TYPES OF NON-SCOTIAN LOGIC

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During the last thirty years, beginning with Jaskowski's paper of 1948, several systems of logic have been proposed which are currently classified as 'paraconsistent' (for a bibliography, see Arruda [1980] and Marconi [1979]). The notion of paraconsistency is still quite vague. N. da Costa and his school, who designed many systems of 'paraconsistent logic' and spread both the use of the name and the interest in the topic, mean by 'paraconsistent system of logic' a system which can be used as a suitable underlying logic for inconsistent, non-trivial theories (see Alves [1976], Arruda [1977]). If we regard a theory as a triple T = {L, C, A}, where {L, C} is a deductive system in the sense of Rasiowa and Sikorski [1970] and A is a (possibly empty) set of non-logical axioms, we can say that T is trivial if every formula of L is a theorem of T; otherwise, T is non-trivial. T may be said to be inconsistent in either of two cases: if for some formula A of L both A and ~A (the formula which is intended as the negation of A) are theorems of T; or if for some A, A & ~A is a theorem of T (where A & B is intended as the conjunction of A and B). The two cases do not necessarily coincide, though they do coincide in any theory based on classical logic.

Just for the purposes of the following discussion I shall label inconsistencies of the first kind 'weak', and inconsistencies of the second kind 'strong'. These words have no mathematical meaning in the present context. They hint at the intuitions of certain philosophers (e.g. Jaskowski [1948], Rescher [1979]) according to whom inconsistencies of the second kind are somewhat less 'tolerable' than those of the first kind.

Some more terminology. Let us call 'Scotian' a system of logic C in L iff it allows us to derive every formula of L from a contradiction, or from a pair of contradictory formulas. The obvious reference is to the classical principle – known as the principle of pseudo-Scotus – according to which from a contradiction everything follows. A system of logic C will then be called 'non-Scotian' if it does *not* allow us to derive every formula from a contradiction or from a pair of contradictory formulas: 'strongly non-Scotian' in the former case, 'weakly non-Scotian' in the latter. In a strongly non-Scotian system, the rule $\frac{A \& \sim A}{B}$ fails, i.e. it is neither primitive nor derived; in a weakly non-Scotian system it is $\frac{A, \sim A}{B}$ which fails.

Now, if a system C is (for instance) Scotian in that it allows us to derive every formula from a contradiction and the theory T, based on C, is strongly inconsistent then T is trivial. In other words C cannot be used as a basis for a (strongly) inconsistent, non-trivial theory, so that according to the intuitive definition C is not paraconsistent. A paraconsistent logic cannot be Scotian: it must be non-Scotian, either strongly or weakly depending on whether we want it not to trivialize a strongly or a weakly inconsistent theory.

One may wonder whether the converse is also the case, i.e. whether every non-Scotian logic is intuitively paraconsistent. If this were so, the precisely definable notion of non-Scotian logic could be used to explicate the intuitive notion of paraconsistent logic. And indeed, one of the founding fathers of what is nowadays called paraconsistent logic, S. Jaśkowski, appears to have identified the two problems, of finding a suitable logical basis for inconsistent theories and of finding a system of logic which would not allow the derivation of any and every formula from a contradiction, i.e. a non-Scotian logic. «The task – Jaśkowski wrote – is to find a system of the sentential calculus which: 1) when applied to contradictory systems would not always entail their over-completeness, 2) would be rich enough to enable practical inference, 3) would have an intuitive justification» (Jaskowski [1969], p. 145). Disregarding conditions (2) and (3), which characterize any interesting system of logic, and given that Jaskowski's 'overcompleteness' is the same as our triviality we can see that his intuition was that 'the problem of the logic of contradictory systems', as he called it, would be solved by any non-Scotian logic.

However, another father of paraconsistent logic (N. da Costa) has objected to this explication of paraconsistency by pointing out that there are systems, such as minimal intuitionistic logic, which do not permit the derivation of every formula from a contradiction and yet could not be considered as intuitively paraconsistent, for in such systems we can derive *the negation* of every formula from a contradiction. According to da Costa's intuitions, this should not happen in a paraconsistent logic. Presumably, the idea is that a paraconsistent system should not justify any derivation which appeals – in one way or another – to the alleged 'untenability' of contradiction: it ought not to be possible to derive a formula from a contradiction *just because* it is a contradiction. Moreover, an inconsistent theory in which every negative formula (though not every formula) is a theorem is not trivial *stricto sensu*, but comes very close to being trivial. As a theory it is uninteresting for it excludes everything (even though it does not assert everything). A similar case could be made against systems which derive every conditional formula from a contradiction, etc.

If these intuitions look sound, as they do to me, the simple concept of a (weakly or strongly) non-Scotian logic must be ruled out as an explication of the idea of paraconsistent logic. The failure of the rule $\frac{A \,\&\, \sim A}{B}$, or of the rule $\frac{A, \sim A}{B}$, is a necessary but not a sufficient condition of paraconsistency.

On the other hand, the intuitions which led us to repudiate non-Scotianism as an *explicatum* of paraconsistency are not so easy to capture in a simple definition. It is fair to say that the problem of an adequate explication of paraconsistency is still open. In what follows, I shall set down some elementary results concerning non-Scotian logic. Secondly, I shall attempt a taxonomy of the existant paraconsistent systems based on the different ways in which strong non-Scotianism, weak non-Scotianism, and a third feature which I shall label 'thesis-non-Scotianism' can combine among themselves. This is clearly possible, for all paraconsistent systems are either weakly non-Scotian, or strongly non-Scotian, or both. Before introducing such a discussion of non-Scotian systems, however, I shall briefly review some of the problems we have to face in looking for a better definition of paraconsistent logic.

As being non-Scotian is a necessary but insufficient condition of paraconsistency one obvious course of action would consist in strenghthening the conditions that make a system non-Scotian. Thus we could require that not only $\frac{A \,\&\, \sim A}{B}$ (respectively $\frac{A, \sim A}{B}$)

but $\frac{A \& \sim A}{\sim B}$ $(\frac{A, \sim A}{\sim B})$, $\frac{A \& \sim A}{B \rightarrow C}$ $(\frac{A, \sim A}{B \rightarrow C})$, etc. also fail. The problem with this solution is how to interpret 'etc.', i.e. where to end the list of the rules which must fail in a paraconsistent system. For instance, should we require the rule $\frac{A \& \sim A}{B \rightarrow (C \rightarrow D)}$ $(\frac{A, \sim A}{B \rightarrow (C \rightarrow D)})$ also to fail? For in a system lacking contraction (the rule $\frac{A \rightarrow (A \rightarrow B)}{A \rightarrow B}$) $\frac{A \& \sim A}{B \rightarrow (C \rightarrow D)}$ could hold in spite of the failure of $\frac{A \& \sim A}{B \rightarrow C}$. But then the temptation arises of generalizing the condition of paraconsistency by requiring the failure of every rule which would license the derivation from a contradiction of formulas which are irrelevant (in the sense of relevant logic) to that particular contradiction.

The main objection to this proposal is that many of the extant systems of paraconsistent logic have not been conceived in the spirit of relevant logic, and are not relevant. So, first of all, they have rules (such as $\frac{A \& \sim A}{B \rightarrow (A \& \sim A)}$) which do not meet the requirement of relevance. Secondly, it appears to be out of place to build into the definition of paraconsistency a concept which is foreign to many paraconsistent systems.

So it seems that there is no simple way of singling out which rules must fail in order for a system to be paraconsistent. Obviously we cannot characterize paraconsistent logic positively, by specifying which derivations from a contradiction are acceptable in a paraconsistent system. All sorts of different derivations from a contradiction are possible in different paraconsistent systems, which is how it should be. There is no end to the list of derivations which are acceptable in an intuitively paraconsistent system.

This discussion may perhaps be concluded by conjecturing that any criterion of paraconsistency which will be proposed will either rule out some of the existent systems as not paraconsistent after all, or it will consist in a strenghthening of one or the other condition of non-Scotianism. However, any particualar form of such strenghthening will be questionable, at least in principle.

Let us now turn to non-Scotian systems. As I anticipated, we shall add to the two categories of strongly and weakly non-Scotian systems the category of *thesis-non-Scotian* systems of logic. A system will be called 'thesis-non-Scotian' just in case the formula $(A \,\&\, \sim A) \rightarrow B$ is not one of its theorems. One obvious result is the following:

Th. 1. If a system C is strongly non-Scotian, then if it has the rule of simplification ($\frac{A \,\&\, B}{A}, \frac{A \,\&\, B}{B}$) C is weakly non-Scotian; if C is weakly non-Scotian, then if it has the rule of composition ($\frac{A,\ B}{A \,\&\, B}$) C is strongly non-Scotian; if C is thesis-non-Scotian, then if the deduction theorem (Σ, $A \vdash B \Rightarrow \Sigma \vdash A \rightarrow B$) holds for C, C is strongly non-Scotian; if C is strongly non-Scotian, then if it has the rule of modus ponens ($\frac{A, A \rightarrow B}{B}$) C is thesis-non-Scotian.

The proof is obvious.

The three concepts of non-Scotian systems which have been introduced so far are defined in syntactical terms. We can also define the corresponding semantical notions. If the language of C is L, we understand by a valuation a mapping v from the set of the formulas of L into a set of values, some of which are designated. We shall say that C is (semantically) strongly non-Scotian iff it is not the case that for all valuations v, $v\,(A \,\&\, \sim A) = T \Rightarrow v\,(B) = T$, where T is a designated value. C is (semantically) weakly non-Scotian iff it is not the case that for all v, $v\,(A) = v\,(\sim A) = T \Rightarrow v\,(B) = T$. Finally, C is (semantically) thesis-non-Scotian iff it is not the case that for all v, $v\,((A \,\&\, \sim A) \rightarrow B) = T$. We can easily prove the following:

Th. 2. If C is (semantically) strongly non-Scotian, then if it is the case that $(\forall v)(v\,(A \,\&\, B) = T \Rightarrow v\,(A) = v\,(B) = T)$ C is (sem.) weakly non-Scotian; if C is (sem.) weakly non-Scotian then if it is the case that $(\forall v)(v(A) = v\,(B) = T \Rightarrow v\,(A \,\&\, B) = T)$, C is (sem.) strongly non-Scotian; if C is (sem.) thesis-non-Scotian, then if it is the case that $(\forall v)(v\,(A \rightarrow B) \neq T \Rightarrow (v\,(A) = T \,\&\, v\,(B) \neq T))$ C is (sem.) strongly non-Scotian; if C is (sem.) strongly non-Scotian, then if it is the case that $(\forall v)(v(A) = T \,\&\, v(B) \neq T \Rightarrow v(A \rightarrow B) \neq T)$ C is (sem.) thesis-non-Scotian.

We can of course relate the syntactical and semantical concepts which were proposed above. Following the usual terminology, we shall call a system C *sound* provided $\Sigma \vdash_c A \Rightarrow (\forall v)\ ((\forall B \in \Sigma) v(B) = T \Rightarrow v(A) = T)$, where Σ can be empty. C will be called *complete* in case $(\forall v)\ ((\forall B \in \Sigma) v(B) = T \Rightarrow v(A) = T) \Rightarrow \Sigma \vdash_c A$. We can easily establish the following:

Th. 3. If C is (syntactically) strongly non-Scotian (weakly non-Scotian, thesis-non-Scotian), then if C is complete (with respect to a semantics) C is semantically strongly non-Scotian (weakly non-Scotian, thesis-non-Scotian) as evaluated in that semantics. If C is semantically strongly non-Scotian (weakly non-Scotian, thesis-non-Scotian) in a semantics, then if C is sound (with respect to that semantics) C is syntactically strongly non-Scotian (weakly non-Scotian, thesis-non-Scotian).

Clearly, a weakly non-Scotian system is not necessarily strongly non-Scotian or thesis-non-Scotian, and so forth. We can indeed define a typology of non-Scotian systems based on the possible combinations of the presence or absence of each of the three (syntactical) features we have been describing. If we exclude the case of systems which are thesis-non-Scotian, but neither strongly nor weakly non-Scotian (these would be non-Scotian in a very weak sense, and probably not very interesting altogether) we are left with six possibilities:

	I	II	III	IV	V	VI
Strongly n.-S.	+	+	+	+	−	−
Weakly n.-S.	+	+	−	−	+	+
Thesis-n.-S.	+	−	+	−	+	−

As we already remarked, a paraconsistent system must be either strongly or weakly non-Scotian, or both. Consequently, all the paraconsistent systems which have been proposed so far fit in the typology of above. We shall classify some of the best known systems of propositional logic, adding that the typology can easily be extended to cover predicative systems. Relevant logics are not considered, even though most relevant systems are paraconsistent (but not all of them:

see Routley and Meyer [1976], Marconi [1979], Arruda [to appear]).

As one would expect, most of the best known system are of type I: the systems of Asenjo and Tamburino [1975], Thomason [1974], Sette [1973]), the two systems of Routley and Meyer [1976], the system of D'Ottaviano and da Costa [1970], the systems $\{H_p\}$ of Arruda [1967] are all of type I. Da Costa's systems C_n, $1 \leq n < \omega$, are also of type I if contradictions of the appropriate form are considered for each n; otherwise they are not non-Scotian (see da Costa [1974]). Priest's system of [1979] is of type II; in fact, its theorems are all and only the classical tautologies, including the law of pseudo-Scotus considered as a formula. I know of no system of either type III or type IV. This is probably due to the fact that one hardly gives up the rule of simplification (case III) or simplification *and* modus ponens (case IV). Jaskowski's system of [1948] is of type V, if one identifies the pseudo-Scotus with the formula Cd Kd A NA B (which is not a theorem); it is of type VI if one identifies it with the formula Cd K A NA B (which is a theorem). Finally, Rescher's 'U-logic' of [1979] is of type VI, and possibly the weakest paraconsistent system constructed so far.

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