A REGULAR SEQUENT CALCULUS FOR QUANTUM LOGIC IN WHICH $\wedge$ AND $\vee$ ARE DUAL

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Introduction

In the paper [Nishimura (1980)] the author develops sequent calculi GO and GOM for *orthologic* and *orthomodular logic*, the latter usually going under the ambiguous name of *quantum logic*. In Nishimura's calculi there is a fundamental lack of duality in the treatment of the disjunction $\vee$ and the conjunction $\wedge$. Moreover, his calculi are not *regular* in the following sense: a sequent calculus SC is *regular* iff for finite sets (of formulae) Γ and Δ

$$\vdash_{\overline{SC}} \Gamma \to \Delta \quad \text{iff} \quad \vdash_{\overline{SC}} \wedge\Gamma \to \vee\Delta.$$

The non-regularity of GO and GOM follows from the non-classical disjunction property established by Nishimura for these calculi (see § 2 below).

In this paper we develop two regular sequent calculi GO† and GO†M which are related to Nishimura's calculi in the following way. The calculi GO† and GO†M are extensions of GO and GOM in that

$$\vdash_{\overline{GO(M)}} \Gamma \to \Delta \quad \text{implies} \quad \vdash_{\overline{GO\dagger(M)}} \Gamma \to \Delta$$

but not conversely. However, for *normal* sequents $\Gamma \to \Delta'$, in which Δ' has at most one member

$$\vdash_{\overline{GO(M)}} \Gamma \to \Delta' \quad \text{iff} \quad \vdash_{GO\dagger(M)} \Gamma \to \Delta'$$

Besides their intrinsic interest as formulations of orthologic and quantum logic, GO† and GO†M have some relevance to the philosophical discussion of the nature of logic. We say that a sequent calculus SC *tolerates a rule R* when the calculus SC + R, obtained by adding the rule R to SC, has the same class of provable sequents as SC.

Neither GO† nor GO†M tolerates the (full) cut rule of GO(M). There is reason to think that GO†(M) are logics if GO(M) are. Therefore toleration of the (full) cut rule cannot be held to be a necessary condition for a sequent calculus to be a logic. We reserve discussion of this and other philosophical points to our final section.

The organisation of this paper is as follows. § 1 contains some preliminaries. In § 2 we discuss Nishimura's calculus GO. § 3 develops GO†. We prove the regularity of GO† and demonstrate its equivalence to GO for normal sequents. Soundness and completeness for GO† with respect to a version of the relational semantics for orthologic devised by [Goldblatt (1974)] are obtained in § 4. In § 5 we sketch GO†M, our orthomodular extension of GO†, together with its relational semantics. We conclude in § 6 with some philosophical remarks prompted by GO†(M).

§ *1 Preliminaries*

In this paper we are concerned only with propositional sequent calculi. A (propositional) *sequent calculus* SC has a denumerable set of *propositional variables* $\{p_o, p_1, \ldots\}$, a set of *logical connectives* (which in our case will be $\{\neg, \wedge\}$ for GO(M) and $\{\neg, \wedge, \vee\}$ for GO†(M)), and *brackets*), (. The set of wffs is the smallest set containing the propositional variables that is closed under the connectives. We use Greek letters $\alpha, \beta, \ldots$ to denote wffs, the capitals $\Gamma, \Delta \ldots$ to denote sets (possibly empty, possibly infinite) of wffs.

In a sequent calculus the proved objects, namely sequents, are of the form $\Gamma \rightarrow \Delta$. Their intuitive interpretation is either

(i) 'whenever all the wffs in Γ are true, at least one wff in Δ is true', or

(ii) for finite Γ, Δ, 'whenever any conjunction of all the formulae in Γ is true, any disjunction of all the formulae in Δ is true'. (i) and (ii) are equivalent for the classical propositional calculus.

The use of sets of formulae, rather than sequences, trivialises certain structural rules which appear in Gentzen's original sequent calculi, and whenever we apply a trivial rule in a proof we shall cite [df.]. In the usual fashion we write 'Γ, α' for $\Gamma \cup \{\alpha\}$; 'Γ, Δ' for $\Gamma \cup \Delta$ etc. Sequents of the form $\{\alpha\} \rightarrow \Delta$ and $\Gamma \rightarrow \{\alpha\}$ are written as $\alpha \rightarrow \Delta$ and $\Gamma \rightarrow \alpha$ respectively. '$\neg\Gamma$' denotes the set $\{\neg\alpha : \alpha \in \Gamma\}$.

Definitions

(a) A *proof* in a sequent calculus SC is a finite tree of sequents such that

(1) the topmost sequents are axiom sequents;
(2) every sequent in the tree except the topmost sequents is the lower sequent of a rule of SC whose upper sequents are in the tree.

(b) The *sequent proved* by a proof in SC is the unique lowest sequent in the tree.

We write $\vdash_{\overline{SC}} \Gamma \to \Delta$ for $\Gamma \to \Delta$ is provable in SC.

For a finite set of formulae Δ there are many (logically equivalent) ways to define the conjunction of the formulae in Δ. Let $[\wedge\Delta]$ denote the set of all the possible ways of forming conjunctions of all the formulae in Δ. We will write $\wedge\Delta$ to mean any of the formulae in $[\wedge\Delta]$. For a language with $\vee$ (primitive or defined), we define $[\vee\Delta]$ and $\vee\Delta$ similarly. When Δ is empty we understand both $[\wedge\Delta]$ and $[\vee\Delta]$ to mean the empty set.

The following notions enable us to see more clearly the distinction between our calculi and those of Nishimura.

Definitions

(a) A sequent calculus SC is *regular* iff, for all finite Γ and Δ

$$\vdash_{\overline{SC}} \Gamma \to \Delta \quad \text{iff} \quad \vdash_{\overline{SC}} \wedge\Gamma \to \vee\Delta$$

(b) A sequent calculus SC is *dual* iff for all finite Γ and Δ

$$\vdash_{\overline{SC}} \Gamma \to \Delta \quad \text{iff} \quad \vdash_{\overline{SC}} \Delta^* \to \Gamma^*$$

where, for a formula α, we obtain α^* by replacing each occurrence of $\wedge$ by $\vee$, and vice versa; and for a set Γ, we define

$$\Gamma^* = \{\gamma^* \ : \ \gamma \in \Gamma\}.$$

Finally, note that all the sequent calculi considered in this paper possess the following feature.

Theorem 1.1 (i) If $\vdash_{\overline{SC}} \Gamma \to \Delta$, then there exists a finite subsequent $\Gamma' \to \Delta'$ of $\Gamma \to \Delta$ such that $\vdash_{\overline{SC}} \Gamma' \to \Delta'$;

(ii) if a finite sequent $\Gamma \to \Delta$ is provable in SC, each sequent occurring in any proof of $\Gamma \to \Delta$ is finite.

Proof

By induction on the construction of proofs in SC.
[Cp. Nishimura (1980) Theorems 2.1 and 2.2].

§ 2 *The system GO (as developed in* [Nishimura (1980)])

The formal language of GO lacks the logical connective $\vee$, which is introduced as an abbreviation, namely $\alpha \vee \beta$ is an abbreviation for $\neg(\neg\alpha \wedge \neg\beta)$.

Axioms of GO:

$$\alpha \to \alpha$$

Rules of GO:

$$\frac{\Gamma \to \Delta}{\Theta, \Gamma \to \Delta, \Sigma}\ (\text{ext})$$

$$\frac{\Gamma_1 \to \Delta_1, \alpha \qquad \alpha, \Gamma_2 \to \Delta_2}{\Gamma_1, \Gamma_2 \to \Delta_1, \Delta_2}\ (\text{cut})$$

$$\frac{\alpha, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta}\ (\wedge\to) \qquad \frac{\beta, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta}\ (\wedge\to)$$

$$\frac{\Gamma \to \Delta, \alpha \qquad \Gamma \to \Delta, \beta}{\Gamma \to \Delta, \alpha \wedge \beta}\ (\to \wedge)$$

$$\frac{\Gamma \to \Delta, \alpha}{\Gamma, \neg\alpha \to \Delta}\ (\neg\to)$$

$$\frac{\alpha \to \Delta}{\neg\Delta \to \neg\alpha}\ (\to \neg)$$

$$\frac{\alpha, \Gamma \rightarrow \Delta}{\neg \neg \alpha, \Gamma \rightarrow \Delta} (\neg \neg \rightarrow)$$

$$\frac{\Gamma \rightarrow \Delta, \alpha}{\Gamma \rightarrow \Delta, \neg \neg \alpha} (\rightarrow \neg \neg)$$

GO has the following non-classical features.
(i) According to the *normal form theorem* [Nishimura (1980)] Theorem 2.5 if $\vdash_{GO} \Gamma \rightarrow \Delta$, there exists a normal subsequent $\Gamma \rightarrow \Delta'$ of $\Gamma \rightarrow \Delta$ such that $\vdash_{GO} \Gamma \rightarrow \Delta'$, and $\Gamma \rightarrow \Delta'$ has a normal proof, i.e. a proof such that every sequent occurring in it is normal.
(ii) The normal form theorem has the *disjunction theorem* [Nishimura (1980)] as a corollary, according to which if $\vdash_{GO} \Gamma \rightarrow \Delta$ and Δ is non-empty, then for some $\alpha \in \Delta$, $\vdash_{GO} \Gamma \rightarrow \alpha$.

GO may be easily seen to be non-regular in the following way. If GO were regular we would have $\vdash_{GO} \alpha \vee \beta \rightarrow \alpha, \beta$. But the addition of $\alpha \vee \beta \rightarrow \alpha, \beta$ as an axiom to GO results in an expansion to classical logic, as the next theorem shows.

Theorem 2.1 $GO + \vdash \alpha \vee \beta \rightarrow \alpha, \beta$ = classical logic.

Proof
Note that we have $\vdash_{GO} \rightarrow (\alpha \vee \neg \alpha)$. From the additional axiom we have $\vdash \alpha \vee \neg \alpha \rightarrow \alpha, \neg \alpha$ giving $\vdash \rightarrow \alpha, \neg \alpha$ via (cut). In the extended system we can derive the following rule

$$\frac{\alpha, \Gamma \rightarrow \Delta}{\Gamma \rightarrow \Delta, \neg \alpha} (\rightarrow \neg)_c$$

The derivation is as follows.

$$\frac{\alpha, \Gamma \rightarrow \Delta \qquad \rightarrow \alpha, \neg \alpha}{\Gamma \rightarrow \Delta, \neg \alpha} (\text{cut})$$

But $GO + (\rightarrow \neg)_c$ = classical logic as noted in [Nishimura (1980) p. 342].

The non-duality of GO may be easily seen as follows.
Since $\vdash_{GO} \alpha, \beta \to \alpha \wedge \beta$, the duality of GO would imply that $\alpha^* \vee \beta^* \to \alpha^*, \beta^*$ is GO-provable. But $\nvdash_{GO} \alpha^* \vee \beta^* \to \alpha^*, \beta^*$ in general.

In the next section we develop the system GO†, an alternative formulation of orthologic which is closely related to GO yet which is regular and in which $\wedge$ and $\vee$ are treated dually.

§ 3 *The system GO†*

The formal language of GO† contains the full set of logical connectives $\wedge$, $\vee$, and $\neg$.

Axioms of GO†:

$$\alpha \to \alpha$$

Rules of GO†:

$$\frac{\Gamma \to \Delta}{\Theta, \Gamma \to \Delta, \Sigma}\ (\text{ext})$$

$$\frac{\Gamma \to \alpha, \Delta_1 \qquad \alpha \to \Delta_2}{\Gamma \to \Delta_1, \Delta_2}\ (\text{cut-1})$$

$$\frac{\Gamma_1 \to \alpha \qquad \Gamma_2, \alpha \to \Delta}{\Gamma_1, \Gamma_2 \to \Delta}\ (\text{cut-2})$$

$$\frac{\alpha, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta}\ (\wedge\to) \qquad \frac{\beta, \Gamma \to \Delta}{\alpha \wedge \beta, \Gamma \to \Delta}\ (\wedge\to)$$

$$\frac{\Gamma \to \alpha \qquad \Gamma \to \beta}{\Gamma \to \alpha \wedge \beta}\ (\to\wedge)\dagger$$

$$\frac{\alpha \to \Delta \qquad \beta \to \Delta}{\alpha \vee \beta \to \Delta}\ (\vee\to)\dagger$$

$$\frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha \vee \beta}\ (\to\vee)\dagger \qquad \frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta, \alpha \vee \beta}\ (\to\vee)\dagger$$

$$\frac{\Gamma \to \alpha}{\Gamma,\ \neg\alpha \to}\ (\neg\to)\dagger$$

$$\frac{\Gamma \to \Delta}{\neg\Delta \to\ \neg\Gamma}\ (\to\ \neg)\dagger$$

$$\frac{\alpha, \Gamma \to \Delta}{\neg\,\neg\alpha, \Gamma \to \Delta}\ (\neg\,\neg\to) \qquad \frac{\Gamma \to \Delta, \alpha}{\Gamma \to \Delta,\ \neg\,\neg\alpha}\ (\to\ \neg\,\neg)$$

We mark the rules specific to GO† with a '†'.

In GO† the cut-rule is restricted. (1) (Cut-1) and (cut-2) are mutually dual, as are the rules for $\wedge$ and $\vee$.

The rule $(\neg\to)\dagger$ is a restriction of the corresponding rule in GO. But the rule $(\to\ \neg)$ of GO is a restriction of the corresponding GO† rule $(\to\ \neg)\dagger$.

Note that

(i) GO + $(\to\ \neg)\dagger$ is equivalent to the classical propositional calculus;

(ii) GO + $(\vee\to)\dagger$ is equivalent to the classical propositional calculus.

To see (i) note that we have $\vdash_{\overline{GO}}\ \alpha, \neg\alpha\to$. So that in GO + $(\to\ \neg)\dagger$ we have $(\to\ \neg)_c$ as a derived rule, as below.

$$\dfrac{\Gamma, \alpha \to \Delta \qquad \dfrac{\dfrac{\alpha,\ \neg\alpha \to}{\to\ \neg\alpha,\ \neg\,\neg\alpha}\ (\to\ \neg)\dagger \qquad \dfrac{\alpha \to \alpha}{\neg\,\neg\alpha \to \alpha}\ (\neg\,\neg\to)}{\to\ \neg\alpha,\ \alpha}\ (\text{cut})}{\Gamma \to \Delta,\ \neg\alpha}\ (\text{cut})$$

To see (ii) it is sufficient to note that in GO + $(\vee\to)\dagger$ we have $\vdash \alpha \vee \beta \to \alpha, \beta$.

In Theorems 3.1 and 3.10 we further consider the relation between GO and GO† and demonstrate their equivalence for normal sequents.

Theorem 3.1 $\vdash_{\overline{GO}} \Gamma \to \Delta$ implies $\vdash_{\overline{GO\dagger}} \Gamma \to \Delta$

Proof

By the normal form theorem for GO, $\vdash_{\overline{GO}} \Gamma \to \Delta$ implies $\vdash_{\overline{GO}} \Gamma \to \Delta'$ for some normal subsequent $\Gamma \to \Delta'$ of $\Gamma \to \Delta$. This normal subsequent

(1) Following a suggestion by Michael Dummett in an unpublished paper entitled 'Introduction to Quantum Logic'.

has a normal proof. But a normal proof in GO is also a (normal) proof in GO† since in a normal proof in GO

(i) any axiom appearing in the GO proof is an axiom of GO†;
(ii) any application of (cut) in GO is an application of (cut-2) in GO†;
(iii) any application of ($\neg\rightarrow$) in GO is an application of ($\neg\rightarrow$)†;
(iv) any application of ($\rightarrow \wedge$) in GO is an application of ($\rightarrow \wedge$)†;
(v) the remaining GO rules are all GO† rules.

Hence $\Gamma \rightarrow \Delta'$ is provable in GO†, and $\Gamma \rightarrow \Delta$ may be proved in GO† by (ext).

Before we prove the converse of Theorem 3.1 for *normal* sequents we prove that GO† is regular. We require the following lemmas.

Lemma 3.2 For finite Γ,

$$\vdash_{\overline{GO\dagger}} \Gamma \rightarrow \Delta \quad \text{iff} \quad \vdash_{\overline{GO\dagger}} \wedge\Gamma \rightarrow \Delta$$

Proof

This is trivial if Γ has less than two members. For larger Γ, the 'only if' part of the result is obtained by repeated application of the derived rule (DR-1)

$$\frac{\alpha, \beta, \Gamma \rightarrow \Delta}{\alpha \wedge \beta, \Gamma \rightarrow \Delta}\ \text{(DR-1)}$$

which is easily seen to be a derived rule of GO†.

For the 'if' part, use repeated application of the following rule (DR-2) which is seen below to be a derived rule of GO†.

$$\frac{\alpha \wedge \beta, \Gamma \rightarrow \Delta}{\alpha, \beta, \Gamma \rightarrow \Delta}\ \text{(DR-2)}$$

Derivation of (DR-2)

$$\cfrac{\cfrac{\cfrac{\alpha \rightarrow \alpha}{\alpha, \beta \rightarrow \alpha}\ \text{(ext)} \quad \cfrac{\beta \rightarrow \beta}{\alpha, \beta \rightarrow \beta}\ \text{(ext)}}{\alpha, \beta \rightarrow \alpha \wedge \beta}\ (\rightarrow \wedge)\dagger \quad \alpha \wedge \beta, \Gamma \rightarrow \Delta}{\alpha, \beta, \Gamma \rightarrow \Delta}\ \text{(cut-2)}$$

Lemma 3.3 For finite Δ,

$$\vdash_{\overline{GO\dagger}} \Gamma \rightarrow \Delta \quad \text{iff} \quad \vdash_{\overline{GO\dagger}} \Gamma \rightarrow \vee\Delta$$

Proof

If Δ has fewer than 2 members this is trivial. For larger Δ, proceed by repeated application of the derived rule (DR-3) of GO† to obtain the 'only if' part of lemma 3.3.

$$\frac{\Gamma \to \Delta, \alpha, \beta}{\Gamma \to \Delta, \alpha \vee \beta}\ \text{(DR-3)}$$

Derivation of (DR-3)

$$\cfrac{\cfrac{\cfrac{\Gamma \to \Delta, \alpha, \beta}{\Gamma \to \Delta, \alpha, \alpha \vee \beta}\ (\to \vee)\dagger}{\Gamma \to \Delta, \alpha \vee \beta, \alpha \vee \beta}\ (\to \vee)\dagger}{\Gamma \to \Delta, \alpha \vee \beta}\ \text{(df.)}$$

For the 'if' part proceed by repeated application of (DR-4), the following derived rule of GO†.

$$\frac{\Gamma \to \Delta, \alpha \vee \beta}{\Gamma \to \Delta, \alpha, \beta}\ \text{(DR-4)}$$

Derivation of (DR-4)

$$\cfrac{\Gamma \to \Delta, \alpha \vee \beta \qquad \cfrac{\cfrac{\alpha \to \alpha}{\alpha \to \alpha, \beta}\ \text{(ext)} \qquad \cfrac{\beta \to \beta}{\beta \to \alpha, \beta}\ \text{(ext)}}{\alpha \vee \beta \to \alpha, \beta}\ (\vee \to)\dagger}{\Gamma \to \Delta, \alpha, \beta}\ \text{(cut-1)}$$

Theorem 3.4 (The regularity of GO†) For finite Γ, Δ

$$\vdash_{GO\dagger} \Gamma \to \Delta \quad \text{iff} \quad \vdash_{GO\dagger} \wedge\Gamma \to \vee\Delta$$

Proof

Directly from lemmas 3.2 and 3.3.

Theorem 3.5 (The duality of GO†)

$$\vdash_{GO\dagger} \Gamma \to \Delta \quad \text{iff} \quad \vdash_{GO\dagger} \Delta^* \to \Gamma^*$$

Proof

Since $\Gamma^{**} = \Gamma$ and $\Delta^{**} = \Delta$ it is sufficient to prove that

$$\vdash_{GO\dagger} \Gamma \to \Delta \quad \text{implies that} \quad \vdash_{GO\dagger} \Delta^* \to \Gamma^*$$

Let us call $\Delta^* \to \Gamma^*$ the *dual* of $\Gamma \to \Delta$. The *dual* of an instance of a rule is obtained by taking the dual of each of the sequents occurring in that rule.

Now observe that the dual of an axiom of GO† is an axiom of GO†, and the dual of an instance of a GO† rule is also an instance of a GO† rule except for $(\neg\to)\dagger$. However, the dual of a finite instance of $(\neg\to)\dagger$ is an instance of the derived rule of GO† for finite Δ

$$\frac{\alpha \to \Delta}{\to \neg\alpha, \Delta}$$

which has the following derivation. It is sufficient, using the derived rules (DR-3) and (DR-4) of Lemma 3.3 to consider the case $\Delta = \{\delta\}$.

$$\dfrac{\dfrac{\dfrac{\alpha \to \delta}{\alpha, \neg\delta \to}\,(\neg\to)\dagger}{\to \neg\alpha, \neg\neg\delta}\,(\to\neg)\dagger \qquad \dfrac{\delta \to \delta}{\neg\neg\delta \to \delta}\,(\neg\neg\to)}{\to \neg\alpha, \delta}\,(\text{cut-1})$$

Hence any GO† proof of a sequent $\Gamma \to \Delta$ (with finite sequents throughout) can be converted to a GO† proof of $\Delta^* \to \Gamma^*$, as required.

We now prove the converse of Theorem 3.1, and hence the equivalence of GO and GO† for normal sequents. We begin with the following lemmas.

Lemma 3.6 The following is a derived rule in both GO and GO†:

$$\frac{\neg\neg\alpha \to \Delta}{\alpha \to \Delta}\,(\text{DR-5})$$

Proof

$$\dfrac{\neg\neg\alpha \to \Delta \qquad \dfrac{\alpha \to \alpha}{\alpha \to \neg\neg\alpha}\,(\to\neg\neg)}{\alpha \to \Delta}\,(\text{cut}(\text{-}1))$$

Lemma 3.7 The following are derived rules of GO:

(a) $$\frac{\Gamma \to \alpha \vee \beta \qquad \alpha \to \gamma}{\Gamma \to \beta \vee \gamma}\,(\Gamma \text{ finite});$$

(b) $$\frac{\alpha \to \delta \qquad \beta \to \delta}{\alpha \vee \beta \to \delta}$$

Proofs

(a) Note first that lemma 3.2 for GO† holds in GO since (DR-1) and (DR-2) are both derived rules of GO.

Thus we need only consider (a) in the following form:

$$\frac{\wedge\Gamma \to \alpha \vee \beta \qquad \alpha \to \gamma}{\wedge\Gamma \to \beta \vee \gamma}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\wedge\Gamma \to \neg(\neg\alpha \wedge \neg\beta)}{\neg\neg(\neg\alpha \wedge \neg\beta) \to \neg\wedge\Gamma}\,(\to \neg)}{(\neg\alpha \wedge \neg\beta) \to \neg\wedge\Gamma}\,(\text{DR-5})}{\neg\alpha, \neg\beta \to \neg\wedge\Gamma}\,(\text{DR-2}) \qquad \dfrac{\alpha \to \gamma}{\neg\gamma \to \neg\alpha}\,(\to \neg)}{\neg\beta, \neg\gamma \to \neg\wedge\Gamma}\,(\text{cut})}{\neg\beta \wedge \neg\gamma \to \neg\wedge\Gamma}\,(\text{DR-1})}{\neg\neg\wedge\Gamma \to \neg(\neg\beta \wedge \neg\gamma)}\,(\to \neg)}{\wedge\Gamma \to \neg(\neg\beta \wedge \neg\gamma)}\,(\text{DR-5})$$

(b) $$\dfrac{\dfrac{\dfrac{\dfrac{\alpha \to \delta}{\neg\delta \to \neg\alpha}\,(\to \neg) \qquad \dfrac{\beta \to \delta}{\neg\delta \to \neg\beta}\,(\to \neg)}{\neg\delta \to \neg\alpha \wedge \neg\beta}\,(\to \wedge)}{\alpha \vee \beta \to \neg\neg\delta}\,(\to \neg)}{\alpha \vee \beta \to \delta}\,(\text{DR-5})$$

Lemma 3.8 The following hold

(a) $\vdash_{GO} \delta \to \delta \vee \alpha$,

(b) for finite Δ (with $\Delta \neq \emptyset$) $\vdash_{GO} \neg\Delta \to \neg(\vee\Delta)$,

(c) (i) $\vdash_{GO} (\alpha \vee \beta) \vee \gamma \to \alpha \vee (\beta \vee \gamma)$,

(ii) $\vdash_{GO} \alpha \vee (\beta \vee \gamma) \to (\alpha \vee \beta) \vee \gamma$.

Proofs

(a)
$$\dfrac{\dfrac{\dfrac{\neg\delta \rightarrow \neg\delta}{\neg\delta \wedge \neg\alpha \rightarrow \neg\delta}\,(\wedge\rightarrow)}{\neg\neg\delta \rightarrow \neg(\neg\delta \wedge \neg\alpha)}\,(\rightarrow\neg)}{\delta \rightarrow \delta \vee \alpha}\,(\text{DR-5})$$

(b) By induction on the size of Δ. Note first that the result is trivial for Δ with one member.

Suppose that Δ has at least one member; as the induction hypothesis assume that

$$\vdash_{\overline{GO}} \neg\Delta \rightarrow \neg(\vee\Delta)$$

Let $\delta = \vee\Delta$; then using the associative laws (c) it is sufficient to show that for any α,

$$\vdash_{\overline{GO}} \neg\Delta, \neg\alpha \rightarrow \neg(\delta \vee \alpha)$$

as follows

$$\dfrac{\dfrac{\begin{matrix}\vdots \\ \neg\Delta \rightarrow \neg\delta\end{matrix} \qquad \begin{matrix}\vdots \\ \neg\delta, \neg\alpha \rightarrow \neg\delta \wedge \neg\alpha\end{matrix}}{\neg\Delta, \neg\alpha \rightarrow \neg\delta \wedge \neg\alpha}\,(\text{cut})}{\neg\Delta, \neg\alpha \rightarrow \neg\neg(\neg\delta \wedge \neg\alpha)}\,(\rightarrow\neg\neg)$$

(c) We leave the associative laws as an exercise for the reader.

In the next theorem we must take into account the fact that $\vee$ is primitive in GO† but not in GO. For any wff γ we write $\bar{\gamma}$ for the corresponding GO formula in which all occurrences of $\vee$ are abbreviations, and similarly for sets of formulae.

Theorem 3.9

$$\vdash_{GO\dagger} \Gamma \rightarrow \Delta \quad \text{implies} \quad \vdash_{\overline{GO}} \overline{\Gamma} \rightarrow \vee\overline{\Delta}, \text{ for finite } \Delta.$$

Proof

By induction on the construction of the proof in GO† of $\Gamma \rightarrow \Delta$. We may assume that all sequents occurring in the proof are finite.

Basis step

Trivial for proofs of unit length; the axioms of GO† are axioms of GO.

Induction Step

Let the last step in the proof of $\Gamma \to \Delta$ be an application of

(i) (ext)

i.e. $$\frac{\Sigma \to \Pi}{\Gamma \to \Delta}\text{(ext)} \qquad \Sigma \subseteq \Gamma, \Pi \subseteq \Delta$$

By the induction hypothesis

$$\vdash_{GO} \overline{\Sigma} \to \vee \overline{\Pi}$$

So from lemma 3.8 (a), and the associative laws (lemma 3.8 (c)) as required

$$\vdash_{GO} \overline{\Gamma} \to \vee \overline{\Delta};$$

(ii) (cut-1)

i.e. $$\frac{\Gamma \to \alpha, \Delta_1 \qquad \alpha \to \Delta_2}{\Gamma \to \Delta_1, \Delta_2}\text{(cut-1)}$$

By the induction hypothesis

$$\vdash_{GO} \overline{\Gamma} \to \bar{\alpha} \vee (\vee \overline{\Delta}_1) \text{ and}$$

$$\vdash_{GO} \bar{\alpha} \to \vee \overline{\Delta}_2.$$

By lemma 3.7 (a) and 3.8 (c) as required

$$\vdash_{GO} \overline{\Gamma} \to (\vee \overline{\Delta}_1) \vee (\vee \overline{\Delta}_2)$$

(iii) (cut-2)

i.e. $$\frac{\Gamma_1 \to \alpha \qquad \Gamma_2, \alpha \to \Delta}{\Gamma \to \Delta}\text{(cut-2)}$$

where $\Gamma = \Gamma_1 \cup \Gamma_2$,
obvious since (cut-2) is a special case of (cut);

(iv) $\underline{(\wedge\rightarrow)}$

trivial;

(v) $\underline{(\rightarrow\wedge)\dagger}$

trivial;

(v) $\underline{(\rightarrow\wedge)\dagger}$

trivial;

(vi) $\underline{(\vee\rightarrow)\dagger}$

i.e.
$$\frac{\alpha\rightarrow\Delta \qquad \beta\rightarrow\Delta}{\alpha\vee\beta\rightarrow\Delta}(\vee\rightarrow)$$

By the induction hypothesis

$$\vdash_{\overline{GO}} \bar{\alpha}\rightarrow \vee\overline{\Delta} \text{ and } \vdash_{\overline{GO}} \overline{\beta}\rightarrow \vee\overline{\Delta},$$

so, by lemma 3.7 (b)

$$\vdash_{\overline{GO}} \overline{\alpha\vee\beta}\rightarrow \vee\overline{\Delta};$$

(vii) $\underline{(\rightarrow\vee)\dagger}$

i.e.
$$\frac{\Gamma\rightarrow\Pi,\alpha}{\Gamma\rightarrow\Pi,\alpha\vee\beta}(\rightarrow\vee);\ \Pi\cup\{\alpha\vee\beta\}=\Delta$$

By the induction hypothesis

$$\vdash_{\overline{GO}} \overline{\Gamma}\rightarrow (\vee\overline{\Pi})\vee\bar{\alpha}$$

and so by lemma 3.8 (a)

$$\vdash_{\overline{GO}} \overline{\Gamma}\rightarrow (\vee\overline{\Pi}\vee\bar{\alpha})\vee\overline{\beta};$$

(viii) $\underline{(\neg\rightarrow)\dagger}$

trivial;

(ix) $\underline{(\rightarrow\neg)\dagger}$

The last line in the proof is

$$\frac{\Sigma \to \Pi}{\neg\Pi \to \neg\Sigma}$$

where $\Gamma = \neg\Pi$ and $\Delta = \neg\Sigma$. By the induction hypothesis

$\vdash_{\overline{GO}} \overline{\Sigma} \to \vee\overline{\Pi}.$

By $(\neg\neg\to)$ we have $\vdash_{\overline{GO}} \neg\neg\overline{\Sigma} \to \vee\overline{\Pi}$, and using lemma 3.2 for GO we have $\vdash_{\overline{GO}} \wedge(\neg\neg\overline{\Sigma}) \to \vee\overline{\Pi}$.

Using $(\to\neg)$ $\vdash_{\overline{GO}} \neg\vee\overline{\Pi} \to \neg\wedge(\neg\neg\overline{\Sigma})$

i.e. $\vdash_{\overline{GO}} \neg\vee\Pi \to \vee(\neg\overline{\Sigma})$

Now use lemma 3.8 (b) to obtain

$$\vdash_{\overline{GO}} \neg\overline{\Pi} \to \vee(\neg\overline{\Sigma})$$

as required.

(x) $\underline{(\neg\neg\to)}$

trivial;

(xi) $\underline{(\to\neg\neg)}$

i.e. $\dfrac{\Gamma \to \Pi, \alpha}{\Gamma \to \Pi, \neg\neg\alpha}(\to\neg\neg)$ where $\Delta = \Pi \cup \{\neg\neg\alpha\}$

By the induction hypothesis

$\vdash_{\overline{GO}} \overline{\Gamma} \to (\vee\overline{\Pi}) \vee \bar{\alpha}$, and $\vdash_{\overline{GO}} \bar{\alpha} \to \neg\neg\bar{\alpha}$,

so by lemma 3.7 (a)

$\vdash_{\overline{GO}} \overline{\Gamma} \to (\vee\overline{\Pi}) \vee (\neg\neg\bar{\alpha})$ as required.

Theorem 3.10 (The equivalence of GO and GO† for normal sequents)
For normal GO sequents $\Gamma \to \Delta$

$\vdash_{\overline{GO}} \Gamma \to \Delta$ iff $\vdash_{GO\dagger} \Gamma \to \Delta$

Proof

'Only if' part from Theorem 3.1; 'if' part from Theorem 3.9.

§ 4 *The Semantics of GO†: Soundness and Completeness*

We now prove the soundness and (extended) completeness of GO† with respect to the relational semantics due to [Goldblatt (1974)]. The following account is similar to that of [Nishimura (1980) § 3], but with a modified definition of the validity of sequents.

GO†: Soundness

A *GO†-frame* is a pair $<X, \perp>$ where

(1) X is a non-empty set,
(2) $\perp$ is an orthogonality relation on X. That is, $\perp \subset X \times X$,

and is irreflexive and symmetric.

For $x \in X$ and $Y \subseteq X$ we say (i) $x \perp Y$ iff for every $y \in Y$, $x \perp y$;

(ii) $Y^* = \{x : x \perp Y\}$

A subset Y of X is *⊥-closed* (or simply *closed*) iff $Y^{**} = Y$.

A *GO†-model* is a triple $<X, \perp, D>$ where

(1) $<X, \perp>$ is a GO†-frame;
(2) D is a function assigning to each propositional variable p a closed subset D(p) of X.

Given a GO†-model $\mathcal{M}$, the notation $\|\alpha\|_{\mathcal{M}}$ where α is a wff, is defined as follows. Note that for any $Y \subseteq X$, Y^* is closed, so $\|\alpha\|_{\mathcal{M}}$ is always closed.

(1) $\|p\|_{\mathcal{M}} = D(p)$
(2) $\|\neg\alpha\|_{\mathcal{M}} = (\|\alpha\|_{\mathcal{M}})^*$
(3) $\|\alpha \wedge \beta\|_{\mathcal{M}} = \|\alpha\|_{\mathcal{M}} \cap \|\beta\|_{\mathcal{M}}$
(4) $\|\alpha \vee \beta\|_{\mathcal{M}} = (\|\alpha\|_{\mathcal{M}}^* \cap \|\beta\|_{\mathcal{M}}^*)^*$

For arbitrary Γ (not necessarily finite) it is convenient to define:

$\| \wedge\Gamma \|_{\mathscr{M}} = \bigcap_{\alpha\in\Gamma} \| \alpha \|_{\mathscr{M}}$. Similarly, $\| \vee\Delta \|_{\mathscr{M}} = [\bigcap_{\delta\in\Delta} \| \delta \|_{\mathscr{M}}{}^*]^*$.

Definition

Define $\text{Val}_{\mathscr{M}}(.\ ;x)$ for a GO†-model $\mathscr{M} = \langle X, \perp, D\rangle$ and $x \in X$ as follows:

(1) *for formulae*

$$\text{Val}_{\mathscr{M}}(\alpha\ ;x) = \begin{cases} 1 \text{ if } x \in \| \alpha \|_{\mathscr{M}}, \\ 0 \text{ otherwise.} \end{cases}$$

(2) *for sequents*

$$\text{Val}_{\mathscr{M}}(\Gamma \to \Delta\ ;x) = \begin{cases} 0 \text{ if } x \in \| \wedge\Gamma \|_{\mathscr{M}} \text{ and } x \notin \| \vee\Delta \|_{\mathscr{M}}, \\ 1 \text{ otherwise.} \end{cases}$$

Definitions

We say $\Gamma \to \Delta$ is *GO†-realisable* iff for some GO†-model $\mathscr{M} = \langle X, \perp, D\rangle$ and some $x \in X$, $\text{Val}_{\mathscr{M}}(\Gamma \to \Delta\ ;x) = 0$.

A sequent which is not GO†-realisable is said to be *GO†-valid*. Thus $\Gamma \to \Delta$ is GO†-valid iff $\| \wedge\Gamma \|_{\mathscr{M}} \subseteq \| \vee\Delta \|_{\mathscr{M}}$ for all $\mathscr{M}$.

The following will be useful in discussing the semantics of GO†. It is easily seen that for any closed sets $Y_1, Y_2 \subseteq X$

(i) $Y_1 \subseteq Y_2$ implies $Y_2{}^* \subseteq Y_1{}^*$;
(ii) $Y_1 \cup Y_2 \subseteq (Y_1{}^* \cap Y_2{}^*)^*$.

Hence

(i) $\| \alpha \|_{\mathscr{M}} \subseteq \| \beta \|_{\mathscr{M}}$ implies $\| \beta \|_{\mathscr{M}}{}^* \subseteq \| \alpha \|_{\mathscr{M}}{}^*$;
(ii) $\| \wedge(\Gamma_1, \Gamma_2) \|_{\mathscr{M}} = \| \Gamma_1 \|_{\mathscr{M}} \cap \| \Gamma_2 \|_{\mathscr{M}}$;
(iii) $\| \vee\Delta_1 \|_{\mathscr{M}} \cup \| \vee\Delta_2 \|_{\mathscr{M}} \subseteq \| \vee(\Delta_1, \Delta_2) \|_{\mathscr{M}}$
(iv) $\| \vee\Delta_1 \|_{\mathscr{M}} \subseteq \| \vee\Delta_2 \|_{\mathscr{M}}$ implies
$\| \vee(\Delta_1, \Delta_3) \|_{\mathscr{M}} \subseteq \| \vee(\Delta_2, \Delta_3) \|_{\mathscr{M}}$

Theorem 4.1 (Soundness of GO†)
Every GO†-provable sequent is GO†-valid.

Proof

By induction on the construction of proofs in GO†.

Every axiom of GO† is GO†-valid. The rules of GO† preserve GO†-validity. Demonstration of this is routine, but we illustrate some of the more awkward cases. The other cases are left to the reader.

(cut-1)

Suppose $\| \wedge\Gamma \|_{\mathcal{M}} \subseteq \| \vee(\alpha, \Delta_1) \|_{\mathcal{M}}$ and $\| \alpha \|_{\mathcal{M}} \subseteq \| \vee\Delta_2 \|_{\mathcal{M}}$; then $\| \vee(\alpha, \Delta_1) \|_{\mathcal{M}} \subseteq \| \vee(\Delta_1, \Delta_2) \|_{\mathcal{M}}$ by the above, and so $\| \wedge\Gamma \|_{\mathcal{M}} \subseteq \| \vee(\Delta_1, \Delta_2) \|_{\mathcal{M}}$ as required.

(cut-2)

Suppose that $\| \wedge\Gamma_1 \|_{\mathcal{M}} \subseteq \| \alpha \|_{\mathcal{M}}$ and $\| \wedge(\Gamma_2, \alpha) \|_{\mathcal{M}} \subseteq \| \vee\Delta \|_{\mathcal{M}}$.

$$
\begin{aligned}
\text{Then } \| \wedge(\Gamma_1, \Gamma_2) \|_{\mathcal{M}} &= \| \wedge\Gamma_1 \|_{\mathcal{M}} \cap \| \wedge\Gamma_2 \|_{\mathcal{M}} \\
&\subseteq \| \alpha \|_{\mathcal{M}} \cap \| \wedge\Gamma_2 \|_{\mathcal{M}} = \| \wedge(\Gamma_2, \alpha) \|_{\mathcal{M}} \\
&\subseteq \| \vee\Delta \|_{\mathcal{M}} \text{ as required.}
\end{aligned}
$$

($\vee\rightarrow$)†

Suppose that $\| \alpha \|_{\mathcal{M}} \subseteq \| \vee\Delta \|_{\mathcal{M}}$, and $\| \beta \|_{\mathcal{M}} \subseteq \| \vee\Delta \|_{\mathcal{M}}$.
Then $\| \vee\Delta \|_{\mathcal{M}}^* \subseteq \| \alpha \|_{\mathcal{M}}^*$ and $\| \vee\Delta \|_{\mathcal{M}}^* \subseteq \| \beta \|_{\mathcal{M}}^*$
Therefore $\| \vee\Delta \|_{\mathcal{M}}^* \subseteq \| \alpha \|_{\mathcal{M}}^* \cap \| \beta \|_{\mathcal{M}}^*$, and
$(\| \alpha \|_{\mathcal{M}}^* \cap \| \beta \|_{\mathcal{M}}^*)^* \subseteq \| \vee\Delta \|_{\mathcal{M}}^{**} = \| \vee\Delta \|_{\mathcal{M}}$.
Therefore whenever both $\alpha \rightarrow \Delta$ and $\beta \rightarrow \Delta$ are GO†-valid, so is $\alpha \vee \beta \rightarrow \Delta$.
This completes the soundness proof for GO†.

GO†: Completeness

The following is similar to Nishimura's treatment for GO.

Definitions

A sequent $\Gamma \rightarrow \Delta$ is *consistent* iff $\nvdash_{GO\dagger} \Gamma \rightarrow \Delta$.
The set Γ is *consistent* iff for some α, $\nvdash_{GO\dagger} \Gamma \rightarrow \alpha$.
The set Γ is *complete* iff (a) Γ is consistent
and (b) $\Gamma = \{\gamma : \vdash_{GO\dagger} \Gamma \rightarrow \gamma\}$.

Lemma 4.2 [Lindenbaum's Lemma] Any consistent set Γ can be extended to some complete set Γ'.

Proof

Put $\Gamma' = \{\gamma : \vdash_{GO\dagger} \Gamma \to \gamma\}$.

(i) Suppose $\vdash_{GO\dagger} \Gamma' \to \beta$. Then for some finite subset $\{\gamma_1, \gamma_2, .. \gamma_m\}$ of Γ' $\vdash_{GO\dagger} \gamma_1, \gamma_2, \ldots, \gamma_m \to \beta$ and $\vdash_{GO\dagger} \Gamma \to \gamma_i$ for $i = 1, 2, \ldots m$.
By m applications of (cut-2) $\vdash_{GO\dagger} \Gamma \to \beta$. Hence $\beta \in \Gamma'$.

(ii) Since Γ is consistent, then for some α, $\nvdash_{GO\dagger} \Gamma \to \alpha$.
Therefore $\alpha \notin \Gamma'$, so by (a) $\nvdash_{GO\dagger} \Gamma' \to \alpha$ and Γ' is consistent.

Hence Γ' is complete.

Lemma 4.3

(a) If $\Gamma \to \alpha$ is consistent, then

(i) $\gamma \in \Gamma$ implies $\neg\alpha \to \neg\gamma$ is consistent,
(ii) $\neg\delta \in \Gamma$ implies $\neg\alpha \to \delta$ is consistent;

(b) if $\Gamma \to \neg\alpha$ is consistent, then

(i) $\gamma \in \Gamma$ implies $\alpha \to \neg\gamma$ is consistent,
(ii) $\neg\delta \in \Gamma$ implies $\alpha \to \delta$ is consistent.

Proofs

We do (a)(i) and leave the rest as an exercise for the reader.
Since $\gamma \in \Gamma$, if $\vdash_{GO\dagger} \neg\alpha \to \neg\gamma$ then $\Gamma \to \alpha$ can be proved as follows.

$$\dfrac{\Gamma\to\gamma \qquad \dfrac{\dfrac{\dfrac{\gamma\to\gamma}{\gamma\to\neg\neg\gamma}(\to\neg\neg) \quad \dfrac{\neg\alpha\to\neg\gamma}{\neg\neg\gamma\to\neg\neg\alpha}(\to\neg)\dagger}{\gamma\to\neg\neg\alpha}(\text{cut-1}) \qquad \dfrac{\alpha\to\alpha}{\neg\neg\alpha\to\alpha}(\neg\neg\cdot}{\gamma\to\alpha}(\text{cut-1})}{\Gamma\to\alpha}(\text{cut-1})$$

Hence $\neg\alpha \to \neg\gamma$ is consistent.

Definition

The GO†-canonical model $\mathscr{M}^\dagger$ is defined as follows.

$\mathscr{M}^\dagger = \langle X, \perp, D \rangle$ where

(1) $X = \{\Gamma : \Gamma \text{ is complete}\}$

(2) $\Gamma \perp \Gamma'$ iff for some α, either
(a) $\alpha \in \Gamma$ and $\neg\alpha \in \Gamma'$,
or (b) $\alpha \in \Gamma'$ and $\neg\alpha \in \Gamma$;

(3) for propositional variables p
$D(p) = \{\Gamma : p \in \Gamma\}$.

Lemma 4.4 $\mathscr{M}^\dagger$ is a GO†-model.

Proof

(a) $\perp$ is an orthogonality relation.

Symmetry is obvious from the definition. Irreflexivity follows as below.

If $\Gamma \perp \Gamma$ then for some α, $\alpha \in \Gamma$ and $\neg\alpha \in \Gamma$. Therefore $\vdash_{GO\dagger} \Gamma \to \beta$ for all β and Γ is not consistent, as follows:

$$\dfrac{\dfrac{\alpha \to \alpha}{\neg\alpha, \alpha \to}\ (\neg\to)\dagger}{\Gamma \to \beta}\ (\text{ext})$$

(b) D(p) is closed, i.e. $D(p) = D(p)^{**}$

Since $Y \subseteq Y^{**}$ for all Y, it is sufficient to show that $D(p)^{**} \subseteq D(p)$. Suppose therefore that some $\Gamma \notin D(p)$. $p \notin \Gamma$ and $\Gamma \to p$ is consistent.

Let $\Sigma = \{\alpha : \vdash \neg p \to \alpha\}$.

Then $\Sigma \perp D(p)$. That is, $\Sigma \in D(p)^*$.

But $\Gamma \not\perp \Sigma$ since (i) if $\gamma \in \Gamma$, $\neg p \to \neg\gamma$ is consistent by lemma 4.3 (a) (i) and therefore $\neg\gamma \notin \Sigma$;

(ii) if $\neg\delta \in \Gamma$, $\neg p \to \delta$ is consistent by lemma 4.3 (a)(ii) and therefore $\delta \notin \Sigma$.

Hence $D(p)^{**} \subseteq D(p)$, and therefore D(p) is closed.

Before we prove the Main Theorem for $\mathscr{M}^\dagger$ we require the following lemma.

Lemma 4.5 (a) (i) $\vdash_{GO\dagger}\alpha\vee\beta\to\neg(\neg\alpha\wedge\neg\beta)$
(ii) $\vdash_{GO\dagger}\neg(\neg\alpha\wedge\neg\beta)\to\alpha\vee\beta$
(b) The following are derived rules of GO†

$$\frac{\Gamma\to\alpha\wedge\beta}{\Gamma\to\alpha}\qquad\qquad\frac{\Gamma\to\alpha\wedge\beta}{\Gamma\to\beta}$$

Proofs

(a)

$$\dfrac{\dfrac{\alpha\to\alpha}{\alpha\to\neg\neg\alpha}(\to\neg\neg)\qquad\dfrac{\dfrac{\neg\alpha\to\neg\alpha}{\neg\alpha\wedge\neg\beta\to\neg\alpha}(\wedge\to)}{\neg\neg\alpha\to\neg(\neg\alpha\wedge\neg\beta)}(\to\neg)^\dagger}{\alpha\to\neg(\neg\alpha\wedge\neg\beta)}(\text{cut-1})$$

$$\dfrac{\alpha\to\neg(\neg\alpha\wedge\neg\beta)\qquad\dfrac{}{\beta\to\neg(\neg\alpha\wedge\neg\beta)}}{\alpha\vee\beta\to\neg(\neg\alpha\wedge\neg\beta)}(\vee\to)^\dagger$$

We leave (ii) as an easy exercise for the reader.
(b) Use $(\wedge\to)$ and (cut-1).

Theorem 4.6 [Main Theorem for $\mathscr{M}^\dagger = <X, \perp, D>$]
For any $\Gamma\in X$, $\gamma\in\Gamma$ iff $\Gamma\in\|\gamma\|\mathscr{M}^\dagger$.

Proof (Throughout this proof we write $\|\alpha\|$ for $\|\alpha\|\mathscr{M}^\dagger$.)
By induction on the length of the wff γ.

(1) $\underline{\gamma = p\text{, some propositional variable}}$

$p\in\Gamma$ iff $\Gamma\in D(p)$, by the definition of $D(p)$;

(2) $\underline{\gamma = \alpha\wedge\beta}$

$\alpha\wedge\beta\in\Gamma$ iff $\vdash\Gamma\to\alpha\wedge\beta$, since Γ is complete,
iff $\vdash\Gamma\to\alpha$ and $\vdash\Gamma\to\beta$, by lemma 4.5 (b)(i) and $(\to\wedge)\dagger$
iff $\alpha\in\Gamma$ and $\beta\in\Gamma$, since Γ is complete
iff $\Gamma\in\|\alpha\|$ and $\Gamma\in\|\beta\|$, by the induction hypothesis,
iff $\Gamma\in\|\alpha\wedge\beta\|$, by the definition of $\|.\|$;

(3) $\underline{\gamma = \neg\alpha}$

There are two cases to consider:

(a) $\underline{\text{suppose } \neg\alpha \in \Gamma}$

Then $\Gamma \perp \Gamma'$ for every Γ' such that $\alpha \in \Gamma'$.

i.e. $\Gamma \perp \{\Gamma' : \alpha \in \Gamma\} = \|\alpha\|$ by the induction hypothesis.

Hence $\Gamma \in \|\alpha\|^*$.

(b) $\underline{\text{suppose } \neg\alpha \notin \Gamma}$

Then $\Gamma \to \neg\alpha$ is consistent.

Let $\Delta = \{\delta : \vdash_{GO\dagger} \alpha \to \delta\} \in X$. Clearly $\alpha \in \Delta$, so $\Delta \in \|\alpha\|$ by the induction hypothesis. But $\Gamma \not\perp \Delta$, since if $\gamma \in \Gamma$, then $\alpha \to \neg\gamma$ is consistent by lemma 4.3 (b)(i) and therefore $\neg\gamma \notin \Delta$, and if $\neg\gamma \in \Gamma$, then $\alpha \to \gamma$ is consistent by lemma 4.3 (b)(ii), and therefore $\gamma \notin \Delta$. So $\Gamma \notin \|\alpha\|^*$, therefore $\Gamma \notin \|\neg\alpha\|$;

(4) $\underline{\gamma = \alpha \vee \beta}$

$\alpha \vee \beta \in \Gamma$ iff $\vdash_{GO\dagger} \Gamma \to \alpha \vee \beta$, since Γ is complete,
iff $\vdash_{GO\dagger} \Gamma \to \neg(\neg\alpha \wedge \neg\beta)$ by lemma 4.5 (a)(i) and (ii)
iff $\neg(\neg\alpha \wedge \neg\beta) \in \Gamma$,
iff $\Gamma \in \|\neg(\neg\alpha \wedge \neg\beta)\|$, from steps (2) and (3) above,
iff $\Gamma \in \|\alpha \vee \beta\|$.

Theorem 4.7 (Completeness of GO†)

If $\Gamma \to \Delta$ is GO†-valid, then $\vdash_{GO\dagger} \Gamma \to \Delta$

Proof

Suppose $\nvdash_{GO\dagger} \Gamma \to \Delta$, i.e. that $\Gamma \to \Delta$ is consistent. Then we shall show that $\|\wedge\Gamma\|_{\mathcal{M}\dagger} \not\subseteq \|\vee\Delta\|_{\mathcal{M}\dagger}$.

Let $\hat{\Gamma} = \{\alpha : \vdash_{GO\dagger} \Gamma \to \alpha\}$. Then $\hat{\Gamma}$ is complete, and it is easily checked that $\hat{\Gamma} \to \Delta$ is consistent. Then it is easy to show that $\neg\Delta \to \neg\hat{\Gamma}$ is consistent.

Let $\Sigma = \{\alpha : \vdash_{GO\dagger} \neg\Delta \to \alpha\}$; then Σ is complete.

We claim that

(1) $\hat{\Gamma} \in \|\wedge\Gamma\|_{\mathcal{M}\dagger}$

(2) $\Sigma \in \|\wedge\neg\Delta\|_{\mathcal{M}\dagger}$

(3) $\hat{\Gamma} \not\perp \Sigma$

(1) and (2) are immediate, using Theorem 4.6 and the fact that $\Gamma \subseteq \hat{\Gamma}$ and $\neg\Delta \subseteq \Sigma$. Consider claim (3). Suppose that $\gamma \in \hat{\Gamma}$ and $\neg\gamma \in \Sigma$. Then there is a finite $\Delta' \subseteq \Delta$ with $\vdash_{GO\dagger} \neg\Delta' \rightarrow \neg\gamma$.
Hence $\vdash_{GO\dagger} \neg\neg\gamma \rightarrow \neg\neg\Delta'$, so $\vdash_{GO\dagger} \gamma \rightarrow \Delta'$, which is a contradiction. Alternatively, suppose that $\neg\gamma \in \hat{\Gamma}$ and $\gamma \in \Sigma$. Then there is a finite $\Delta' \subseteq \Delta$ with $\vdash_{GO\dagger} \neg\Delta' \rightarrow \gamma$. So $\vdash_{GO\dagger} \neg\gamma \rightarrow \neg\neg\Delta'$, hence $\vdash_{GO\dagger} \neg\gamma \rightarrow \Delta'$, which again contradicts the consistency of $\hat{\Gamma} \rightarrow \Delta$.

Thus claim (3) is verified.

By (2) and (3) together, $\hat{\Gamma} \notin \| \wedge \neg\Delta \|_{\mathcal{M}^*\dagger}$, i.e. $\hat{\Gamma} \notin \| \vee\Delta \|_{\mathcal{M}\dagger}$ Hence $\hat{\Gamma} \varepsilon \| \wedge\Gamma \|_{\mathcal{M}\dagger} \setminus \| \vee\Delta \|_{\mathcal{M}\dagger}$ and we have shown that $\Gamma \rightarrow \Delta$ is not valid.

The following corollary is immediate.

Theorem 4.8 (Compactness of GO†)

For arbitrary Γ and Δ, $\Gamma \rightarrow \Delta$ is GO†-realisable iff every finite subsequent of $\Gamma \rightarrow \Delta$ is GO†-realisable.

Remark If $\Gamma \rightarrow \Delta$ is a *finite* GO† consistent sequent, the completeness proof above can be modified to provide a finite GO† model that realises $\Gamma \rightarrow \Delta$. This is done as in [Nishimura (1980) § 4] by restriction to a suitable finite admissible set of formulae Ω containing Γ and Δ. In our case Ω is *admissible* means

(a) Ω is closed under subformulae;
(b) if $p \in \Omega$ then $\neg p \in \Omega$;
(c) if $\alpha \vee \beta \in \Omega$ then $\neg(\neg\alpha \wedge \neg\beta) \in \Omega$.

Thus

Theorem 4.9 GO† has the finite model property.

§ 5 *GO†M: Orthomodular Logic*

We obtain GO†M, the orthomodular extension of GO†, in a manner similar to Nishimura, by adding to GO† the rule (OM)

$$\frac{\neg\beta \rightarrow \neg\alpha \qquad \neg\alpha, \beta \rightarrow}{\neg\alpha \rightarrow \neg\beta} \text{(OM)}$$

The resulting calculus GO†M = GO†+ (OM) is equivalent to GOM for normal sequents in virtue of the following theorem.

Theorem 5.1 Let R′ be a rule whose upper and lower sequents are all normal. Let SC_1 and SC_2 be two sequent calculi which are equivalent for normal sequents. The $SC_1 + R'$ and $SC_2 + R'$ are equivalent for normal sequents.

Proof

By induction on the length of proofs

GO†M: Soundness and Completeness

We proceed by analogy with GOM. Thus
a *GO†M-frame* is a triple $<X, \perp, \Psi>$ where

(1) $<X, \perp>$ is a GO†-frame

(2) Ψ is a non-empty set of $\perp$-closed subsets of X such that
  (a) Ψ is closed under set-theoretic intersection and the operation $*$

  (b) for any $Y, Z \in \Psi$, $Y \subseteq Z$ and $Y^* \cap Z = \emptyset$ implies $Y = Z$

A *GO†-model* is a 4-tuple $<X, \perp, \Psi, D>$ such that

(1) $<X, \perp, \Psi>$ is a GO†M-frame;
(2) D is a function assigning to each propositional variable p a closed subset in Ψ.

The notations $\|\alpha\|_{\mathcal{M}}$, $Val_{\mathcal{M}}(. \; ; x)$ are defined for a GO†M-model as before. The expressions *GO†M-realisable* and *GO†M-valid* are defined similarly.

Theorem 5.2 (Soundness of GO†M)
Every GO†M-provable sequent is GO†-valid.

Proof

By induction on the construction of proofs in GO†M: as in the GO† case but with the following addition. (Let $\mathscr{M}$ be a GO†M-model, and write $\|\alpha\|$ for $\|\alpha\|_{\mathscr{M}}$.)

(OM)

Suppose $\|\neg\beta\| \subseteq \|\neg\alpha\|$, so that $\|\alpha\| \subseteq \|\beta\|$.
Suppose also that $\|\neg\alpha\| \cap \|\beta\| = \emptyset$, so that $\|\alpha\|^* \cap \|\beta\| = \emptyset$.
Then $\|\alpha\| = \|\beta\|$, so that $\|\beta\| \subseteq \|\alpha\|$ and hence $\|\neg\alpha\| \subseteq \|\neg\beta\|$.

We now sketch the proof of the Completeness Theorem for GO†M. We say a sequent $\Gamma \to \Delta$ is *GO†M-consistent* iff $\nvdash_{GO\dagger M} \Gamma \to \Delta$. Similarly, a set Γ is *GO†M-consistent* iff for some α, $\nvdash_{GO\dagger M} \Gamma \to \alpha$.

The set Γ is *GO†M-complete* iff (a) Γ is GO†M-consistent and (b) $\Gamma = \{\gamma : \vdash_{GO\dagger M} \Gamma \to \gamma\}$.

Lindenbaum's Lemma for GO†M can then be proved exactly as for GO† in lemma 4.2.

Definition The GO†M-canonical model $\mathscr{M}^\ddagger_0$ is defined as follows.

$\mathscr{M}^\ddagger_0 = \langle X, \perp, \Psi, D \rangle$ where

(1) $X = \{\Gamma : \Gamma \text{ is GO†M-complete}\}$;

(2) $\Gamma \perp \Gamma'$ iff for some α, either
(a) $\alpha \in \Gamma$ and $\neg\alpha \in \Gamma'$,
or (b) $\neg\alpha \in \Gamma$ and $\alpha \in \Gamma'$;

(3) $\Psi = \{\|\alpha\| : \alpha \text{ is a formula}\}$, where $\|\alpha\|$ is defined using (4);

(4) for propositional variables p,
$D(p) = \{\Gamma : p \in \Gamma\}$.

Theorem 5.3 $\mathscr{M}^\ddagger_0$ is a GO†M-model and realises every GO†M-consistent sequent.

Proof

We sketch the proof. First show that $\langle X, \perp, D \rangle$ is a GO†-model exactly as in lemma 4.4 for $\mathscr{M}^\dagger$, and then establish that

(5.4) $\quad \Gamma \,\varepsilon\, \|\alpha\| \;$ iff $\; \alpha \,\varepsilon\, \Gamma$

exactly as in Theorem 4.6 (where we write $\|\alpha\|$ for $\|\alpha\|_{\mathscr{M}\ddagger}$).

Now, exactly as in the proof of completeness for GO†, we show that if $\Gamma \to \Delta$ is GO†M-consistent, then

(5.5) $\quad \| \wedge\Gamma \| \not\subseteq \| \vee\Delta \|.$

To complete the proof we have to show that $\mathscr{M}\ddagger_0$ is a GO†M-model.

Suppose that $\|\alpha\| \subseteq \|\beta\|$, and $\|\alpha\|^* \cap \|\beta\| = \emptyset$.

Then by (5.5) $\vdash_{\mathrm{GO\dagger M}} \alpha \to \beta$ and $\vdash_{\mathrm{GO\dagger M}} \neg\alpha, \beta \to$, and using (OM) we obtain $\vdash_{\mathrm{GO\dagger M}} \neg\alpha \to \neg\beta$ and thus $\vdash_{\mathrm{GO\dagger M}} \beta \to \alpha$.

Using (5.4) this means that $\|\beta\| \subseteq \|\alpha\|$. We then have the following theorem immediately.

Theorem 5.6 (Completeness of GO†M)
If $\Gamma \to \Delta$ is GO†M-valid, then $\vdash_{\mathrm{GO\dagger M}} \Gamma \to \Delta$.

§ 6 *Concluding Remarks*

As noted in § 1, for finite Γ and Δ, the intuitive interpretation of the sequent $\Gamma \to \Delta$ in classical logic may be rendered equivalently as

(i) whenever all the formulae in Γ are true, at least one of the formulae in Δ is true;
(ii) whenever a conjunction of all the formulae in Γ is true, any disjunction of all the formulae in Δ is true.

The interpretation of finite GO(M) sequents follows (i), that of finite GO†(M) sequents (ii). Since (i) and (ii) are not equivalent for both GO(M) and GO†(M) there is reason to think that the *meanings* of sequents in these calculi differ from their meanings in classical logic. But there seems to be no conclusive reason for regarding either (i) or (ii) as the more fundamental intuitive interpretation.

The primitive rules for the $\wedge$ - $\neg$ fragment of GO† naturally differ from those of GO. But the most interesting difference between the systems occurs in their structural rules. GO(M) has the *full* cut rule among its primitives. Owing to the normal form theorem GO(M) requires only the restricted cut-rules of GO†(M). But clearly GO(M) tolerates full cut.

GO†(M) does not tolerate full cut. GO†(M) + (cut) = classical logic, as in shown in the lemma below. There is in the literature the suggestion ([2]) that one can expect only elastic constraints on what is to count as a logic but that toleration of full cut is one constraint required to reflect the necessary transitivity of implication. But GO†(M) illustrates the fact that one can reflect the transitivity of implication with weaker rules. If GO(M) are to count as logics, there seems to be no good reason for denying that status to the extended calculi GO†(M).

Lemma 6.1 GO†(M) + (cut) = classical logic.

Proof

It is sufficient to obtain the result for finite sequents. We are required to show that $(\neg\rightarrow)$, $(\rightarrow\neg)_c$ and $(\rightarrow\wedge)$ are derived rules of GO† + (cut).

<u>$(\neg\rightarrow)$</u>

$$\dfrac{\Gamma\rightarrow\Delta,\alpha \qquad \dfrac{\alpha\rightarrow\alpha}{\alpha,\neg\alpha\rightarrow}\,(\neg\rightarrow)\dagger}{\Gamma,\neg\alpha\rightarrow\Delta}\,(\text{cut})$$

<u>$(\rightarrow\neg)_c$</u>

$$\dfrac{\Gamma,\alpha\rightarrow\Delta \qquad \begin{array}{c}\vdots\\ \rightarrow\alpha,\neg\alpha\end{array}}{\Gamma\rightarrow\Delta,\neg\alpha}\,(\text{cut})$$

<u>$(\rightarrow\wedge)$</u>

([2]) Cf. Hacking (1979) § XIV.

for finite Δ

$$\dfrac{\Gamma \rightarrow \Delta, \alpha}{\quad} \qquad\qquad \dfrac{\Gamma \rightarrow \Delta, \beta}{\quad}$$

$$\vdots \qquad\qquad \vdots \qquad \text{finite number of applications of } (\neg\rightarrow)$$

$$\dfrac{\dfrac{\Gamma, \neg\Delta \rightarrow \alpha \qquad \Gamma, \neg\Delta \rightarrow \beta}{\Gamma, \neg\Delta \rightarrow \alpha \wedge \beta}\,(\rightarrow \wedge)\dagger}{\quad}$$

$$\vdots \qquad \text{finite number of applications of } (\rightarrow \neg)_c$$

$$\dfrac{\Gamma \rightarrow \neg\neg\Delta, \alpha \wedge \beta}{\quad}$$

$$\vdots$$

$$\Gamma \rightarrow \Delta, \alpha \wedge \beta \quad .$$

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