A NOTE ON THE BARCAN FORMULA AND SUBSTITUTIONAL QUANTIFICATION

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When Ruth Barcan Marcus pioneered quantified modal logic she chose Lewis' S2 as a modal basis and added various axioms, amongst which was $\Diamond\exists x\Phi x. \prec \exists x\Diamond\Phi x$ (Barcan, 1946). Subsequently dubbed the Barcan formula, this axiom was questioned by many writers. Prior, for example, suggested that one cannot correctly infer that there exists someone who will possibly reach the moon from the premiss that possibly someone will reach the moon (Prior, 1957, p. 26). Finally Kripke showed formally that the Barcan formula is valid only given the rather unlikely assumption that there are no worlds possible relative to any given world which contain objects not existing in the given world (Kripke, 1963).

Marcus' response to all this was to maintain that her axiom looks doubtful only if one insists on interpreting the existential quantifier *referentially*. She claims that if one interprets the quantifiers *substitutionally* – an interpretation which she has independent reasons for favouring – the Barcan formula is unproblematic. On the substitutional interpretation, she says, the Barcan formula is read as follows: if it is logically possible that a substitution instance of Φx is true, then it is true that a substitution instance of Φx is logically possible. She claims that this reading, although clumsy, is not paradoxical: it does not, she says, make «that mysterious move from possibility to actuality». (1)

In this paper it is established that Marcus' manœuver to save her axiom is unsuccessful. Even in an S5 setting the use of the substitutional interpretation of the quantifiers does not guarantee the validity of the Barcan formula. Various S5 strength semantics which use the substitutional interpretation can be constructed (2), and in some of these the Barcan formula is valid, whilst in others it is not. Moreover, the conditions which must be built into an S5 strength substitutional semantics in order to validate the Barcan formula are precisely analogous to those which must be built into the corresponding referential semantics in order to validate the formula, namely constancy of domain of quantification in all mutually possible worlds.

For the present purposes a *model* is defined as an ordered triple $<K, Q, v>$, where K is a set of «possible worlds», v is a function which assigns a truth value to each formula in each world, and Q is a function which, so to speak, tells you which objects exist in any given world. In standard referential semantics such a function would straightforwardly assign to each world a set of objects to be used as the domain of quantification in that world. ([3]) In the substitutional case Q assigns a set (possibly null) of *terms* to each member of K, which are viewed as the only terms of the language which denote in that world. Where $w \in K$, Q(w) is used as the substitution class in evaluating the truth of quantified formulae in w, thus tying quantification to existence. If Q is made to take the same value at all worlds in K, the following semantics validates the Barcan formula; otherwise the formula is invalid.

Given K and Q, v is defined recursively as follows. For all $w \in K$, all variables x of the language([4]), all formulae A, B, Cx: $v(A, w)$ is arbitrary if A is atomic; $v({\sim}A, w) = T$ iff $v(A, w) = F$; $v(A \supset B, w) = T$ iff either $v(A, w) = F$ or $v(B, w) = T$; $v(\Box A, w) = T$ iff $v(A, w') = T$ for every $w' \in K$; $v(\exists x Cx, w) = T$ iff $v(Ct, w) = T$ for some term t in Q(w). ([5]) Derivatively we have: $v((x)Cx, w) = T$ iff $v(Ct, w) = T$ for every term t in Q(w); $v(\Diamond A, w) = T$ iff $v(A, w') = T$ for some $w' \in K$; $v(A \mathbin{-\!3} B, w) = T$ iff for every $w' \in K$ either $v(A, w') = F$ or $v(B, w') = T$. A formula A is true in a model $<K, Q, v>$ iff $v(A, w) = T$ for every $w \in K$; and a formula is valid iff true in all models.

The Barcan formula is valid if for every model $<K, Q, v>$, $Q(w) = Q(w')$ for every $w, w' \in K$; when this condition is not satisfied the Barcan formula is invalid (but each axiom of Kripke's (1963) version of quantified S5 is valid). The following construction yields a model in which the Barcan formula is false. Let $<\{w_1, w_2\}, Q, v>$ be a model such that $Q(w_1) = \{a\}, Q(w_2) = \{b\}$, a and b are distinct, $v(\Phi a, w_1) = T$, $v(\Phi b, w_1) = F$ and $v(\Phi b, w_2) = F$. Here $v(\exists x \Diamond \Phi x, w_2) = F$ and, since $v(\exists x \Phi x, w_1) = T$, $v(\Diamond \exists x \Phi x, w_2) = T$.

The paper concludes with some remarks on the treatment of atomic formulae. As with Kripke's treatment of atomics (1963, p. 66) there is no requirement in the foregoing semantics that an atomic formula

$Ft_1 \dots t_n$ be true in a world only if all of $t_1, \dots, t_n$ are amongst the denoting terms associated with that world. However, Kripke remarks in a footnote (op. cit.) that it is more natural to assume an atomic formula is *false* in a world in which some or all of its terms do not refer. But Kripke offers no argument for the correctness of this assumption, which prima facie seems to succumb to counterexamples. The sentence «France needs de Gaulle» is arguably atomic and arguably true, yet the term «de Gaulle» does not denote an actually existing object. Be that as it may, the assumption under discussion can readily be incorporated into the foregoing semantics by rewriting the clause for atomics in the definition of v as: $v(Ft_1 \dots t_n, w) = F$ if $\{t_1, \dots, t_n\} \not\subseteq Q(w)$, and otherwise $v(Ft_1 \dots t_n, w)$ is arbitrary. The construction given earlier continues to yield a model which falsifies the Barcan formula.

Amended in this way the semantics validates formulae of the form $\exists x \Box Cx \supset \Box \exists x Cx$ when (and only when) Cx is atomic. Some writers have expressed doubts about such formulae: for example, Hintikka suggests that even if it were admitted for the sake of argument that every wheel is necessarily round, one would not wish to say that «since there happen to be wheels in existence, the existence of round objects is ... necessary and unavoidable». (1961, p. 124) However, if by «A is necessarily true» one means «A is true in all possible worlds» then these formulae are completely unproblematic. To reject the validity of the formulae one must have in mind a different account of necessity, perhaps «A is true in all possible worlds in which all terms occurring in A denote». Obviously such a concept of necessity may be adopted in the foregoing semantics by rewriting the clause for $\Box$ in the definition of v as: $v(\Box A, w) = T$ iff $v(A, w') = T$ for every $w' \in K$ such that all terms occurring in A are members of $Q(w')$. No formula of the form under discussion is then valid. However, it is noteworthy that the logic of this $\Box$ is non-normal, the semantics no longer validating the Gödel-Feys-von Wright axiom $\Box(A \supset B) \supset (\Box A \supset \Box B)$. [6]

(1) All references in this paragraph are to Barcan Marcus (1962).

(2) An assortment is presented in Copeland (1978).

(3) See, for example, Hughes and Cresswell (1968), p. 171.

(4) In matters of syntax I follow Kripke (1963).

(5) The notation Cx, Ct presupposes the usual conventions concerning the accidental binding of variables.

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