being replica's of one another. (10) Likewise, Parsons has suggested,

> The relation 'x is of the same type as y' is a relation of physical

(9) Saul KRIPKE, 'Is There a Problem about Substitutional Quantification?' In Evans and McDowell, eds. *Truth and Meaning*, (Oxford: Oxford University Press, 1976), p. 341.

(10) Nelson GOODMAN, *Languages of Art*, (Indianapolis, Hackett, 1976), p. 131.

> things (e.g. ink marks). It seems that it might be explicable independently of the notion of types, at least of types as kinds of entities... In general, two inscriptions in a linear notation are of the same type if they can be decomposed into sequences of primitive signs of the same length such that the corresponding signs are of the same type.
>
> This characteristic seems not to involve any non-physical *entities*, but the *notions* of inscriptions which can be constructed by *successive* addition of signs, and of sequences *being capable of* being placed in a spacial one-to-one correspondence are perhaps peculiarly mathematical. ([11])

The program suggested by Goodman and Parsons seems plausible, but there are those who think it a misguided program. ([12]) Let us examine some of the difficulties which such a program will encounter.

If we follow Goodman and regard tokens of the same type as merely being replicas for one another, we presumably cannot regard two tokens as being 'exact' replicas of one another. We shall have to allow that a replica of a token t be slightly different from t. But this raises the question of whether resemblence is to be a transitive relation (of course, on Parsons' proposal, the notion of relation is not invoked). If so, it would seem that the two tokens below,

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could be related by a finite chain of slightly different inscriptions, in which each inscription resembles the previous inscription in the chain.

There are two responses to this problem with transitivity. The first response is to have each token of the same type resemble some canonical token. The second response is to require that a token resemble every other token of the same type. The first response suggests a program which, though it could be done, does not reflect

([11]) Charles PARSONS, 'Ontology and Mathematics,' *Philosophical Review*, 80 (1971), pp. 158-159.

([12]) For example: Nicholas WOLTERSTORFF, *Works and Worlds of Art*, (Oxford, Oxford University Press, 1980), pp. 339-340.

how we come to fix the notion of an expression type. Clearly, there is no cannonical token for the expression type '418'. Where would we find it? And if we did establish a canonical token for '418' (say the ink marking in the upper corner of the preceding page) would it reflect how we come to use and understand 'the expression '418''? Clearly not.

It is also problematic to regaıd tokens of the same type as being tokens all of which resemble each other. There are cases where tokens of the same type do not resemble each other (consider the two tokens immediately below).

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One can specify similarities between these tokens, but one can arguably specify more similarities between the two tokens below.

 

Presumably, we would not want to say that these tokens are of the same type.

The objections made to Goodman's program apply to Parsons' as well. For Parsons, two tokens might be construed as of the same type if they could be brought into spacial correspondence. This is problematic even in the trivial example of the two tokens immediately below.

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Parenthetically, there is the question of whether in forsaking relations for modality, Parsons does not commit us to an ontology of possible worlds. Can Parsons be paraphrased as saying 'two tokens are of the same type if there is a possible world in which those tokens are brought into a spacial correspondence'? And if so, are we committed to an ontology of possible worlds?

Of course Parsons could be interpreted as merely suggesting that such a construal of types can be worked out in a single standardized notation. Even in standardized notations, however, problems can arise. Consider the following two tokens.

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Clearly they can be brought into a spacial correspondence (by inverting one of the tokens) but we would not want to say that they are of the same type. Thus a qualification must be added. Perhaps we could define tokens as being of the same type if they can be brought into spacial correspondence without a change in the intended frame of reference. That is, tokens are of the same type if they can be brought into spacial correspondence without being turned upside down.

Let us grant that in a suitably restricted, standardized notation, types can be analyzed without recourse to abstract entities. The question now arises as to whether such a limited claim is useful. Recall that the substitutionalist's concern is to rid our ontology of numbers. To accomplish this task, quantification over numbers must be substitutional not only in limited realms of discourse, but in the language in which mathematicians do mathematics. Clearly mathematicians do not specify a notation when they work, nor should they be expected to. Mathematical discourse takes place verbally, on blackboards, in notebooks, and scholarly journals. To specify one notation for all these realms of discourse would clearly be impossible.

While in some restricted circumstances committment to expression types as abstract entities may be avoided, generally such committment will be necessary.

(iii)

The final argument to be considered is that though substitutional quantification requires an ontology of expression types, expression types are less abstract than (and are therefore better than) numbers. Gottleb argues,

> The use of abstracta in semantics may well be more tractable to explanation than the use of abstract numbers in mathematics. Semantics will require expression types as its abstracts; these may be construed as the shapes (visual, auditory, etc.) of tokens. Thus they are properties instantiated directly by objects, only one step removed from the concrete. Some contend that they will

> play a role in an account of perception. Numbers, by contrast, are at least two steps from the concrete: numbers register the multiplicity of *classes* of things. They thus require a double abstraction. Nor are they likely to be needed in an account of perception. And the usual set-theoretical reductions of numbers make matters worse; they employ pure sets which have no ground whatsoever in the concrete. Thus there is some hope that the problem of abstracta in semantics may receive an independent solution. ([13])

To begin with, let us consider Gottlieb's remarks concerning the role types might play in perception. It is not at all clear what such a role might be. The only clue which Gottlieb gives us is a footnote to Parsons, ([14]) but this reference seems to be at cross-purposes to Gottlieb's program. Parson's point was that tokens might play a role in the perception of abstract entities. Thus,

> There are perceptions of linguistic expressions [for abstract entities] and perception of *concrete instances* of mathematical structures. Either one might play the role, for 'deferred ostension' of mathematical objects, which ordinary perception plays for direct ostension of physical objects. ([15])

Clearly Gottlieb would not want to say that expression types play this sort of role in perception. If expression types do play such a role, it may well be to the benefit of the Platonist, rather than Gottlieb.

We might also question whether it is sensible to talk of *relative* abstractness. We have some intuition as to what abstract objects might be, but do we have an intuition as to what it would be for one abstract object to be more abstract than another? Gottlieb suggests that numbers are more abstract than types because they are classes of classes of objects, as opposed to classes of objects. Suppose this is an adequate characterization of the notion of relative abstractness. The question then arises as to whether it is significant that numbers are

([13]) GOTTLIEB, p. 109.

([14]) Charles PARSONS, 'Ontology and Mathematics,' section III is footnoted.

([15]) Charles PARSONS, 'Ontology and Mathematics,' p. 158.

more abstract than types. What difference does it make if numbers are twice removed from particulars and types are once removed? There certainly seems to be no epistemological difference. Whether referring to or denothing either numbers or types we seem to encounter the same epistemological difficulties. Further, once we have allowed that classes of objects are permissible within our ontology, how can we possibly say that classes of classes are impermissible (or undesireable)? To admit classes within an ontology is to consider them kinds of objects. To that end, classes of classes are nothing more than classes of certain kinds of objects.

If classes of classes are worse than classes, Gottlieb must demonstrate why. It is difficult to envision what form such a demonstration could take.

Of course it might be argued that numbers are not sets of sets of concrete entities, but are pure sets. The distinction between pure and impure sets is made by Jubien as follows.

> By definition, any set with a concrete member is impure; and inductively, any set with an impure member is *impure*. *Pure* sets, then, are sets that are not impure; they are the sort of sets apparently dealt with, for example, in Zermelo-Fraenkel set theory. ([16])

In what sense is a pure set worse than an impure set? Epistemological considerations come immediately to mind. For example, can one be causally related to a pure set? If we are forced to construe numbers as pure sets, Gottlieb's argument may hold. But are we forced to construe numbers in such a fashion?

The argument can be made that numbers cannot be associated with sets of sets of objects because sets of physical objects can never be of sufficient cardinality to account for all of the natural numbers. There are three ways to get around this problem. First, one can introduce the notion of modality, associating numbers with sets of sets of possible numbers or events. Second, one can regard each point of physical space-time as being a physical object. Third, one can postulate an infinite number of physical objects (such as stars). In each case one

([16]) Michael JUBIEN, 'Ontology and Mathematical Truth,' *Nous*, 11 (1977), p. 146.

can form sets of sufficient cardinality to account for all of the natural numbers. Numbers, it would seem, need not be identified with pure sets.

We have seen that substitutional arithmetic does commit us to an ontology of expression types, that these expression types must be construed as abstract entities, and that there is no clear sense in which expression types are 'better' than numbers. There is, however, an argument which has not been considered here. This argument, also from Gottlieb, is that 'even if overall nominalism is unworkable, there is no point in freely expanding the pernicious effects of abstracta in an area for which a nominalist alternative is available.' ([17]) The argument relies heavily on those epistemological concerns raised by Jubien ([18]) and Benacareff. ([19]) For example, Benacareff argues that an ontology of abstracta in mathematics prevents the formulation of mathematical epistemology because knowledge of objects must be mediated by causal interraction with those objects (and because causal interraction with mathematical objects is impossible).

Does substitutional quantification provide advantages for mathematical epistemology? It has already been hinted that it does not. The question is important, but beyond the scope of our current inquiry. Accordingly, a full discussion shall have to wait until a later date.

Columbia University Peter LUDLOW

([17]) GOTTLIEB, p. 110.

([18]) JUBIEN, 'Ontology and Mathematical Truth.'

([19]) Paul BENACERAFF, 'Mathematical Truth,' *Journal of Philosophy*, 72 (1973), pp. 661-679.