S5 AND THE PREDICATE CALCULUS

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1. It has often been remarked that there is a formal similarity between modalities and quantification (possibility, M, corresponding to $\exists$, while necessity, L, corresponds to $\forall$), and more especially between S5 and the first-order predicate calculus. See for instances Von Wright[21], Montague[7], Prior[12], (pp. 185-193), Thomas[20] or Kuhn[5]. This paper aims to point at several precise relationships between S5 and the (first-order) predicate calculus. (The words 'first-order' will be dropped in what follows).

2. A WELL KNOWN RESULT – The following theorem may be considered 'folklore', for practically every logician knows it, but it is seldom clearly stated in the literature – although it was actually the chief point of Wajsberg[22] and of Parry[9]. See, however, Feys-Dopp[3] (sections 21-23 and 73) or Prior[13] (p. 25).

Theorem 1 – Let us consider the restriction of the predicate calculus to unary (or 'monadic') predicates and to only one individual variable, and let us suppress every occurrence of the variable; then we obtain a system which is isomorphic to S5, predicates becoming propositional variables, $\forall$ becoming L, $\exists$ becoming M, and the usual propositional connectives being unchanged.

Proof: Compare the decision method for S5 in Carnap[2], and the decision method for the 'monadic' predicate calculus in Quine[16].

An alternetive proof consists in looking for what an axiomatization of the predicate calculus reduces to, when we consider only the said restricted class of formulas. A key point is that, when there is only one individual variable, v_1, (and no propositional variable) the only means for a formula not to contain any free occurrence of v_1 is to be closed.

3. THE NOTION OF 'CLOSURE' – In Quine[17] the closure of a formula of the predicate calculus is the formula which is obtained by prefixing to the given formula, X, a row of universal quantifiers in the anti-alphabetic order (which suppose that an 'alphabetic order' of the variables is given in the definition of the formal system), these quantifiers corresponding to all the variables which have at least one free occurrence within X. Of course, when X is a closed formula, its closure is X itself.

In what follows, the closure of X will be denoted by '$\mathscr{C}X$'.

Remark: Quine[17] (as well as Quine[15]) considers 'well-formed' only closed formulas. This feature allows him to write '$\vdash X$' instead of '$\vdash \mathscr{C}X$'. That will not be possible in what follows, where we consider 'well-formed' such a formula as

$$\forall v_1 \, P(v_1) \rightarrow P(v_2)$$

(where P is a unary predicate, and v_1 and v_2 are different variables).

It is easy to prove that the classical predicate calculus (with non-closed formulas) is obtained by adding to Quine's system the axiom schema

$$\forall x A \rightarrow A$$

(where x is an arbitrary individual variable, and A is an arbitrary formula).

Then it is very easy to prove that, in this system, we have

A1: $\mathscr{C}T$, if T is a substitution instance of a classical tautology
A2: $\mathscr{C}\,(\mathscr{C}\,(A \rightarrow B) \rightarrow (\mathscr{C}A \rightarrow \mathscr{C}B))$
A3: $\mathscr{C}\,(\neg \mathscr{C}A \rightarrow \mathscr{C} \neg \mathscr{C}A)$
(I) A4: $\mathscr{C}\,(\mathscr{C}A \rightarrow A)$
A5: $\mathscr{C}A \rightarrow A$
R1: $A, A \rightarrow B \;/\; B$

(where A, B are arbitrary formulas).

4. *Theorem 2* – Let every instance of $\mathscr{C}$ be replaced by L, and every formula of the predicate calculus (such as A, B, etc... be replaced by a formula of a modal propositional system – i.e. by a formula constructed with $\neg$, $\to$ and L, where L replaces $\mathscr{C}$, and an infinite denumerable list of propositional variables: p_1, p_2, etc... – And let us consider *a formula of this modal system to be 'acceptable' iff all the formulas it replaces are theses of the predicate calculus* – Then, this modal system is S5.

On one hand it is obvious from A1-A5 and R1 that the theses of the modal system contain all the theses of S5.

On the other hand, let us suppose that the new system contains an 'acceptable' formula, X, which is not an S5-thesis. Let us translate it into a thesis X' of the predicate calculus by replacing the different propositional variables by different subformulas, all containing only unary predicates and one individual variable, v, and by replacing L by $\forall v$. Then the modal translation of X' defined in Theorem 1 (dropping every occurrence of v, and the predicates becoming propositional variables) is a substitution instance of X. But, by Theorem 1, this translation of X' should be an S5 – thesis – which contradicts the hypothesis.

Intuitively speaking we can say that the meaning of Theorem 2 is that S5 says all we can say about the predicate calculus when we use only the propositional connectives and the notion of 'closure'.

In a more abstract way, we can express Theorem 2 as follows:

Let us consider the mathematical structure

$$\mathscr{M} = \langle \mathscr{F}, \mathscr{T}, \neg^*, \to^*, L^* \rangle$$

where

$\mathscr{F}$ is the set of formulas of the predicate calculus
$\mathscr{T}$ is the set of theses of the predicate calculus
$\neg^* = X \mapsto \neg X$
$\to^* = (X, Y) \mapsto X \to Y$
$L^* = X \mapsto \mathscr{C}X$

Then *$\mathscr{M}$ is a characteristic matrix for S5* ($\mathscr{T}$ being the set of designated elements).

5. In what precedes, I have used the definition of 'closure' given in Quine[17] (who had borrowed it from Berry[1]).

There are other kinds of 'closures'. One is original Quine's notion (see Quine[15]): the variables of the prefix are put into alphabetic order rather than into anti-alphabetic order. Berry's closure is more useful than original Quine's one in order to obtain a simple axiomatization of the predicate calculus. But it would play exactly the same role in the proof of system (I) and in Theorem 2.

Another kind of 'closure' is the one of Fitch[4] – which indeed offers the same advantages than Berry's in order to get a simple axiomatization of the predicate calculus.

Fitch's closure is constructed in a way similar to Berry's but taking into account the order in which the free occurrence of the variables appears within the formula under study, rather than their alphabetic order within the system – Indeed Fitch's work does not need that any 'alphabetic order' be defined.

Fitch's closure could be used in place of Berry's in system (I) and Theorem 2 would stay true.

There are other possible definitions of 'closure' – indeed an infinity. If we substitute any of these definitions to the one of Berry, does Theorem 2 remain true? A necessary and sufficient condition is that the results expressed in system (I) (A1-A5, R1) remain true.

6. Reasoning exactly as in § 4 we can prove

Theorem 3 – Let us consider the mathematical structure

$$\mathcal{M}_1 = <\mathcal{F}, \mathcal{C}, \neg^*, \rightarrow^*, L^*>$$

where everything is defined as in $\mathcal{M}$, except that now

$$L^* = X \mapsto \forall v_1 X$$

– Then $\mathcal{M}_1$ *is a characteristic matrix for S5*.

It is obvious by symmetry that v_1 does not play any special role and that using other variables will yield other characteristic matrices for S5 – all being isomorphic.

7. The foregoing results suggest generalizations leading to open problems.

If, in a way similar to the line which led to Theorem 1 let us consider the part of predicate calculus whose atomic formulas are of the form $P(v_1, v_2)$, where P is any binary predicate while v_1 and v_2 are particular individual variables. Then again, by the transformation which consists of suppressing every occurrence of v_1 and v_2, and replacing $\forall v_1$ by L_1 and $\forall v_2$ by L_2, we obtain a propositional calculus, whose connectives are $\neg$, $\rightarrow$, and two unary ones, L_1 and L_2 (the predicates becoming propositional variables).

Let 2S5 be that new system. Starting from an axiomatization of the predicat e calculus with rules of modus ponens and necessitation, and proceeding as in the second proof of Theorem 1, we are led to the following axiomatization of 2S5 – where $i = 1$ or 2, A and B being any formulas:

Ax 1 – T, if T is a subsitution instance of a classical tautology
Ax 2 – $L_i(A \rightarrow B) \rightarrow (L_i A \rightarrow L_i B)$
Ax 3 – $L_i A \rightarrow A$
Ax 4 – $A \rightarrow L_i A$, if A is fully L_i-modalized
R1 – $A, A \rightarrow B \,/\, B$
R2 – $A \,/\, L_i A$

– A formula, such as A, is *fully L_i-modalized* ($i = 1$ or 2) if every occurrence of a propositional variable is within the scope of an occurrence of L_i in A.

A few open problems are the following ones:

(i) Is 2S5 a conservative extension of S5? (Conjecture: yes. Suggestion: use the results of Scroggs[18]).

(ii) Is 2S5 decidable? (Conjecture: yes).

(iii) Is there a finite standard axiomatization of 2S5? i.e. an axiomatization by a finite number of sequential rules (see Łoś and Suszko[6] – those rules are called 'connective rules' in [10] or [11]), and a finite number of axiom schemas? (Conjecture: no).

2S5 will be a kind of 'bidimensional modal system', but different from those which have been studied by Prior (the 'tense logics' of [12]) or Segerberg[19].

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