VON WRIGHT'S «AND NEXT» VERSUS A SEQUENTIAL TENSE-LOGIC

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0. *Introduction*

A general theory of action presupposes a logic of change. A change in time takes place when the present state of affairs p is transformed into a state of affairs q, say, where q = p is not excluded: p then continues to obtain. In order to capture the formal properties of «p change into q», G.H. von Wright introduced a tense-logical binary constant T which was meant to play the rôle of a non-commutative connective, and pTq is read as «p and next q». The resulting calculus, the T- calculus, was presented as a formalized axiomatic structure in [19], whose axioms are the following:

A0. Any set of axioms of classical two-valued Propositional Logic (PL).

A1. $(p \vee qTr \vee s) \leftrightarrow (pTr) \vee (pTs) \vee (qTr) \vee (qTs)$. Distributivity.

A2. $(pTq) \vee (rTs) \leftrightarrow (p \wedge rTq \wedge s)$. Co-ordination.

A3. $p \leftrightarrow (pTq \vee \neg q)$. Redundancy.

A4. $\neg(pTq \wedge \neg q)$. Impossibility.

The rules of inference for the T-calculus are

R1. Substitution (of T-expressions for variables)

R2. Detachment (i.e. modus ponens)

R3. Extensionality (i.e. intersubstitutability of provably equivalent T-expressions).

The T-calculus has subsequently played an important part also in studies in deontic logic (v. Wright [20], Åqvist [1]) and has been complemented by a «logic of the past» (Clifford [4]). More important from a logician's point of view, its *soundness* and *completeness* have been demonstrated by means of the method of semantic tableaux (Åqvist [1]) or by the normal-form method already suggested in v.Wrights pioneering paper «And next» in [19]. Furthermore, L. Åqvist has attempted a solution for «Chisholm's Puzzle of Contrary-to-Duty Imperatives» via his deontic tense-logical system DDT.

In this paper, however, we are only interested in the formal, not the philosophical, aspect of the T-calculus. Nevertheless, we hold that the philosophical impact of a theory is enhanced by a convincingly adequate and *cogent* set of axioms rather than a merely intuitively appealing formalization which otherwise is more or less arbitrary. We content that the T-calculus is still too arbitrary, too general and not uniquely determined by its axioms; it is, in short, not canonically adapted to its underlying tense-logical ideas. Moreover, one would like to see a realization of the T-calculus where, e.g. the non-commutativity of T is shown by examples in that realization, not stipulated a priori. Does the T-calculus have a model besides PL where T collapses into classical conjunction; is, in other words, the T-calculus *consistent* and *new*? Are the axioms *independent?* Or is there an axiom which has to be omitted because it «leads to counterintuitive results» as was the case with the axiom of associativity (pTq) Tr $\leftrightarrow$ pT (qTr) (cf. the footnote on page 297 of [19]).

In proposing a «sequential logic »S» we shall in fact only retain analogues to A3. and A4. as trivial consequences. In addition to A3. we *will* have an analogue to p $\leftrightarrow$ qvq$^{\perp}$Tp, which is *not* a theorem in the T-calculus. Distributivity and co-ordination will *not* be valid in our sequential calculus S which is also not associative and not commutative.

S is closely connected to quantum probability theory (viz. the «conditional probability operator»); it has been interpreted in the context of «counterfactuals»; and S plays a major rôle in non-Boolean lattices, Baer-Semi-Groups and general quantum logic.

At last we shall show how the S-calculus can be used to provide necessary and sufficient conditions for qvq$^{\perp}$Tp to be a necessity operator and for the T-calculus to be nothing but classical PL.

1. von Wright's «and next» is a Boolean homomorphism

Because of axiom A0 we may assume that the «states of affair»p, q,... are elements of a Boolean lattice $L = L_B$, which contains, together with p and q, also $p \wedge q$, $p \vee q$, $\neg p$, $\neg p \vee q$ (written as $p \supset q$), $p \mathrm{T} q$, and finite combinations of these expressions. Let us write $q^{\perp}$ for $\neg q$, and abbreviate $1 := q \vee q^{\perp}$, $0 := q \wedge q^{\perp}$. 1 is the largest, 0 is the smallest element of L_B. We shall rewrite here v. Wright's theorems T5 and T6 in terms of the above notation, using one further abbreviation:

$$f(p) := 1 \mathrm{T} p$$

Then T5. $q \wedge f(p) = q \mathrm{T} p$

T6. $f(p) \mathrm{T} q = f(p \wedge q)$.

From these equalities follows

T7. $f(p) \wedge f(q) = f(p \wedge q)$

and from A1 we get

T8. $f(p) \vee f(q) = f(p \vee q)$.

Also, from v. Wright's theorem T2 and his discussion at the bottom of p. 296 in [19] we have:

T9. $f(p^{\perp}) = f(p)^{\perp}$

and hence T10. $f(0) = 0$

T11. $f(1) = 1$.

In summa, we see that T defines a homomorphism $f(p) = 1 \mathrm{T} p$ in L_B. Conversely, it is an easy exercise to verify that every homomorphism f in L which satisfies T7 and T8 defines a binary connective T through

$$1 \mathrm{T} p := f(p)$$

which in turn also satisfies A1 through A4. In other words:

(1.1) Theorem: von Wright's T defines a Boolean homomorphism in L_B; and conversely, every Boolean homomorphism defines a T à la von Wright's calculus.

This theorem expresses the fact that there are as many possible connectives «and next» as there are homomorphism in L_B, and none of the (possibly infinitely many) T's has an edge over the others.

The notation $f(p) = 1 \mathrm{T} p$ makes it especially transparent what commutativity and associativity of T would mean: T is commutative if and only if for all p and q in L_B the equality $p \wedge f(q) = q \wedge f(p)$ holds.

This implies, for $q = 1$, that $p = f(p)$ for every p, so that from T6 we infer

$$pTq = p \wedge q,$$

i.e., T is nothing but the classical PL-conjunction $\wedge$. Associativity appears in f-notations as

$p \wedge f(q) \wedge f(r) = p \wedge f(q \wedge f(r))$, which for $p = q = 1$ becomes

$$f(r) = f(f(r)),$$

i.e., f is idempotent. Interpreted in terms of v. Wright's «and next»-language, this idempotency would mean that after the change of r into $f(r) = 1\,Tr$ (the state «next to r»), nothing more will happen to f(r) (since $f(r) = f(f(\ldots(r))\ldots)$ for any finite number of repetitions of f). This is certainly «conterintuitive» to a concept of change.

If, moreover, f is injective, associativity entails $f(r) = r$ for all $r \in L_B$, and so again, as with commutativity, T boils down to $\wedge$.

Let us investigate the question: when is f injective? In order to lay down the ground for the more general discussion in Sections 2 seq. we assume that L is a (not necessarily Boolean) orthomodular lattice and $f: L \to L$ a homomorphism: f fulfills as axioms

T8. $f(p) \vee f(q) = f(p \vee q)$

and T9* $p \perp q \Rightarrow f(p) \perp f(q)$

($r \perp s$ is the symmetrical predicate defined by $r \leq s^{\perp}$ or by $s \leq r^{\perp}$).

From T8 and T9* follows

T10. $f(0) = 0$

since $f(0) \perp f(0)$. f is also isotone:

if $p \leq q$, then $f(p) \leq f(p) \vee f(q) = f(p \vee q) = f(q)$.

Finally, since $f(p) \perp f(p^{\perp})$, we have

$f(1) \wedge f(p)^{\perp} = [f(p) \vee f(p^{\perp})] \wedge f(p)^{\perp} = f(p^{\perp}) \wedge f(p)^{\perp} = f(p^{\perp})$, so that, *if* $f(1) = 1$, we get

T9. $f(p^{\perp}) = f(p)^{\perp}$.

But *when* is $f(1) = 1$? Certainly when f is an epimorphism because then there is a $p \in L$ such that $f(p) = 1$; consequently $f(1) \geq f(p) = 1$ from isotony, which gives $f(1) = 1$. If T9 is valid, then we have also

T7. $f(p) \wedge f(q) = f(p \wedge q)$.

More generally, T7 holds for every homomorphism f such that $f(1) = 1$, as the following Lemma shows, which also answers our initial question: when is f injective?

(1.2) Lemma: Let $f: L \rightarrow L$ be a homomorphism such that $f(1) = 1$. Then

(1) $f(p \wedge q) = (p) \wedge f(q)$ for all $p, q \in L$

(2) f is injective (a «monomorphism») if and only if $f(p) = 0$ implies $p = 0$.

Proof (1) $f(p \wedge q) = f(1) \wedge f(p^{\perp} \vee q^{\perp})^{\perp}) = f(1) \wedge f(p^{\perp})^{\perp} f(1) \wedge f(q^{\perp})^{\perp}$

$= f(p) \wedge f(q)$.

(2) From $f(p) = f(q)$ we have
$f(p \wedge (p \wedge q)^{\perp}) = f(p \wedge (p^{\perp} \vee q^{\perp})) = f(p) \wedge (f(p^{\perp}) \vee f(q^{\perp}))$

$= f(p) \wedge f(q^{\perp})$ from T9 and orthomodularity

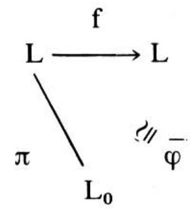
$= f(1) \wedge f(p) \wedge f(q^{\perp})$

$= f(1) \wedge f(p) \wedge f(p)^{\perp} = 0$.

Therefore $p \wedge (p \wedge q)^{\perp} = 0$. Since $p \wedge q \leq p$, it follows that $p = p \wedge q$, and so $p \leq q$. Similarly $q \leq p$, so $p = q$.

To sum up: Every homomorphism f of an orthomodular lattice such that $f(1) = 1$ defines a tense-logical v. Wright-connective T. If $f(p) = 0$ (i.e., $\neg(1\,Tp)$) implies $p = 0$ (i.e., impossibility $\neg p\,T1$) and if T is associative, then $T = \wedge$, and the T-calculus collapses into classical PL, which latter also happens when T is commutative.

Remark: If an epimorphism $f: L \rightarrow L$ is not injective, f may at least be factorized through the so-called canonical projection π such that the diagram



commutes: $f = \bar{f}\pi$.

L_0 is the orthomodular lattice which consists of equivalence classes, defined by the equivalence relation

$$p \sim q :\Rightarrow f(p) = f(q),$$

and $\bar{f}$ is injective and surjective (an «isomorphism»).

Thus, if f is not injective (but still surjective), substitute L_0 for L and $\bar{f}$ for f, and L_0 is still a truthfull (viz. isomorphic) image of L.

In terms of v. Wright's interpretation of T it seems natural anyhow that, if «next p» is false for every p ($\neg(1\,Tp)$) then p itself is false – which is just injectivity of f.

2. *Residuated maps*

We assume L to be an ortholattice, and f is an automorphism of L. In this section we shall discuss the classical conjunction $\wedge$, then f and our sequential connective π (defined below) in terms of the unifying concepts of residuated maps.

Let us begin with the PL-conjunction $\wedge$, and let $L = L_B$ be Boolean. Then, for any p, q, $r \in L_B$,

$$\text{MP.} \quad p \wedge (p \supset q) \leq q$$

$$\text{MAX.} \quad p \wedge r \leq q \Rightarrow r \leq p \supset q,$$

where $\supset$ denotes «material» implication: $p \supset q := p^{\perp} \vee q$, and $\Rightarrow$ stands for the usual meta-logical «if ... then».

MP is of course the *modus ponens* of PL, and MAX together with MP define $p \supset q$ uniquely as the maximal element r fulfilling MAX.

We see therefore, that the map $\varphi_p(r) := p \wedge r$ is an example of a residuated map according to the following definition.

(2.1) Definition: A mapping $\varphi: L \to L$ is residuated if

(1) φ is isotone, and

(2) if $q \in L$, then the subset

$$\{r \in L \mid \varphi(r) \leq q\}$$

possesses a greatest element.

The automorphism f of $L = L_B$ also is a residuated map: f is isotone (see section 1), and the maximum of $\{r \in L_B \mid f(r) \leq q\}$ is $r_{max} = f^{-1}(q)$. Therefore, also $r \to p \wedge f(r) = p\,T r$ is, for every fixed p, a residuated map with $f^{-1}(p \supset q)$ being the greatest element of $\{r \in L_B \mid p \wedge f(r) \leq q\}$ as we have seen above. If f is the identity, $\varphi_p(r)$ and $p\,T r$ coincide; but for more general f there is more than one residuated map in a Boolean lattice L_B.

What if L is orthomodular, but not Boolean? In this important case, the map φ_p given by

$$\varphi_p(r) := p \wedge (p^\perp \vee r)$$

defines a residuated map, and the corresponding maximal element is $r_{max} = p^\perp \vee (p \wedge q)$. Thus, writing $\varphi_p(r) =: p \pi r$ and $r_{max} =: p \to q$ we introduce the sequential connective π (read: «and then») and a so-called conditional implication or quasiimplication $\to$ by the stipulation: $p \to q$ is the unique solution of MP_s and MAX_s:

$$MP_s \quad p \wedge (p \to q) \leq q$$

$$MAX_s \quad p \wedge r \leq q \Rightarrow r \leq p \to q.$$

For this quasiimplication $\to$, the following remarkable uniqueness property holds:

(2.2) Theorem (Mittelstaedt) In an orthomodular lattice L there exists for any two elements $p, q \in L$ one and only one element $p \to q$ satisfying the modus ponens MP_s and QI: $p \wedge r \leq q \Rightarrow p^\perp \vee (p \wedge r) \leq p \to q$ and which is given by

$$p \to q = p^{\perp} \vee (p \wedge q) = \varphi_p(q^{\perp})^{\perp}.$$

Proof [11], Theorems 2.28 and 2.33, p. 38-39. Notice also ibid. p. 41, Theorem 2.40 (b) which shows that $p\,\pi\,(p \to q) = p \wedge (p \to q)$. What is so special about the conditions MP_s and QI is the fact that they are the strongest conditions which can be postulated for $p \to q$ as the existence of $p \to q$ already implies the quasi-modularity of L:

(2.3) Theorem (Mittelstaedt) An ortholattice L with the property that for any two elements $p, q \in L$ there exists an element $p \to q$ satisfying MP_s and QI, is orthomodular.

Proof [11] Theorem (2.36), p. 40 and Theorem (2.9), p. 31. The quasiimplication can, on the other hand, also been viewed as an implication in an ortholattice L subject to the following three minimal implicative conditions:

a. if p semantically entails q then $p \to q$ is true, i.e.

$$p \leq q \Rightarrow p \to q = 1;$$

b. modus ponens:

$$p \wedge (p \to q) \leq q;$$

c. modus tollens:

$$q^{\perp} \wedge (p \to q) \leq p^{\perp}.$$

It can be shown that only three implications or «conditionals» C_1, C_2, C_3 satisfy the conditions. They are given by (Hardegree [7]):

$$C_1(p, q) = p^{\perp} \vee (p \wedge q)$$

$$C_2(p, q) = (p^{\perp} \wedge q^{\perp}) \vee q$$

$$C_3(p, q) = (p \wedge q) \vee (p^{\perp} \wedge q) \vee (p^{\perp} \wedge q^{\perp}).$$

We see that $C_1(p, q) = \varphi_p(q^{\perp})^{\perp} = p \to q$. C_1 is sometimes called the Sasaki arrow. For fixed p, the function $C_1(p,.)$ is usually written

$\varphi_p^+(\cdot)$ i.e.,

$$\varphi_p^+(q) := \varphi_p(q^{\perp})^{\perp}.$$

The Sasaki arrow φ_p^+ is dual to the Sasaki projection φ_p (which is our π-sequential (function) and named «residual» (for more information see [5]).

The following uniqueness property for the Sasaki arrow, i.e. for our $r_{max} = p \rightarrow q$, provides another uniqueness proof for the Sasaki projection φ_p:

(2.4) Theorem (Hardegree) $\varphi_p^+(\cdot) = C_1(p, \cdot)$ is the only conditional in L that is both residual and locally Boolean.

Proof. Hardegree [7].
(To say that a conditional is locally Boolean is to say that it agrees with the classical PL-material implication $\supset$ on all Boolean sub-ortholattices $L_B \subset L$.)

3. The sequential calculus S

We took pains (and the reader's patience) in stating the above uniqueness results using residuated maps as a unifying thread in order to motivate and justify the introduction of the connective π («and then») as the building block of a tense logic different from Prior's [13] and v. Wright's T-calculus. To fix ideas and also to have a sufficiently rich model at hand we choose as our propositional lattice the ortholattice $L = L_H$ of closed subspaces of a complex Hilbert space H.

This choice sounds rather special; but Piron [12] and Jauch [8] have shown, following Mackey [10] and Birkhoff – von Neumann [3] that L_H is *the* realization of non-classical logic at least for physical systems: L_H is not distributive, and so a fortiori not Boolean. Closed subspaces, or equivalently their projections P, Q,... represent «measurable properties» of a (physical) system «in action», and the intrinsic non-commutativity of projections, e.g. $PQ \neq QP$ unless P and Q are «jointly measurable» without interfering with each other, makes L_H the canonical candidate for the object language of a theory of change and action.

There is a superficial technical hitch, however, in the fact that the product PQ of projections is in general not a projection anymore, and no propositional (or physical) meaning can be given to these products.

This seeming disadvantage has been overcome as early as 1936 by John v. Neumann in his famous projection rule ([9] p. 200-201): he took the self-adjoint operator («observable») PQP as the (unnormalized) conditional probability operator for a physical system in a state ψ to have the property Q after the (earlier) property P had obtained. The corresponding probability for this «joint» event is given by the inner product $<\psi, PQP\psi>$ in H, and the corresponding proposition, also called yes-no event by Jauch, is the projection $P\pi Q$ onto the closed range of PQP. It can indeed been shown that this $P\pi Q$ coincides with our earlier residuated map $\varphi_P(Q)$:

$$\varphi_P(Q) = P \wedge (P^{\perp} \vee Q) = P\pi Q$$

is the projection onto range of PQP.

Reading $P\pi Q = P \wedge (P^{\perp} \vee Q)$ as *P and from P follows materially Q* provides a certain justification for the suggested metalinguistic expression «P and then Q».

The next step is the introduction of «or then» via

$$P \amalg Q := (P^{\perp} \pi Q^{\perp})^{\perp}$$

and «if (first) ... then»

$$P \rightarrow Q := P^{\perp} \amalg Q$$

which is nothing but the above quasiimplication or Sasaki arrow. Beltrametti and Cassinelli read $P \rightarrow Q$ as follows:
«$P \rightarrow Q$ is true iff either $P^{\perp}$ is true or the occurrence of the yes-outcome of P leaves the system in a state that makes true Q. In other words, it is true that P quasi-implies Q iff either $P^{\perp}$ is true or the conditional probability of Q given P is equal to 1». ([2], p. 378).

Although this is not the place to study these new connectives in detail (see Rehder [14]), we must mention a few differences to v. Wright's «and next»:

L_H is *not* a lattice with respect to $\pi, \amalg, \leq$;

φ_P is *not* a homomorphism with respect to π or $\amalg$ (f is !);

L_H is *not* a commutative propositional system although there are several suggestive conditions for $P\pi Q = Q\pi P$ to hold (the physical condition being commensurability; the probabilistic condition being

$PQP = QPQ$, i.e., the respective conditional probabilities are the same).

Furthermore, our sequential system $S = (L_H, \pi)$ displays a curious duality between the logical constants $\wedge$ and π:

$$P \pi Q = P \wedge (P^{\perp} \vee Q)$$

$$P \wedge Q = P \pi (P^{\perp} \rotatebox[origin=c]{180}{\(\pi\)} Q) .$$

It need not be stipulated but *can be proven* that the laws of associativity, commutativity and contraposition ($P \rightarrow Q = Q^{\perp} \rightarrow P^{\perp}$) are not valid.

As for completeness of the proposed sequential logic as well as its semantical foundation by Lorenzen's dialog-games see Stachow's [16] and also his recent survey Logical Foundation of Quantum Mechanics [17].

For the philosophical relevance of $\rightarrow$ as a formalization of counterfactual implication see v. Fraassen [6], Stalnaker and Thomason [18], and Hardegree [7].

4. F as a necessity operator, and the case $f = id$

We have seen in section 1 that associativity of T implies idempotency of $f(p) = 1\,Tp : f(p) = f(f(p))$. On the other hand, always $f(p \vee q) = f(p) \vee f(q)$. In this context it would be interesting to know under what condition $f(p) \leq p$ is valid, for then the associative homomorphism f is a S4 necessity operator.

In fact the following Theorem is true even without f being idempotent.

(4.1) Theorem If $f : L \rightarrow L$ is a homomorphism is an ortholattice L and $p \in L$, then (1) and (2) below are equivalent:

(1) $f(p) \leq p$

(2) $f \circ \varphi_p = \varphi_p \circ f \circ \varphi_p$

Proof: If $f(p) \leq p$ then $\varphi_p \circ f \circ \varphi_p(q) = p \pi f(p \pi q) = f(p \pi (p \pi q)) = f(p \pi q) = f \circ \varphi_p(q)$ for all $q \in L$.

Note that f is a homomorphism with respect to π !
Conversely, if (2) holds then

$$f(p) = f \circ \varphi_p(p) = \varphi_p \circ f \circ \varphi_p(p) \leq p,$$

since

$$\varphi_p(p) = p \text{ and } \varphi_p(r) \leq p \text{ for every } r \in L.$$

Can we interpret the criterion in (4.1) (2)? Put $q := p$, then (2) says $f(p) = p\pi f(p)$, i.e. if we look at 'p at the next moment' ($= f(p)$) then this is the same as looking at 'p, and then next p' ($= p\pi f(p)$). So we do not know more of 'p at the next moment' by knowing about the present of p as well: knowledge of p is redundant – or implicit in the knowledge of $f(p)$.

In this instance it may be tempting to suggest an interpretation of $f(p)$ as 'p is necessary', which is the interpretation of f as a necessity operator.

With the help of Theorem 4.1 it is easy to characterize the identity among all automorphisms f of L. If $f(p) = f(f(p))$ (from associativity of T) we know already that then f is idempotent.

The following Theorem is more general:

(4.2) Theorem Let $f: L \to L$ be an automorphism of the orthomodular ortholattice L and $p \in L$. Then $f(p) = p$ if and only if $f \circ \varphi_p = \varphi_p \circ f$.

Proof: For necessity, since $f(p) \leq p$, we have by (4.1) (2) that $f \circ \varphi_p = \varphi_p \circ f \circ \varphi_p$.

Since $f^{-1}(p) \leq p$ we also have $f^{-1} \circ \varphi_p = \varphi_p \circ f^{-1} \circ \varphi_p$.

Taking the involution * on both sides gives $(f^{-1} \circ \varphi_p)^* = \varphi_p \circ f$, $(\varphi_p \circ f^{-1} \circ \varphi_p)^* = \varphi_p \circ f \circ \varphi_p$ (here we have used the fact that in an orthomodular ortholattice $\varphi_p^* = \varphi_p$ holds).

For sufficiency, $f \circ \varphi_p = f \circ \varphi_p \circ \varphi_p = \varphi_p \circ f \circ \varphi_p$, so by Theorem 4.1 $f(p) \leq p$.

Since $f^{-1} \circ \varphi_p = \varphi_p \circ f^{-1}$, we also have $f^{-1}(p) \leq p$, so $p \leq f(p)$ which completes the proof.

If v. Wright's $f(p) = 1\,Tp$ claims to capture the main features of «change», it must be tested first, how f fares as a mapping in the propositional system L_H of projections in a complex Hilbert space H, because where else is «change» as essential as in physics, or its «object»: Nature? So let us assume f to be an automorphism of L_H. Then the following amazing fact is true: there exists at least one proposition ($\neq 0,1$) in L_H, which does not change at all under f, i.e., $f(P) = P$. I do not know if this is a welcome result for the T-calculus or another counterintuitive consequence.

5. *Conclusion*

It was the limited purpose of this paper to outline the general framework of mathematization in which an adequate tense logic may be formulated. Sequential logic, I claim, is a legitimate competitor of v. Wright's T-calculus, and I have given some mathematical (but almost no philosophical) reasons for this claim.

A sound principle in philosophy is, that a good idea must be expressible in a formal language; an extraneous formalization, however, is but a fancy ornament.

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