BOLL-REINHART MODAL LOGIC

Jean PORTE

1. In 1948, Marcel Boll published a new idea for the foundation of a «modal logic», considering it to be a particular aspect of the first-order predicate calculus. See Boll[1], also Boll and Reinhart[2]. Boll credited Jacques Reinhart (unpublished papers of 1944) for this idea.

Boll's «logic» was not presented as a formal system – and indeed the very notion of a formal system seems to have been completely alien to Boll's thought. But it is possible to extract from Boll's book an idea leading to the construction of a modal propositional logic as a translation of a part of the first-order predicate calculus. This is the aim of the present paper(1).

The «new» modal system, provisionally called BR, will be shown to be identical with the system E5 of Lemmon[10].

2. *The basic idea* – It seems to have been an attempt to use the notion of «scientific certainty» (but see below, section 6). In the Predicate Calculus (PdC), let p be any unary predicate, ω be a particular unary predicate, j be an individual variable, and a an individual constant (this departure from more usual notations is motivated by the need of simplifying the translations defined in next sections), then a proposition $p(a)$ will be «certain» (or «necessary») relative to ω if the proposition «a is ω and all ω are p» is true. We are thus led to define the necessity of $p(a)$ by

$$\mathrm{L}(p(a)) \quad \text{for} \quad \omega(a) \wedge \forall j(\omega(j) \rightarrow p(j)) \tag{1}$$

– and possibility will be defined as usual by means of necessity and negation.

That may be illustrated by describing the behavior of an astronomer who has just discovered a comet: He considers «necessary» that the comet obeys the law of gravitation, while the various constants which describe its orbit can «possibly» have such or such values, to be determined from observations. This does not mean that astronomers think the law of gravitation is known a priori, but only that they do not intend to verify it by observations of a newly discovered comet.

In that illustrative example, $\omega(a)$ means that a (the comet) obeys the law of gravitation. In other similar examples, $\omega(a)$ will mean that individual a obeys the laws of science – the science of nature rather than logic.

Now let us recall the well-known relationship which exists between S5 and a part of PdC: If we start from PdC-formulas which contain only unary predicates and one variable (say: j), and transform them by suppressing all the occurrences of the variable (together whith the parentheses which enclosed it), and putting L (respectively M) instead of $\forall$ (respect. $\exists$), we get formulas of a propositional modal logic, where (unary) predicates have become propositional variables – and the formulas which were valid in PdC become exactly the S5-theses. See Wajsberg[14], Parry[12], Carnap[4], Feys and Dopp[5] (pp. 31-32 and 73).

After such a transformation, modalities are attached to predicates rather than to propositions: A property is «necessary» when it hold, of all objects – and this idea is in agreement with modern semantics for S5, a formulation of which consists of saying that Lp is true if p is true in every alternative «possible world».

Let us try to apply such a process to the basic idea of Boll's modalities. We will have first to forget the constants, since we are not interested in any particular interpretation of PdC; then a will be replaced by a variable, which may well be j (for, in PdC, if we have only unary predicates, restricting the set of variables to one of them is not an essential change). Then the variable will be suppressed, unary predicates being turned into propositional variables, and necessity (which was an abbreviation in (1)) will be now defined by a translation, the formula

$$\omega(j) \wedge \forall j(\omega(j) \rightarrow p(j))$$

being translated into Lp. Thus we obtain a modal propositional calculus as a translation of a part of PdC, the theses being the translations of valid formulas.

Now, that translation is a little more complex than the one which leads to S5. A part of PdC is translated; but exactly which part? This will be seen more easily if we examine the converse translation. The change of point of view is mathematically indifferent (since the translation is one-one), and leads to simpler results (since every modal formula has a converse translation into PdC). System BR will be defined in next section by means of that converse translation – which will be called simply «translation».

3. *The system BR* – The translation which defines it, T′, will be described as a modification of the translation, T, which converts S5 into a part of PdC.

Modal formulas will be considered to be constructed in the usual way, from a denumerable list of propositional variables, $q_1, q_2, \ldots,$ with connectives $\neg$ (negation), $\rightarrow$ (implication), and L (a unary connective called «necessity»). Other connectives are defined as usual in classical Propositional Calculus (PC). Possibility, M, is defined as $\neg \mathrm{L} \neg$.

Translation T is defined inductively by

$$\mathrm{T}(p) = p(j) \tag{2}$$

for every propositional variable p

$$\mathrm{T}(\neg x) = \neg \mathrm{T}(x) \tag{3}$$

$$\mathrm{T}(x \rightarrow y) = \mathrm{T}(x) \rightarrow \mathrm{T}(y) \tag{4}$$

$$\mathrm{T}(\mathrm{L}x) = \forall j \mathrm{T}(x) \tag{5}$$

for all formulas x, y.

As has been recalled above, we have

$$\vdash_{\mathrm{S5}} x \quad \text{iff} \quad \vdash_{\mathrm{PdC}} \mathrm{T}(x) \tag{6}$$

Translation T′ is defined inductively by

$$\mathrm{T}'(p) = p(j) \tag{7}$$

for every propositional variable

$$\mathrm{T}'(\neg x) = \neg \mathrm{T}'(x) \tag{8}$$

$$\mathrm{T}'(x \rightarrow y) = \mathrm{T}'(x) \rightarrow \mathrm{T}'(y) \tag{9}$$

$$\mathrm{T}'(\mathrm{L}x) = \omega(j) \wedge \forall j(\omega(j) \rightarrow \mathrm{T}'(x)) \tag{10}$$

for all formulas x, y, ω being a fixed unary predicate.

System BR is defined by fixing its set of theses:

$$\vdash_{\mathrm{BR}} x \quad \text{iff} \quad \vdash_{\mathrm{PdC}} \mathrm{T}'(x) \tag{11}$$

By elementary calculations in PdC, we find

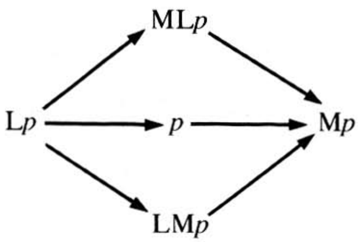
$$T'(Lp) = \omega(j) \wedge \forall j(\omega(j) \rightarrow p(j)) \tag{12}$$
$$\vdash T'(Mp) \leftrightarrow (\omega(j) \rightarrow \exists j(\omega(j) \rightarrow p(j))) \tag{13}$$
$$\vdash T'(LMp) \leftrightarrow (\omega(j) \wedge \exists j(\omega(j) \rightarrow p(j))) \tag{14}$$
$$\vdash T'(MLp) \leftrightarrow (\omega(j) \rightarrow \forall j(\omega(j) \rightarrow p(j))) \tag{15}$$

It follows at once

$$\nvdash_{BR} Lx \tag{16}$$

for any formula x - a result in agreement with the properties of the epistemic E-systems of Lemmon (see [9] and better [10]), as well as with the interpretation of modalities suggested above (section 2 - see also section 6).

It is easy to prove in a similar manner that there is no other positive modality, and that the five positive modalities are distinct and can be placed in the following diagram, which shows all the implications which hold between them:



The implications are familiar, but the fact that the diagram admits neither reductions nor complications separates BR from S4, S5, and the intermediate systems (see for instance Hughes and Cresswell[7], chapter 14).

All these last results were, more or less explicitly, in Boll's book.

4. *BR, Ł, and S5*. – If we suppress the quantifiers, (12) has the form

$$\tau(Lp) = \Omega \wedge p \tag{17}$$

which defines necessity in the «Ω-system» (see Porte[13]), which is another formulation of the Ł-system (see Łukasiewicz[11], or Smiley[14]). On the other hand, if we suppress ω, T′ reduces to T, and BR collapses into S5.

It is possible to put those remarks into more precise form.

A formula of PdC which is valid in every universe is valid as well in a universe of only one individual. But, if there is only one individual, $\forall j p(j)$, $\exists j p(j)$, and $p(j)$ are true or false together; in such a universe, every PdC-formule is equivalent to a quantifier-free formula. Let us call BR_1 the set of modal formulas whose translations are valid in a universe of one individual. We have

$$BR \subset BR_1 \tag{18}$$

But for BR_1-formulas, T′ reduces to

$$\vdash T'(Lp) \leftrightarrow (\omega(j) \wedge (\omega(j) \rightarrow p(j))) \tag{19}$$

whence

$$\vdash T'(Lp) \leftrightarrow (\omega(j) \wedge p(j)) \tag{20}$$

In the translations of modal formulas, j plays no role, while $\omega(j)$ plays the same role as Ω in (17). It follows (identifying systems with their set of theses)

$$BR_1 = Ł \tag{21}$$

whence, by (18)

$$BR \subset Ł \tag{22}$$

On the other hand, every formula valid in every universe is valid as well in a universe in which every individual possesses property ω – whence it follows that the free formula $\omega(j)$ is true. Let us call BR_2 the set of modal formulas whose translations are valid in such a universe. We have

$$BR \subset BR_2 \tag{23}$$

But when $\omega(j)$ is true, T′ reduces to

$$\vdash T'(Lp) \leftrightarrow \forall j p(j)$$

i.e. T′ reduces to T, and

$$BR_2 = S5 \tag{24}$$

whence

$$BR \subset S5 \tag{25}$$

From (22) and (25), we get

$$\mathrm{BR} \subset \text{Ł} \cap \mathrm{S5} \tag{26}$$

It is possible to prove directly that

$$\mathrm{BR} = \text{Ł} \cap \mathrm{S5} \tag{27}$$

But it will be easier to prove it by using known properties of Lemmon's system E5.

5. *Proof that BR = E5* – Let us consider the following axiomatization of E5 (see Lemmon[10], p. 214):

There are five axiom schemas and two rules:

A1 – t if t is a substitution instance of a tautology
A2 – $\mathrm{L}(x \to y) \to (\mathrm{L}x \to \mathrm{L}y)$
A3 – $\mathrm{L}x \to x$
A4 – $\mathrm{L}x \to \mathrm{LL}x$
A5 – $\mathrm{LML}x \to \mathrm{L}x$
R1 – $x,\ x \to y \ / \ y$
R2 – $x \to y \ / \ \mathrm{L}x \to \mathrm{L}y$
(for all formulas x, y)

It is easy to prove, by elementary calculations in PdC, that A1-A5 hold (are schemas of theses) and that R1 and R2 hold (are admissible rules) in BR. Indeed those results are again, more or less explicitely, in Boll's book. It follows

$$\mathrm{E5} \subset \mathrm{BR} \tag{28}$$

Now, if, following Kripke[8], we call «Trivial» the system obtained by adding the axiom schema

$$x \to \mathrm{L}x \tag{29}$$

to S5 (Lemmon [10] called it «PC»), and we call «Falsum» the system obtained by adding the axiom schema

$$\neg \mathrm{L}x \tag{30}$$

to S5 (Lemmon [10] called it «E»), and if we call «Ł» Łukasiewicz Ł-system without functorial variable (Lemmon [10] called it «L»), we have

$$\text{E5} = \text{Falsum} \cap \text{S5} \tag{31}$$

after Lemmon [10] (Theorem 64), and

$$\text{Ł} = \text{Falsum} \cap \text{Trivial} \tag{32}$$

after Kripke[8], p. 210 – whence

$$\text{E5} = \text{Ł} \cap \text{S5} \tag{33}$$

From (26), (28) and (33), we get

$$\text{E5} \subset \text{BR} \subset \text{E5} \tag{34}$$

Q.E.D.

An alternative proof would have been to deduce

$$\mathrm{L}(q \rightarrow q) \rightarrow \mathrm{L}(\mathrm{M}p \rightarrow \mathrm{LM}p)$$

(where p and q are two different propositional variables) from A1-A5, R1-R2, then to remark that E5 possesses Halldén's property (see [6], and apply Theorem 2 of Kripke[8], using a Carnap-style axiomatization of S5 (i.e. an axiomatization with R1 as sole rule, as in [3]).

6. *Further remarks* – As for axiomatization of E5, it may be recalled that two tentative definitions of E5 had encountered unhappy fate: The definition given in Lemmon[9] collapses into S5, while the «improved» definition of Kripke[8] (p. 209) is too weak, being unable to yield A4 as a set of theses (that fact has been signalled to me by Kripke in a letter of January 14, 1981 – it can also be proved by a four-elements matrix), an occurrence of L having been inadvertently omitted. The really improved definition of Kripke had indeed been published in Yonemitsu[16].

As for motivation, it seems that Boll looked for a definition of «scientific certainty», an evidence being his use of the word «certain» instead of the more usual «nécessaire». He wrote (p. 314) «Une modalité est une interdépendance entre un fait et un certain jeu de postulats» (a modality is an interdependence between a proposition and a set of postulates). The conjunction of such a set of postulates is represented by a predicate, e (called ω in section 2 above). Necessity is always relative to a certain set of postulates in a science other than logic itself ([1], p. 313).

But is $x \vee \neg x$ scientifically certain? It is! whatever may be the postulates of the non-logical science in question. And indeed Boll seemed to think that $L(x \vee \neg x)$ ought to be deducible ([1], p. 320, footnote) – not seeing that it is not deducible in his own system.

Lemmon's motivation was similar, but more precise. In [9], p. 183, he wrote that L means «it is scientifically but not logically necessary that». From that point of view it is easier to understand that $L(p \vee \neg p)$ is not a thesis. But, as was remarked by Hughes and Cresswell ([7], p. 302, footnote) we have

$$\vdash Lq \rightarrow L(p \rightarrow p)$$

in all the E-systems, and that result can hardly be read: «If anything is scientifically but not logically certain, then $p \rightarrow p$ is scientifically but not logically certain».

In fact, the «not logically necessary» part of Lemmon's interpretation applies only to complete formulas – not to subformulas.

The interpretation of epistemic system E5 may become a little clearer by the predicate calculus translation of section 2. In the real world, scientific truths are true, i.e. $\omega(a)$ is true for every individual a, and E5 collapses into S5. The proper role of E5 is to try to put apart, as far as it is possible, the relationships holding between scientific laws which should hold in worlds where scientific laws would be different from the real ones.

Now, there is an obvious shortcoming in the predicate calculus interpretation: It uses only unary predicates – and it is not possible to reduce an arbitrary «set of postulates» to unary predicates. To do that would mean to return to the aristotelian world – where everything is decidable!

For instance, can we express by a formula such as $\omega(a)$ the fact that comet a obeys the law of gravitation? If we restrict the work of astronomers to the simple job of computing a provisional orbit, that expression is acceptable. But Newton's law itself is expressed by a system of second order differential equations, which cannot by any reasonable means be reduced to a unary predicate.

Indeed Boll himself seems to have perceived the need of going outside the domain of unary PdC: in [1], p. 314, he mentioned unpublished works of «J. Reinhart 1944» in this direction; but he did not pursue the idea; and anyway such a job might be exceedingly difficult.

Université des Sciences et de la Technologie
Houari Boumedienne
Algiers, Algeria

Jean PORTE

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REFERENCES

[1] BOLL, M., *Manuel de logique scientifique*, Dunod, Paris, 1948.

[2] BOLL, M., and J. REINHART, «Une interprétation des modalités aristotéliciennes», *Congrès International de Philosophie des Sciences, 1949*, Hermann, Paris, 1951, pp. 15-17.

[3] CARNAP, R., «Modalities and quantification», *The Journal of Symbolic Logic*, vo. 11 (1946), pp. 33-46.

[4] CARNAP, R., *Meaning and Necessity*, Chicago, The University Press, 1947.

[5] FEYS, R., and J. DOPP, *Modal Logics*, Nauwelaerts, Leuven, and Gauthier-Villars, Paris, 1965.

[6] HALLDÉN, S., «On the semantic non-completeness of certain Lewis calculi», *The Journal of Symbolic Logic*, vol. 16 (1951), pp. 127-129.

[7] HUGHES, G.E., and M.J. CRESSWELL, *An Introduction to Modal Logic*, Methuen, London, 1968.

[8] KRIPKE, S.A., «Semantical analysis of modal logic, II. Non-normal modal propositional calculi», *Symposium on the Theory of Models, Berkeley 1963*, North-Holland, Amsterdam, 1965, pp. 206-220.

[9] LEMMON, E.J., «New foundations for Lewis modal systems», *The Journal of Symbolic Logic*, vol. 22 (1957), pp. 176-186.

[10] LEMMON, E.J., «Algebraic semantics for modal logics II», *The Journal of Symbolic Logic*, vol. 31 (1966), pp. 191-218.

[11] ŁUKASIEWICZ, J., «A system of modal logic», *The Journal of Computing Systems*, vol. 1 (1953), pp. 111-149.

[12] PARRY, W.T., «Zum Lewisschen Aussagenkalkül», *Ergebnisse eines mathematisches Kolloquiums*, vol. 4 (1933), (für Jahr 1931-32), pp. 5-6.

[13] PORTE, J., «The Ω-system and the Ł-system of modal logic», *Notre Dame Journal of Formal Logic*, vol. 20 (1979), pp. 915-920.

[14] SMILEY, T., «On Łukasiewicz's Ł-modal system», *Notre Dame Journal of Formal Logic*, vol. 2 (1961), pp. 149-153.

[15] WAJSBERG, M., «Ein erweiterter Klassenkalkül», *Monatshefte für Mathematik und Physik*, vol. 40 (1933, pp. 113-126.

[16] YONEMITSU, N., Review of [9], *The Journal of Symbolic Logic*, vol. 23 (1958), pp. 346-347.