A NOTE ON LEŚNIEWSKI AND FREE LOGIC

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The aim of this note is to correct a misconception which may arise from a paper by Karel Lambert and Thomas Scharle(1) in which systems of free logic as they have been developed in the past quarter century or so in America are compared with the logic, in particular the Ontology, of Leśniewski. I shall draw out some consequences for what I believe is a correct view of the relationship between free logic and Ontology.

Lambert and Scharle provide a system of rules for translating from a system FL^t of free logic to the first order fragment(2) L4′ of a Leśniewskian system L4, formulated by Czesław Lejewski(3). They show that the translation of every axiom of FL^t into L4′ is a theorem of L4′, and that the translation of every axiom of L4′ into FL^t is a theorem of FL^t; further that the equivalence of the two translations is provable in both systems, so they are isomorphic. They claim that the translation

> provides for the first time, so far as we know, a way of interpreting at least the first order fragment of one version of Leśniewski's system called *Ontology* in more general parlance(4).

The claim is false, because L4 is not the full system of Ontology, but rather a proper fragment of it, so that L4′ is not the first order fragment of Ontology, but only a proper fragment of this. The nominal terms (nominal constants and nominal variables) of L4 are intended to be so interpreted that a term either designates a single individual on an interpretation, in which case we call it a *singular* term (using this expression for the semantic property of a term, not its syntactic category), or else it designates nothing at all on an interpretation, in which case we call it an *empty* term. No other possibilities are envisaged in L4. However, in Leśniewski's Ontology, a third semantic possibility for terms exists, namely that a term designate more than one individual on an interpretation, in which case we call it a *plural* term (where again this terminology is semantic rather than syntactic). Thus in Ontology two different binary semantic divisions may be made among interpreted terms: empty versus designating (i.e., singular and plural taken together) and plural versus non-plural (i.e. empty and singular taken together).

To elucidate further the relationship between American free logic and Leśniewskian Ontology we introduce the following notation in Ontology: we let a, b, c, d be nominal variables, the sentential connectives be $\sim$, $\wedge$, $\vee$, $\supset$, $\equiv$, which are presumed to bind, unless parentheses dictate to the contrary, in decreasing order of strength as listed. We use '[ ]' and '[∃]' for Leśniewskian quantifiers, and mark quantifier scope by upper corners. Universal quantifiers whose scope is the whole of a formula are conventionally omitted. The single primitive predicate is Leśniewski's 'ε', which is understood as follows: '$a\,\varepsilon b$' is true if and only if 'a' designates exactly one thing, and 'b' also designates this thing, and perhaps in addition others besides. The single intuitive axiom for Ontology is

$$\text{LA1}\quad a\,\varepsilon b \equiv [\exists c]\ulcorner c\,\varepsilon a\urcorner \wedge [c]\ulcorner c\,\varepsilon a \supset c\,\varepsilon b\urcorner \wedge [cd]\ulcorner c\,\varepsilon a \wedge d\,\varepsilon a \supset c\,\varepsilon d\urcorner$$

and Ontology is further provided with rules for manipulating the quantifiers, and presupposes Protothetic, a propositional calculus with quantifiers binding propositional and functorial variables. We omit this background here for brevity (5). We define several predicates as follows (where we follow Leśniewski's practice of expressing definitions as equivalences, but mark their status with a 'D' in the nomenclature):

LD1	$a \circ b$	$\equiv [c]\ulcorner c\,\varepsilon a \equiv c\,\varepsilon b\urcorner$	(Identity)
LD2	$\mathrm{E}a$	$\equiv [\exists c]\ulcorner c\,\varepsilon a\urcorner$	(Existence)
LD3	$!a$	$\equiv [bc]\ulcorner b\,\varepsilon a \wedge c\,\varepsilon a \supset b\,\varepsilon c\urcorner$	(Non-plurality)
LD4	$\mathrm{E}!a$	$\equiv \mathrm{E}a \wedge !a$	(Singular existence)
LD5	$a = b$	$\equiv a \circ b \wedge !a$	(Non-plural identity)
LD6	$a \doteq b$	$\equiv a\,\varepsilon b \wedge b\,\varepsilon a$	(Singular identity)
LD7	$a \Box b$	$\equiv a \circ b \wedge \mathrm{E}a$	(Designating identity)

The notation here is slightly divergent from that normally used by

Leśniewskian logicians, and is chosen to harmonise with Lambert and Scharle's paper.

Lejewski's system L4 *can* be seen as that obtained from this system by stipulating that every term is non-plural, i.e. by adding as axiomatic the formula

LF1 $b\,\varepsilon a \wedge c\,\varepsilon a \supset b\,\varepsilon c$ (*i.e.*: $[a]\ulcorner !a \urcorner$)

The effect of assuming LF1 is to collapse the distinction between 'ε' and '$\doteq$', as can be seen by proving, assuming LF1:

LF2 $a\,\varepsilon b \supset b\,\varepsilon a$

We give a somewhat abbrieviated natural deduction proof:

1. $a\,\varepsilon b$	Assumption
2. $[\exists c]\ulcorner c\,\varepsilon a \urcorner$	1, LA1
3. $[c]\ulcorner c\,\varepsilon a \supset c\,\varepsilon b \urcorner$	1, LA1
4. $[cd]\ulcorner c\,\varepsilon a \wedge d\,\varepsilon a \supset c\,\varepsilon d \urcorner$	1, LA1
5. $[\exists c]\ulcorner c\,\varepsilon b \urcorner$	2,3
6. $[cd]\ulcorner c\,\varepsilon b \wedge d\,\varepsilon b \supset c\,\varepsilon d \urcorner$	LF1
7. $c\,\varepsilon b$	Assumption
8. $c\,\varepsilon b \wedge a\,\varepsilon b \supset c\,\varepsilon a$	LF1
9. $c\,\varepsilon a$	1,7,8
10. $[c]\ulcorner c\,\varepsilon b \supset c\,\varepsilon a \urcorner$	7-9
11. $b\,\varepsilon a$	5,6,10,LA1
12. $a\,\varepsilon b \supset b\,\varepsilon a$	1-11

If follows immediately that

LF3 $a\,\varepsilon b \equiv a \doteq b$

The system obtained by adding LF1 to Ontology allows also the proof of the following equivalences (proofs are similarly simple):

LF4 $\mathrm{E}a \equiv \mathrm{E}!a$
LF5 $a \circ b \equiv a = b$
LF6 $\square b \equiv a \doteq b$

Alternatively, if Ontology is not to lose some of its expressive power by the addition of LF2, we can embed L4 directly in an extension of Ontology formed by adding a second run of non-plural variables x, y, z,... for which LA1, LD1-7 hold, and for which we add the axiom

LA2 $[x]^{\ulcorner}[\exists a]^{\ulcorner} x = a^{\urcorner\urcorner}$

from which follow the restrictions of LF1-6 to non-plural variables. Then also every formula in which a non-plural variable occurs is equivalent to one in which only normal Leśniewskian variables occur. This formula will in general be more complicated, since it will have to state explicitly the non-plurality condition «built in» to non-plural variables by virtue of LA2. So every formula of FL^t is also equivalent to one of Ontology, but the converse is not the case.

The differences between Lambert and Scharle's system FL^t and Lejewski's L4′ turn on two basic points: the differences between the quantifiers and the choice of '=' or '≐' as basic identity predicate (the former, but not the latter, is totally reflexive). The function of the Lambert-Scharle translation is to show that the differences, despite their possible philosophical significance, are not logically significant: the two systems are beating about the same logical bush(5). That L4′ is a proper fragment of first order Ontology can be seen by the fact that Ontology + LF1 has theorems which Ontology lacks (such as LF2-6), since there are interpretations of Ontology which falsify these, the simplest being a two-membered domain in which at least one name is singular and one is plural.

The interpretation of L4′ which we have given conforms with the intentions of Lejewski's paper. However another, different interpretation within Ontology is possible, namely that obtained by ignoring the difference between singular and plural terms, and concentrating only on that between empty and designating terms. This fragment consists of all theorems of first order Ontology which contain no subformulas whose truth may depend on whether the terms they contain are singular or plural, which means omitting all formulas containing the symbols 'ε', '!', 'E!', '=' and '≐' from the system we have given, and retaining only those containing 'o', 'E' and '□'. The results of Lambert and Scharle may be repeated by rewriting '=' as 'o', '≐' as '□' and 'E!' as 'E'. This is also a proper fragment of first order Ontology.

This tells us something about the functors of Ontology, in particular about identity. If an identity predicate I is one for which the Leibnizian formula

$$a \, \mathrm{I} \, b \equiv [\Phi]^{\ulcorner} \Phi a \equiv \Phi b^{\urcorner}$$

holds, then in Ontology only 'o' is an identity predicate, and '=', '≐' and '□' are restrictions of the identity predicate. Similarly in FL^t or L4' it is '=' rather than '≐' which is the identity predicate. Now in the fragment of Ontology containing only 'o' as predicate, no distinctions may be made between existence and non-existence, or between singularity and plurality. By adding 'E' we can distinguish existence from non-existence, but not singularity from plurality, which is why the o,E fragment effectively allows of two interpretations; where the terms are singular or empty on the one hand, and where they are designating or empty on the other. The full system of Ontology based on 'ε', or any other functor of equivalent strenghth, allows us to further distinguish singularity from plurality. The fact that two divergent interpretations of the o, E (or =, E!) fragment are possible shows by Padoa's method (7) that 'ε' cannot be defined in terms of 'o' and 'E', and hence that it cannot be defined in terms of 'o' alone. This tells us that the economy of Ontology in using only 'ε' as primitive means that the three different notions of identity, existence and singularity are dealt with at a single blow, whereas it might be philosophically more perspicuous to introduce the notions successively. Indeed identity ('o') alone is barely a concept of Ontology at all: – it may indeed be defined using 'ε', but, given that Leśniewski's apparatus of quantifiers is already present, identity may be *defined* by

LD8 $a \circ b \equiv [\Phi] \ulcorner \Phi a \equiv \Phi b \urcorner$

which shows that it can be dealt with by Leśniewski without introducing any new primitive notions of Ontology at all: all that is required is to introduce variables in the new syntactic category of names, and the functor categories resulting therefrom, and allow these too to be subject to the quantifier rules.

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NOTES

(1) Karel Lambert and Thomas Scharle, «A translation theorem for two systems of free logic», *Logique et Analyse* 10 (1967), 328-341.

(2) The first order fragment of a system is here understood as that wherein only nominal variables are bound by quantifiers. Normally such variables are singular only, but in Leśniewski they need not be.

(3) Czeslaw Lejewski, «A theory of non-reflexive identity and its ontological ramifications», in: P. Weingartner, ed., *Grundfragen der Wissenschaften und ihre Wurzeln in der Metaphysik,* Salzburg: Pustet, 1967, pp. 65-102.

(4) Lambert and Scharle, *op. cit.*, p. 330.

(5) On Leśniewski's logic generally, cf. E. Luschei, *The logical systems of Leśniewski,* Dordrecht: North-Holland, 1962.

(6) For a system which beats about the same bush as ontology in distinguishing singular from plural terms, but whose quantifiers are more like those of American free logic, cf. my «Plural reference and set theory», in: B. Smith, ed., *Parts and Moments,* Munich: Philosophia, forthcoming.

(7) Cf. P. Suppes, *Introduction to Logic*, Princeton: Van Nostrand, 1957, p. 169 ff. I owe the suggestion that Padoa's method is useful in connection with Ontology to Prof. Lejewski.