THE SEMANTICS OF MINIMAL INTUITIONISM

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Introduction

There has existed some controversy in the history of intuitionism concerning the tenth axiom of Heyting's famous formalization of intuitionistic logic. For example, some logicians have suggested that this axiom does not have any intuitive foundation (cf. [5] and [6], p. 421). Even Heyting admitted that the axiom was perhaps not intuitively clear and, as a consequence, believed that counting it among the axioms of intuitionistic logic required some kind of justification (cf. [4], p. 102). Heyting's attempted justification for keeping it involved claiming that it added to the 'precision of the definition of implication' (again cf. [4], p. 102). However, it has been argued more recently by Susan Haack that its inclusion really amounts to an extension of the intuitionist's sense of 'construction' to a point where it no longer seems characteristically intuitionistic. In view of this consideration, she concludes rather emphatically that the system resulting from dropping the tenth axiom represents the set of intuitionistic logical truths better than the original Heyting axiomatization ([2], p. 102).

Whatever the merits of this claim, it shall not be discussed here. We mention it only because it suggests an examination of the system is perhaps worthwile, especially from a semantical point of view. Consequently, the purpose of this paper will be to provide a semantical interpretation of the resulting system. This system is well-known and has come to be called Johansson's 'minimal calculus' (cf. [5]). It has been studied proof-theorretically by Prawitz[9] and algebraically by Rasiowa and Sikorski[10]. Actually, a modeling for the system has been provided by Fitting[1], p. 40, but since the semantics we shall propose differs in many important respects from his we consider it of some interest to present it here.

In formulating 'possible world' semantics for intuitionistic logic, Kripke makes use of semantic tableaux for proving the completeness theorem [7]. The virtue of that approach is that it is more in keeping with the spirit of intuitionistic constructivism. Nevertheless, in proving the completeness theorem for the minimal calculus, we shall employ techniques adapted from Henkin [3], and, accordingly, shall proceed without any intuitionistic scruples. The reason for adopting this approach is that we consider it intrinsically interesting to direct attention to the two kinds of saturated sets of statement-forms in the minimal calculus. In any event, since the interpretation we shall offer involves a rather straight-forward modification of Kripke's semantics for intuitionistic logic, recovering a completeness theorem based on semantic tableaux would be easily obtainable with only a minimum of corresponding changes. It should also be mentioned that we shall only concern ourselves with statement logic: an extension of the modeling to the first-order predicate calculus could be accomplished by following after the manner of either [7] or [11].

Section 1: Syntax

The symbolic language for Johansson's minimal calculus (JMC) is a triple $<P,C,F>$ where P is a denumerably infinite set of statement variables, C is the set whose members are the unary connective $\neg$, the binary connectives $\wedge$, $\vee$, $\rightarrow$ and the punctuation symbols (,), and F is the set of statement-forms built up as usual from the statement variables in P and the connectives and punctuation symbols in C. The axiom schemata of JMC are the following:

A1. $A \rightarrow (A \wedge A)$
A2. $(A \wedge B) \rightarrow (B \wedge A)$
A3. $(A \rightarrow B) \rightarrow ((A \wedge C) \rightarrow (B \wedge C))$
A4. $((A \rightarrow B) \wedge (B \rightarrow C)) \rightarrow (A \rightarrow C)$
A5. $B \rightarrow (A \rightarrow B)$
A6. $(A \wedge (A \rightarrow B)) \rightarrow B$
A7. $A \rightarrow (A \vee B)$

A8. $(A \vee B) \to (B \vee A)$
A9. $((A \to C) \wedge (B \to C)) \to ((A \vee B) \to C)$
A10. $((A \to B) \wedge (A \to \neg B)) \to \neg A$

We note that the axiom-schemata for JMC are the same as for the intuitionistic logic of Heyting except of course for the conspicuous absence of $\neg A \to (A \to B)$.

The only definition and rule of inference of JMC is given by:

D1. $A \leftrightarrow B =$df $(A \to B) \wedge (B \to A)$
R1. From $A \to B$ and A infer B.

The notion of a *deduction* of a statement form A from a set of statement-forms S is defined as in [8]. Accordingly, we will write '$S \vdash A$' to indicate that there exists a deduction of A from S, and '$\vdash A$' as an abbreviation for '$\phi \vdash A$'. We now list, without proof, some metatheorems of JMC for future reference (they are easily proved in the usual sorts of ways):

MT1. If $A \in S$, then $S \vdash A$.
MT2. If $S \vdash A$, then $S \cup S' \vdash A$.
MT3. $\vdash A \to (B \vee A)$
MT4. If $S \cup \{A\} \vdash B$, then $S \vdash A \to B$.
MT5. If $S \vdash A$ and $S \vdash A \to B$, then $S \vdash B$.
MT6. If $S \cup \{A\} \vdash B$ and $S \cup \{A\} \vdash \neg B$, then $S \vdash \neg A$.
MT7. If $S \vdash A$ and $S \vdash B$, then $S \vdash A \wedge B$.
MT8. If $S \vdash A \wedge B$, then $S \vdash A$ and $S \vdash B$.

Because of the absence in JMC of Heyting's tenth axiom, it is possible to distinguish between two kinds of deductive consistency. Let's define a set of S of statement-forms of JMC to be *negation consistent* just in case for some statement-form A of JMC, not both $S \vdash A$ and $S \vdash \neg A$. Correspondingly, a set S of statement-forms of JMC is said to be *absolutely consistent* just in case for some statement-form A, not $S \vdash A$. Of course, the distinction between these two kinds of deductive consistency breaks down in Heyting's intuitionistic logic and in the classical logic, but not so in the minimal calculus.

Section 2: Semantics

Except for two modifications, JMC-models are essentially the same as intuitionistic models. The first modification involves distinguishing between two kinds of possible worlds. In intuitionistic models, there is only one kind of possible world, however, JMC-models have what we shall call 'negation coherent' and 'absolutely coherent' possible worlds. Negation coherent worlds are worlds which are negation consistent; absolutely coherent worlds, however, may or may not be negation consistent. Nevertheless, we require that the latter worlds be characterized by the stipulation that there exists at least one statement-form in them which receive the valuation 'false.' Furthermore, we require that every JMC-model has either one or both of these kinds of possible worlds. The second modification has to do with the valuation of statement-forms of the form $\neg A$. We stipulate that $\neg A$ is true in a given world provided that A is false in all accessible negation coherent worlds.

More formally, we define a JMC-model as a triple $< W,R,V >$ where W is a set of possible worlds, R is a reflexive and transitive relation defined on W and V is a value assignment satisfying the following conditions:

SC1. For any $A \in P$ and for any $w_i \in W$, if both $V(A,w_i) = T$ and w_iRw_j, then $V(A,w_j) = T$.

SC2. For any $A \in F$ of the form $B \wedge C$ and for any $w_i \in W$, $V(A,w_i) = T$ if both $V(B,w_i) = T$ and $V(C,w_i) = T$; otherwise $V(A,w_i) = F$.

SC3. For any $A \in F$ of the form $B \vee C$ and for any $w_i \in W$, $V(A,w_i) = T$ iff $V(B,w_i) = T$ or $V(C,w_i) = T$; otherwise $V(A,w_i) = F$.

SC4. For any $A \in F$ of the form $B \rightarrow C$ and for any $w_i \in W$, $V(A,w_i) = T$ iff for every $w_j \in W$ such that W_iRw_j, $V(B,w_j) = F$ or $V(C,w_j) = T$; otherwise $V(A,w_i) = F$.

SC5. For any $A \in F$ of the form $\neg B$ and for any $w_i \in W$, $V(A,w_i) = T$ iff for every *negation coherent* $w_j \in W$ such that W_iRw_j, $V(B,w_j) = F$; otherwise $V(A,w_i) = F$.

We say that any $w_i \in W$ is *negation coherent* if and only if for no $A \in F$ is it the case that both $V(A,w_i) = T$ and $V(\neg A,w_i) = T$. Any $w_i \in W$ is said to be *absolutely coherent* nust in case there is at least one $A \in F$ such that $V(A,w_i) = F$. Finally, any $A \in F$ is said to be JMC-logically true if and only if for every JMC-model $<W,R,V>$ and for every $w_i \in W$, $V(A,w_i) = T$.

Given our definitions of the two kinds of possible worlds in JMC-models, it is obvious that every negation coherent world is also an absolutely coherent world; but of course the converse doesn't hold. Accordingly, JMC-models allow for the possibility that there are states of affairs which are classically and intuitionistically inconsistent (in the sense of allowing A and $\neg A$ to be present) but without the consequent disaster of allowing everything to be derivable. Resorting to metaphor, we might say that, unlike classical and intuitionistic models, JMC-models permit the existence of states of affairs which are negation inconsistent, but which are, nevertheless, devoid of total chaos and absolute absurdity.

It is an easy matter to prove the soundness theorem for JMC and so we leave it to the reader to verify that all of the axiom-schemata of JMC are JMC-logically true and that the rule of detachment is truth preserving. As we should expect, Heyting's questionable axiom is not JMC-logically true since it isn't true in all JMC-models. This is patently evident once we consider the following JMC-model: let $W = \{w_1, w_2, w_3\}$ where w_1 and w_2 are negation coherent, and w_3 is absolutely coherent. Furthermore, let w_iRw_2 and w_2Rw_3 and, finally, let $V(A,w_2) = F$ and $V(A,w_3) = T$. Clearly, it follows (by SC5) that $V(\neg A,w_2) = T$. Now since w_3 is absolutely coherent it must be the case that there exists some statement-form, say B, such that $V(B,w_3) = F$. But in that case we have (by SC4) that $V(A \rightarrow B,w_2) = F$. Undoubtedly again (by SC4), we have $V(\neg A \rightarrow (A \rightarrow B),w_1) = F$ and so Heyting's questionable axiom is not JMC-logically true.

Section 3: Intuitive Interpretation

Borrowing from Kripke in [7], pp. 97 ff. we shall say that, in general, in a JMC-model $< W,R,V>$, we interpret W as the set of 'evidential situations.' Where w_i, is any situation, we understand w_iRW_j to mean, as far as we know, at time w_i, we may later get enough information to advance to w_j. We take $V(A,w_i) = T$ to mean that at a particular time w_i, we have enough information to prove A; thus, we might alternatively understand $V(A,w_i) = T$ to say that A has been verified at the point w_i in time. $V(A,w_i) = F$, however, means that A has not been verified at w_i. As Kripke observes, T and F do not denote intuitionistic truth and falsity since although $V(A,w_i) = T$ does mean that A has been verified to be true of w_i; the latter only means that A has not, as yet, been verified at w_i, but it might be later.

Intuitionistically speaking, an evidential situation w_i such that $V(A,w_i) = T$ and $V(\neg A,w_i) = T$ would spell disaster since that would entail that any statement whatsoever has been verified to be true at w_i. However, when viewed from the perspective of minimal intuitionism such an evidential situation, although negation inconsistent, would not necessarily entail that any statement whatsoever is verified to be true at w_i. After all, w_i could very well represent an absolutely coherent evidential situation, in which case, there would be at least one statement at w_i which would not have been verified to be true. In intuitionistic logic, to assert $\neg A$ at w_i is tantamount to knowing not only that A has not been verified at w_i, but that it cannot possibly be verified at any later time, no matter how much more information is gained; thus in intuitionistic logic, $V(\neg A,w_i) = T$ iff for every $w_j \in W$ such that w_iRw_j, $V(A,w_j) = F$. As a consequence, a negation inconsistent evidential situation, intuitionistically speaking, is disastrous since such a situation commits us to the absurd view that a given statement has been verified to be true at that situation and not verified to be true at that situation and any other situation which will eventually come at a later time.

However, in JMC-models, since for any $w_i \in W$, $V(A,w_i) = T$ iff for every negation coherent $w_j \in W$ such that w_iRW_j, $V(A,w_j) = F$, to assert $\neg A$ at w_i amounts to asserting that A has not been verified there and cannot possibly be verified at any later negation coherent situation, no matter how much more information is gained. Thus, an evidential situation w_i such that $V(A,w_i) = T$ and $V(\neg A,w_i) = T$ (i.e., a situation which is negation inconsistent, but, nonetheless, absolutely coherent) simply entails that A has been verified to be true at w_i, but not also that A has not been verified to be true at w_i. Clearly, then, the absurdity mentioned above in the case of intuitionistic models, doesn't arise in the case of JMC-models.

Section 4: Completeness

Given that there are two kinds of deductively consistent sets of statement-forms in JMC, it is possible to define two corresponding kinds of saturated sets in JMC. Let's define a set Γ of statement-forms of JMC to be *negation-saturated* just in case (i) Γ is negation consistent; and (ii) if $\Gamma \vdash A$, then $A \in \Gamma$. Accordingly, we define a set Γ of statement-forms of JMC to be *absolutely-saturated* just in case (i) Γ is absolutely consistent; and (ii) if $\Gamma \vdash A$, then $A \in \Gamma$. We now direct our attention to the proof of certain lemmata which will eventually be used in the proof of the completeness theorem.

L1. For every negation consistent set S of statement-forms of JMC there exists some negation-saturated set Γ of statement-forms such that $S \subseteq \Gamma$.

Proof:

Define the sets $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ as follows:

(a) $\Gamma_0 = S$

(b) Enumerate the statement-forms of JMC. For each $i \leq 0$, if $\Gamma_i \cup \{A_i\}$ is negation consistent in JMC, then $\Gamma_{i+1} = \Gamma_i \cup \{A_i\}$; otherwise $\Gamma_{i+1} = \Gamma_i$.

(c) $\Gamma = \bigcup_{0 \leq i < \omega} \Gamma_i$.

Obviously there exists a set Γ such that $S \subseteq \Gamma$.

What we must now show is that conditions (i) and (ii) of a negation-saturated set hold for Γ.

It is easily demonstrated by induction on *i* that if S is negation consistent in JMC then so is each Γ_i, and hence $\bigcup \Gamma_i = \Gamma$. Γ_0 is negation consistent since S is. Assume that Γ_i is negation consistent but that Γ_{i+1} is not. In that case $\Gamma_i \cup \{A_i\}$ is inconsistent and so

$\Gamma_{i+1} = \Gamma_i$. Thus, contrary to the assumption, Γ_{i+1} is negation consistent after all. Therefore, Γ_{i+1} is negation consistent and so is $\cup\Gamma_i = \Gamma$.

We now show that if $\Gamma \vdash A$, then $A \in \Gamma$. Suppose that $\Gamma \vdash A$. Furthermore, suppose for the sake of *reductio* that $A \notin \Gamma$. In view of the way Γ was constructed, $\Gamma \cup \{A\}$ must be negation inconsistent; i.e., for any statement-form B, both $\Gamma \cup \{A\} \vdash B$ and $\Gamma \cup \{A\} \vdash \neg B$. Consequently (by MT6), $\Gamma \vdash \neg A$. But now we have both $\Gamma \vdash A$ and $\Gamma \vdash \neg A$ and so Γ is negation inconsistent contrary to what was established above. Therefore, $A \in \Gamma$.

L2. For every absolutely consistent S of statement forms of JMC there exists some absolutely-saturated set Γ of statement-forms such that $S \subseteq \Gamma$.

Proof:

Define the sets $\Gamma_0, \Gamma_1, \Gamma_2, \ldots$ as in the proof of L1, but this time replace 'negation consistent' by 'absolutely consistent.'

Obviously there exists a set Γ such that $S \subseteq \Gamma$.

$\cup\Gamma_i = \Gamma$ is absolutely consistent by way of the same argument given in the proof of L1.

Now we show that if $\Gamma \vdash A$, then $A \in \Gamma$. Suppose that $\Gamma \vdash A$. Also suppose for the sake of *reductio* that $A \notin \Gamma$. In view of the way Γ was constructed, $\Gamma \cup \{A\}$ must be absolutely inconsistent; i.e., for any statement-form B, $\Gamma \cup \{A\} \vdash B$. Consequently (by MT4), $\Gamma \vdash A \rightarrow B$ and so (by MT5) for any statement-form B, $\Gamma \vdash B$. But in that case Γ is absolutely inconsistent contrary to what was established above. Therefore, $A \in \Gamma$.

The following lemma identifies the distinctive feature of absolutely-saturated sets:

L3. For any absolutely-saturated set Γ of statement-forms of JMC there is some statement-form A of JMC such that $A \notin \Gamma$.

Proof:

Suppose that Γ is any absolutely-saturated set of statement-forms of JMC. Since it is absolutely consistent it must be the case that for some statement-form A of JMC not $\Gamma \vdash A$. But in that case we may conclude that for some statement-form A of JMC, $A \notin \Gamma$ (by MT1).

In most of the remaining lemmata, when we speak of a 'saturated set' we are speaking of any and every saturated set whether it is negation-saturated or absolutely-saturated.

L4. Let A be any statement-form of JMC and Γ any saturated set of statement-forms of JMC. Then $\Gamma \vdash A$ if and only if $A \in \Gamma$.

Proof:

By condition (ii) of either definition of a saturated set and MT1.

L5. Let A be any statement-form of JMC and Γ any saturated set of statement-forms of JMC. Then $\Gamma \cup \{A\}$ is consistent (in the appropriate sense) if and only if $A \in \Gamma$.

Proof:

Suppose $\Gamma \cup \{A\}$ is consistent (in either sense). Hence in wiew of the way Γ was constructed A would have been added to Γ and so $A \in \Gamma$.

Now suppose $A \in \Gamma$. Furthermore, suppose $\Gamma \cup \{A\}$ is inconsistent (in either sense). In that case, in view of the way Γ was constructed, it must be the case that $A \notin \Gamma$ which is contrary to the first supposition. Therefore, $\Gamma \cup \{A\}$ is consistent (in the appropriate sense).

L6. Let A be any statement-form of JMC and Γ any saturated set of statement-forms of JMC. Then if $A \notin \Gamma$, then there exists some saturated set Γ' such that $\Gamma \subseteq \Gamma'$ and $A \notin \Gamma'$.

Proof:

Suppose $A \notin \Gamma$. Then clearly (by L5) $\Gamma \cup \{A\}$ is inconsistent. Now L1 and L2 guarantee that there exists some saturated set Γ' such that $\Gamma \subseteq \Gamma'$. But in that case $\Gamma' \cup \{A\}$ is also inconsistent. Consequently (by L5), $A \notin \Gamma'$.

L7. For any statement-form A of JMC and every saturated set Γ, if $\vdash A$, then $A \in \Gamma$.

Proof:

Suppose $\vdash A$. Also suppose for the sake of *reductio* $A \notin \Gamma$. Now $\vdash A$ is the same as $\phi \vdash A$. Hence (by MT2) $\phi \cup \Gamma \vdash A$ and so, since $\phi \cup \Gamma = \Gamma$, $\Gamma \vdash A$. But (by hypothesis) $A \notin \Gamma$, thus (by L4) not $\Gamma \vdash A$ and so we have a contradiction. Therefore, $A \in \Gamma$.

L8. For any statement-forms A and B of JMC and for every saturated set Γ, if $A \vee B \in \Gamma$, then either $A \in \Gamma$ or $B \in \Gamma$.

Proof:

(a) Suppose that Γ is negation-saturated and that $A \vee B \in \Gamma$. Furthermore, suppose for the sake of *reductio* that neither $A \in \Gamma$ nor $B \in \Gamma$; i.e., $A \notin \Gamma$ and $B \notin \Gamma$. Then (by L5) $\Gamma \cup \{A\}$ and $\Gamma \cup \{B\}$ are both negation inconsistent; in other words:

$$\Gamma \cup \{A\} \vdash C \text{ and } \Gamma \cup \{A\} \vdash \neg C\text{; and}$$
$$\Gamma \cup \{B\} \vdash C \text{ and } \Gamma \cup \{B\} \vdash \neg C.$$

Clearly (by MT4), we have in each case:

$$\Gamma \vdash A \rightarrow C \text{ and } \Gamma \vdash A \rightarrow \neg C\text{; and}$$
$$\Gamma \vdash B \rightarrow C \text{ and } \Gamma \vdash B \rightarrow \neg C.$$

Hence (by MT7) we have:

$$\Gamma \vdash (A \rightarrow C) \wedge (B \rightarrow C)$$
$$\Gamma \vdash (A \rightarrow C) \wedge (B \rightarrow \neg C).$$

Now since both

$$((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C) \text{ and}$$
$$((A \rightarrow \neg C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$$

are axioms of JMC, it follows (by L7) that they are both in Γ. Hence (by MT1):

$$\Gamma \vdash ((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$$
$$\Gamma \vdash ((A \rightarrow \neg C) \wedge (B \rightarrow \neg C)) \rightarrow ((A \vee B) \rightarrow \neg C).$$

Consequently (by MT5):

$$\Gamma \vdash (A \vee B) \rightarrow C$$
$$\Gamma \vdash (A \vee B) \rightarrow \neg C$$

But $A \vee B \in \Gamma$ (by hypothesis), hence (by MT1):

$$\Gamma \vdash A \vee B.$$

Thus (again by MT5):

$$\Gamma \vdash C \text{ and}$$
$$\Gamma \vdash \neg C.$$

Consequently, Γ is negation inconsistent contrary to supposition. Therefore, either $A \in \Gamma$ or $B \in \Gamma$.

(b) Now suppose Γ is absolutely-saturated and $A \vee B \in \Gamma$. Again, for the sake of *reductio*, assume $A \in \Gamma$ and $B \in \Gamma$. Then (by L5) both $\Gamma \cup \{A\}$ and $\Gamma \cup \{B\}$ are absolutely inconsistent; i.e., for any satement-form C,

$$\Gamma \cup \{A\} \vdash C$$
$$\Gamma \cup \{B\} \vdash C.$$

Hence (by MT4 and MT7)

$$\Gamma \vdash (A \rightarrow C) \wedge (B \rightarrow C).$$

But because

$$((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$$

is an axiom of JMC, we have (by L7 and MT1):

$$\Gamma \vdash ((A \rightarrow C) \wedge (B \rightarrow C)) \rightarrow ((A \vee B) \rightarrow C)$$

and so (by MT5):

$$\Gamma \vdash (A \vee B) \rightarrow C.$$

but $A \vee B \in \Gamma$ (by hypothesis). Hence (by MT1):

$$\Gamma \vdash A \vee B$$

Consequently (by MT5) it follows that for any statement-form C,

$$\Gamma \vdash C.$$

Clearly, Γ is absolutely inconsistent contrary to the hypothesis of the proof. Therefore, either $A \in \Gamma$ or $B \in \Gamma$.

On either supposition, then, we have $A \in \Gamma$ or $B \in \Gamma$ and so the lemma is proved.

L9. For any statement-forms A and B of JMC and every saturated set Γ, if $A \vee B \notin \Gamma$, then $A \notin \Gamma$ and $B \notin \Gamma$.

Proof:

Suppose $A \vee B \notin \Gamma$. Furthermore, suppose, for the sake of *reductio*, that not both $A \notin \Gamma$ and $B \notin \Gamma$; i.e., either $A \in \Gamma$ or $B \in \Gamma$. If $A \in \Gamma$, then $\Gamma \vdash A$ (by MT1). But $A \rightarrow (A \vee B)$ is an axiom of JMC, hence (by L7 and MT2) $\Gamma \vdash A \rightarrow (A \vee B)$ and so (by MT5) $\Gamma \vdash A \vee B$. But (by hypothesis and L4) not $\Gamma \vdash A \vee B$. Obviously we have a contradiction and so $A \notin \Gamma$ and $B \notin \Gamma$. If $B \in \Gamma$, similar reasoning together with MT3 yields the same conclusion. Either way, then, we have $A \notin \Gamma$ and $B \notin \Gamma$ and so the lemma is proved.

L10. For any statement-form A of JMC and for every saturated set Γ, if $\neg A \in \Gamma$, then for every negation-saturated set Γ' such that $\Gamma \subseteq \Gamma'$, $A \notin \Gamma'$.

Proof:

Suppose $\neg A \in \Gamma$ and that $\Gamma \subseteq \Gamma'$ where Γ' is negation-saturated. Obviously, $\neg A \in \Gamma'$ and so (by MT1) $\Gamma' \vdash \neg A$. Now suppose for the sake of *reductio* $A \in \Gamma'$. Then (again by MT1) $\Gamma' \vdash A$. Thus Γ' is negation inconsistent contrary to the hypothesis. Therefore $A \notin \Gamma'$.

L11. For any statement form A and every saturated set Γ, if $\neg A \notin \Gamma$, then there exists some negation-saturated set Γ' such that $\Gamma \subseteq \Gamma'$ and $A \in \Gamma'$.

Proof:

Suppose $\neg A \notin \Gamma$. Then (by 16) there exists some saturated set Γ' such that $\Gamma \subseteq \Gamma'$ and $\neg A \notin \Gamma'$. Hence (by L4) not $\Gamma' \vdash \neg A$. Clearly, then, (by MT6) not both $\Gamma' \cup \{A\} \vdash B$ and $\Gamma' \cup \{A\} \vdash \neg B$. Consequently, $\Gamma' \cup \{A\}$ is negation consistent. But in that case (by L5) $A \in \Gamma'$ and Γ' is negation-saturated.

Let us now employ the set Γ as the set of all negation-saturated and absolutely-saturated sets of JMC with set-theoretic inclusion defined on it. We can now easily form a JMC-model $<W,R,V>$ on the basis of Γ. With each $\Gamma_i \in \Gamma$ associate a $w_i \in W$ (where Γ_i is negation-saturated associate a negation coherent world, where Γ_i is absolutely-saturated associate a absolutely coherent world). In other words, let $W = \Gamma$. Let R be the relation such that $w_i R w_j$ if and only if $\Gamma_i \subseteq \Gamma_j$. Let V be the following value assignment: for any statement-variable A and every $w_i \in W$, $V(A,w_i) = T$ if $A \in \Gamma_i$ and $V(A,w_i) = F$ if $A \notin \Gamma_i$. Clearly, $<W,R,V>$ thus defined is a JMC-model. We are now prepared to state and prove the completeness theorem.

(Completeness) Let W, R and V be defined as above. Then for every statement-form A of JMC and for every $w_i \in W, V(A,w_i) = T$ if $A \in \Gamma_i$ and $V(A,W_i) = F$ if $A \notin \Gamma_i$.

Proof:

Case 1: A is a statement-variable. Obviously the theorem holds for A by the initial value assignment to statement variables.

Case 2: A is $B \wedge C$. (a) Let $A \in \Gamma_i$. Then (by MT1) $\Gamma_i \vdash A$. But in that case $\Gamma_i \vdash B$ and $\Gamma_i \vdash C$ (by MT8). Consequently (by L4), $B \in \Gamma_i$ and $C \in \Gamma_i$. Hence (by the hypothesis of induction) $V(B,w_i) = T$ and $V(C,w_i) = T$. Therefore (by SC2), $V(A,w_i) = T$. (b) Let $A \notin \Gamma_i$. In that case not $\Gamma_i \vdash A$ (by L4). Hence (by MT7) either not $\Gamma_i \vdash B$ or not $\Gamma_i \vdash C$. If not $\Gamma_i \vdash B$, then $B \notin \Gamma_i$ (by MT1). Consequently $V(B,w_i) = F$ (by inductive assumption) and so $V(A,w_i) = F$ (by SC2). If not $\Gamma_i \vdash C$, then $C \notin \Gamma_i$ (by MT1). Therefore, $V(C,w_i) = F$ (by induction hypothesis) and so $V(A,w_i) = F$ (by SC2).

Case 3: A is $B \vee C$. (a) Let $A \in \Gamma_i$. Then either $B \in \Gamma_i$ or $C \in \Gamma_i$ (by L8). But in that case, either $V(B,w_i) = T$ or $V(C,w_i) = T$ (by the hypothesis of induction). Therefore, $V(A,w_i) = T$ (by SC3). (b) Let $A \notin \Gamma_i$. Then (by L9) $B \notin \Gamma_i$ and $C \notin \Gamma_i$. Thus (by induction hypothesis) both $V(B,w_i) = F$ and $V(C,w_i) = F$. Therefore (by SC3), $V(A,w_i) = F$.

Case 4: A is $B \rightarrow C$. (a) Let $A \in \Gamma_i$. Obviously for every Γ_i such that $\Gamma_i \subseteq \Gamma_j$, $A \in \Gamma_j$. Hence (by MT1) $\Gamma_j \vdash A$. Consequently (by

MT5) $\Gamma_j \vdash C$ if $\Gamma_j \vdash B$. Clearly, then (by L4), $C \in \Gamma_j$ if $B \in \Gamma_j$. Put differently, either $B \notin \Gamma_j$ or $C \in \Gamma_j$. Therefore (by the hypothesis of induction) either $V(B,w_j) = F$ or $V(C,w_j) = T$, and so (by SC4) $V(A,w_i) = T$. (b) Suppose $A \notin \Gamma_i$. Then (by L4) not $\Gamma_i \vdash A$. Hence (by MT4) not $\Gamma_i \cup \{B\} \vdash C$ and so (by MT1) $C \notin \Gamma_i \cup \{B\}$. Consequently (by L6) there exists some saturated set Γ_j such that $\Gamma_i \cup \{B\} \subseteq \Gamma_j$ and $C \notin \Gamma_j$. Obviously $B \in \Gamma_j$. Thus (by hypothesis of the induction) $V(B,w_j) = T$ and $V(C,w_j) = F$. Therefore (by SC4), $V(A,w_i) = F$.

Case 5: A is $\neg B$. (a) Let $A \in \Gamma_i$. Then (by L10) for every negation-saturated set Γ_j such that $\Gamma_i \subseteq \Gamma_j$, $B \notin \Gamma_j$. Hence (by hypothesis of induction) for every negation coherent world w_j, $V(B,w_j) = F$. Therefore (by SC5) $V(A,w_i) = T$. (b) Let $A \notin \Gamma_i$. Then (by L11) there exists some negation-saturated set Γ_j such that $\Gamma_i \subseteq \Gamma_j$ and $B \in \Gamma_j$. Hence (by the hypothesis of the induction) there exists some negation coherent world w_j such that $V(B,w_j) = T$. Therefore (by SC5) $V(A,w_i) = F$.

Undoubtedly, if we were to define entailment along the lines mentioned in [11], we could also, given our approach, prove the strong semantical completeness theorem without any difficulty.*

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