MODAL TREES FOR MODAL PREDICATE LOGICS *

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This paper describes a method adequate for testing consistency, and hence validity, in the common modal predicate logics. The method is an extension of that given by R. Jeffrey in [1] for the classical sentential and predicate calculi and that given by B. Davidson, F. Jackson and R. Pargetter in [2] for the common modal sentential logics. Familiarity with the basic tree method will be assumed in the subsequent discussion. The systems for which the procedure will be described are LPC + T, T + BF and LPC + S5 of [3]. The method can be adopted for the systems B, S4 and their predicate extensions but will not be explicitly given here.

Syntax is standard. However, in applying the method, we assign all the predicate symbols contained in wffs occurring in a tree some superscript. In any one wff, all predicate symbols must be assigned the same superscript. Thus, every wff occurring as a full line in a tree has an associated superscript. When a wff, Φ, is associated with a superscript, i, Φ is said to be of degree i; written Φ^i. We say that the wff *corresponding to* Φ^i is just the wff, Φ, obtained by deleting all superscripts.

The method is applied to a set of (initial) wffs to test them for consistency. The rules of inference for the non-modal connectives are as given in [1] but with superscripts added. For example, where Φ^i and ψ^i range over wffs of degree i:

(V): $\Phi^i \vee \psi^i$

$\quad\quad\quad\quad\diagup\diagdown$

$\quad\quad\Phi^i \quad\quad \psi^i$

In stating the rules for modal operators and quantifiers, «Φ^i» ranges over wffs of degree i, «β» ranges over individual variables and «α» and «*b*» range over individual constants. «$\Phi^{i\,\beta}/\alpha$» will represent «Φ^i with all free occurrences of β replaced by α». The modal rules for LPC + T, T + BF and LPC + S5 differ with respect to the rules for the necessity and possibility operators, ($\Box$) and ($\Diamond$) respectively. The rules ($\sim\Box$) and ($\sim\Diamond$) apply to all three systems:

$$(\sim\Box): \quad \frac{\sim\Box\Phi^i}{\Diamond\sim\Phi^i} \qquad\qquad (\sim\Diamond): \quad \frac{\sim\Diamond\Phi^i}{\Box\sim\Phi^i}$$

The rules ($\Diamond$T) and ($\Box$T) apply to both systems LPC + T and T + BF while the rules for LPC + S5 are ($\Diamond$S5) and ($\Box$S5).

($\Diamond$T): $\Diamond\Phi^i$
|
Φ^j where, if Φ^i occurs as a full line in the path above Φ^j, j = i; and if not, j > i and j does not occur previously in the path.

($\Box$T): $\Box\Phi^i$
|
Φ^j where j = i or j is the degree of some wff above Φ^j in the path obtained by an application of ($\Diamond$T) to a wff of degree i.

($\Diamond$S5): $\Diamond\Phi^i$
|
Φ^j where, if Φ^k occurs as a full line in the path above Φ^j for some k, j = k; and if not, j > i and j does not occur previously in the path.

($\Box$S5): $\Box\Phi^i$
|
Φ^j for any j occurring in the path.

The rules ($\exists$), ($\sim\exists$) and ($\sim\forall$) are identical for all three systems:

($\exists$): $\exists\beta\Phi^i$
|
$\Phi^{i\beta}/\alpha$ where $\alpha = b$ if $\Phi^{i\beta}/b$ occurs as a full line above

$\Phi^{i\beta}/\alpha$ in the path; otherwise α is new to the path.

$$(\sim\exists):\quad \frac{\sim\exists\beta\Phi^i}{\forall\beta\sim\Phi^i} \qquad\qquad (\sim\forall):\quad \frac{\sim\forall\beta\Phi^i}{\exists\beta\sim\Phi^i}$$

The rule for ($\forall$) for LPC + T differs from that for T + BF and LPC + S5. The rule for LPC + T is ($\forall$T) while that for T + BF and LPC + S5 is ($\forall$S5):

$$(\forall T):\quad \frac{\forall\beta\Phi^i}{\Phi^{i\beta}/\alpha}$$

for all α such that α occurs in a wff of degree i in the path; or α occurs in a wff of degree j in the path *and* a wff of degree i was obtained by an application of ($\Diamond$T) to a wff of degree j in the path; or, if no such individual constants occur in the path, α is new to the path.

$$(\forall S5):\quad \frac{\forall\beta\Phi^i}{\Phi^{i\beta}/\alpha}$$

for all α such that α occurs in the path; or if no individual constants occur in the path, α is new to the path.

The testing procedure is to list the initial wffs, assigning them all degree zero. Apply the rules of inference to lines in the tree according to the following principle and in the order indicated by the flow chart:

> If Φ^i occurs as a line in a tree apply the relevant rule of inference by writing the list(s) of conclusions of that rule at the bottom of every path in which Φ^i occurs. Check (√) Φ^i unless the rule applied was ($\Box$) or ($\forall$). If the rule applied was ($\Box$), write the superscript assigned to the conclusion next to Φ^i and check that superscript. If the rule applied was ($\forall$), write the individual constant that occurs in the conclusion as the result of this application next to Φ^i and check that constant. A rule of inference may not be applied to an already checked line; a line with a checked superscript may not have ($\Box$) applied to it to yield a wff

with the degree of that superscript; and a line with a checked individual constant may not have (∀) applied to it to yield a wff containing that individual constant in place of the relevant variable.

The flow chart, being a modification of that given in [1], is as follows: –

1. List the initial wffs all with degree zero

2. Close any path that contains an atomic wff and its denial. Do all paths close?
   NO — YES
   Stop The initial wffs are inconsistent.

3. Can any rule for the connectives, modal operators or their denials, (~∃) or (~∀) be applied to any wff in an open path?
   NO — YES → Apply it.

4. Is there a wff to which (∃) can be applied?
   NO — YES → Apply it

5. Is there a wff to which (∀) can be applied?
   NO — YES → Apply it

6. Have any significant changes been made in the tree since stage 2 was last entered?
   NO — YES

7. Stop. The initial wffs are consistent.

This method is adequate in the sense that it will produce a tree with all paths closed if and only if the set of initial wffs is inconsistent. It is not, of course, a decision procedure since infinite trees can be formed for some sets of consistent wffs and no purely mechanical procedure can be specified which will identify all such infinite trees. Hence we define a finished tree as one such that all possible applications of the rules have been made, that is, every line is such that either no rule applies to it, or for each rule that applies to it, there is a line below it obtained by that rule from it. In practice, potentially infinite trees are identified and «stopped» if further applications of the rules will not bring about any *significant* change in the tree; for example, when further applications of the rules will result merely in a *continuing* recurrence of wffs differing only in the individual constants they contain. This of course is the same procedure followed for standard predicate trees in [1]. A path in a finished tree is *closed* if it contains both an atomic wff and its denial (with the same superscript) as full lines; and *open* otherwise. A set of initial wffs is inconsistent if and only if the corresponding finished tree contains no open paths.

The method can be used to show, for example, that «$\sim(\forall x \Box Fx \supset \Box \forall x Fx)$» is consistent in LPC + T:

1.	$\surd$	$\sim(\forall x \Box F^{o}x \supset \Box \forall x F^{o}x)$	initial wff.
2.	$\surd b$	$\forall x \Box F^{o}x$	from 1 by $(\sim\supset)$
3.	$\surd$	$\sim\Box \forall x F^{o}x$	from 1 by $(\sim\supset)$
4.	$\surd$	$\Diamond\sim\forall x F^{o}x$	from 3 by $(\sim\Box)$
5.	$\surd$	$\sim\forall x F^{1}x$	from 4 by $(\Diamond T)$
6.	$\surd$	$\exists x \sim F^{1}x$	from 5 by $(\sim\forall)$
7.		$\sim F^{1}\alpha$	from 6 by $(\exists)$
8.	$\overset{\surd}{o}\ \overset{\surd}{1}$	$\Box F^{o}b$	from *2* by $(\forall T)$
9.		$F^{o}b$	from 8 by $(\Box T)$
10.		$F^{1}b$	from 8 by $(\Box T)$

This completed tree has an open path, and hence the initial wff is consistent in LPC + T. But notice that this wff is not consistent in T + BF or LPC + S5. The tree for these systems consists of lines 1-7 above and:

8′.	$\overset{\surd}{1}$	$\Box F^{o}\alpha$	from 2 by $(\forall S5)$

9′.	$F^1\alpha$	from 8′ by (□S5)
10′.	X	by 7 and 9′.

All paths close in the T + BF and LPC + S5 tree, since (∀S5) permits the step from 2 to 8′, but not in the LPC + T tree since (∀T) does not.

Adequacy: The proof of adequacy outlined here is of the same general form as that given in [1]. It is given in terms of possible world semantics. Interpretations for LPC + T, T + BF and LPC + S5 satisfy the requirements specified in [3] for LPC + T –, T + BF – and LPC + S5 – models respectively.

Adequacy is proved by proving

(A): A set of sentences is consistent if and only if the finished modal tree constructed on these sentences is not closed.
It is sufficient to prove both (B) and (C).

(B): The interpretation described by an open path in a finished tree is one which makes all the initial wffs true in the corresponding world;

where an interpretation which makes the wff corresponding to a wff of degree i true (false) in w_i is said to make that wff true (false) in the corresponding world.

(C): If there is an interpretation which makes all the initial wffs of a tree true in some given world then there is an open path through the finished tree.

Let I_0 be the interpretation described by an open path in a finished tree. I_0 is defined thus:–

a. $W = \{w_i$: a wff of degree i occurs in the path$\}$
b. R is a dyadic relation satisfying for LPC + T and T + BF:
   (i) $w_i R w_i$ for all $w_i \in W$; and
   (ii) $w_i R w_j$ whenever a wff of degree j is obtained as a full line by an application of (◇T) to a wff of degree i in the path,
   while for LPC + S5:
   $w_i R w_j$ for all w_i, $w_j \in W$.
c. D is the set such that $D = \{\alpha : \alpha$ occurs in the path$\}$
d. Q is a function from W to D satisfying for LPC + T: for all $w_i \in W$, $Q(w_i) = D_i = \{\alpha : \alpha$ occurs in a wff of degree i in the path or occurs in a wff of degree j in the path where $w_j R w_i\}$

while for T + BF and LPC + S5:
$Q(w_i) = D_i = D$ for all $w_i \in W$.

e. The valuation function V satisfies for each of the systems: for all $w_i \in W$,
   I. $V(\alpha, w_i) = \alpha$ for each $\alpha \in D_i$.
   II. for every sentence letter (o-adic predicate), p, that occurs in the path $V(p, w_i) = T$ whenever p^i occurs as a full line in the path; otherwise $V(p, w_i) = F$.
   III. for every monadic predicate symbol, F, that occurs in the path, $V(F, w_i) = \{\alpha : \alpha \in D_i$ and $F^i\alpha$ occurs as a full line in the path$\}$
   IV. for every n-adic predicate symbol, G, that occurs in the path, $V(G, w_i) = \{<a_1, \ldots, a_n> : a_1, \ldots, a_n \in D_i$ and $G^i a_1 \ldots a_n$ occurs as a full line in the path$\}$

To prove (B) assume its negation.

It is a feature of the rules of inference that if the wffs corresponding to all the conclusions of a rule are true in the corresponding world on the interpretation described by the path in which they occur, then the wff corresponding to the premise of the rule is true in the corresponding world on that interpretation. Consider, for example, ($\forall$T). If Φ^β/α is true in w_i on I_o for all α satisfying the conditions stipulated by that rule, that is, for all $\alpha \in D_i$, then, by the valuation rules for I_o, $\forall\beta\Phi$ will be true in w_i on I_o. Similarly for all the rules. Hence, by contraposition, if I_o makes the wff corresponding to the premise of a rule false in the corresponding world then the wff corresponding to at least one of the conclusions of that rule is false in the corresponding world on I_o.

By assumption, I_o makes at least one initial wff of the tree false in the corresponding world.

If the line in the tree corresponding to this initial wff *has not* been used as the premise of a rule of inference then it must be an atomic wff or the denial of one of degree zero. If this line *has* had a rule of inference applied to it then the wff corresponding to at least one of its conclusions that occurs in the open path will be false in the corresponding world on I_o. This conclusion will be either an atomic wff or the denial of one of some degree, or will itself have been used as the premise of a rule of inference. If the latter, the wff corresponding to at least one of *its* conclusions that occurs in the path will be false in the corresponding world on I_o and will be either an atomic wff or the denial of one of some degree or will itself have been used as the premise of a rule of inference.

Another feature of the rules of inference is that their continued application results in shorter wffs occurring as full lines in any path. This feature, together with the flow chart, ensures that every full line in a finished tree is either an atomic wff or the denial of one of some degree or has had all possible applications of the rules made to it. Hence the above reasoning continues until eventually the wff corresponding to an atomic wff or the denial of one occuring as a full line in the path is false in the corresponding world on I_o. This contravenes the conditions on I_o since these conditions are such as to ensure that the wff corresponding to an atomic wff or the denial of one occurring as a full line in a path is true in the corresponding world in I_o. Hence (B) is proved.

Note that the proof holds for infinite as well as finite trees.

(C) is proved by proving:

(D): If after n applications of the rules there is a path, P_n, and an interpretation, I, such that (i) all wffs corresponding to full lines of P_n are true in some world on I, and (ii) wffs corresponding to wffs of the same degree are true in the same world on I; then there is a path, P_{n+1}, and an interpretation, I′, such that this is true after n + 1 applications.

When the (n + 1)st application is of a non-modal rule or of ($\sim\Diamond$), ($\sim\Box$), ($\sim\exists$) or ($\sim\forall$), (D) is obviously true taking I = I′ and P_{n+1} to be P_n plus the appropriate list of conclusions.

If the (n + 1)st application is of ($\Box$T), P_n must contain a full line of the form $\Box\Phi^i$. Let P_{n+1} be P_n plus the conclusion Φ^j say added to P_n as the result of this application. If j = i, (D) is true since every interpretation which makes $\Box\Phi$ true in a world also makes Φ true in that world. If $j \neq i$, P_n must contain a line ψ^j, obtained by ($\Diamond$T) from $\Diamond\psi^i$. Hence I must assign *T* to $\Diamond\psi$ and $\Box\Phi$ in the same world and, further, must assign *T* to Φ in *every* world accessible to that world, and *T* to ψ in some world accessible to that world. Hence I assigns *T* to Φ and ψ in the same world. Φ and ψ are wffs corresponding to wffs of the same degree, so again (D) is true with I = I′ and P_{n+1} as given.

Again similar remarks apply if ($\Diamond$S5) is the rule applied.

If the (n + l)st rule applied was (∃) then P_n must contain a full line of the form $\exists\beta\Phi^i$. Let P_{n+1} be P_n plus the conclusion, $\Phi^{i\,\beta}/\alpha$ say, added to P_n as the result of this application. If $\Phi^{i\,\beta}/\alpha$ already occurs as a full line in P_n prior to this application then (D) is true with P_{n+1} as given and I = I′. If not, α is new to the path. Since $\exists\beta\Phi$ is true in w_i on I, I makes Φ^{β}/α true in w_i for some α with a referent in the domain of w_i. Hence P_{n+1} is such that the wffs corresponding to all full lines of P_{n+1} are true in some world on I and all wffs corresponding to wffs of the same degree are true in the same world on I. So (D) is true.

Finally, if the (n + l)st application was of (∀T) then P_n must contain a full line of the form $\forall\beta\Phi^i$. Take P_{n+1} to be P_n plus the conclusion $\Phi^{i\,\beta}/\alpha$ say, added to P_n as the result of this application. Either (a) P_n contains a wff of degree i in which α occurs; or (b) P_n contains a wff of degree j in which α occurs and a wff of degree i was obtained as a full line in P_n by an application of (◇T) to a wff of degree j; or (c) α is new to the path and neither (a) nor (b). I makes $\forall\beta\Phi$ true in w_i, hence Φ^{β}/α is true in w_i on I for all α which refers to an element in the domain of w_i on I. α does so refer if either (a) or (b) occur. If (c) occurs then α so refers since the domain of w_i is non-empty. Hence Φ^{β}/α is true in w_i on I and (D) is true with P_{n+1} as given and I = I′.

Similarly if (∀S5) is the rule in question.

Hence, by induction, if after no applications of the rules of inference there is a P_o and an I such that (i) all wffs corresponding to full lines of P_o are true in some world on I; and (ii) wffs corresponding to wffs of the same degree are true in the same world on I, this is true after n applications of the rules, for all n. If P is such a resulting path through a finished tree then P must be open since an atomic wff and its denial are not assigned *T* in the same world on any interpretations. But P_o is just the initial lines of the tree, so (C), and hence adequacy, is proved.

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NOTES

* I am indebted to Robert Pargetter, Frank Jackson and Barry Taylor for their many helpful remarks and suggestions.

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